$f(18) = 108217$. A rather convincing example of this kind corresponds to $a = 5$, $H = 22$: here the smallest n for which the case $d = 1$ can be applied is $n = 204$, yielding $f(204) = 1732807009$, whereas Theorem 1 proves the irreducibility with $n = 30$, which gives $f(30) = 7 \cdot 118543$, and with more than forty other values of n less than 204.

Theorem 1 can also be applied if all values $f(n)$ for n in $\mathbb{Z}$ are divisible by a common factor $d > 1$. But in this case a straightforward transformation of $f(x)$ into a polynomial without this property may be more advisable than the direct application of the theorem. For instance, $d = 2$ divides all values of the polynomial $f(x) = x^4 + 9x^2 + 4$, and the fact that $f(17) = 2 \cdot 43063$ establishes its irreducibility. On the other hand, the substitution $x \mapsto 2x$ transforms this polynomial into $g(x) = 4x^4 + 9x^2 + 1$, which is irreducible because $g(7) = 2 \cdot 5023$.

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On a "Singular" Integration Technique of Poisson

Robert J. MacG. Dawson

Most undergraduates will at some point come across the famous trick, popularly associated with Gauss but attributed to Poisson by his contemporary Sturm [**4**, vol. 2, p. 16], for evaluating $\int_{-\infty}^{\infty} e^{-x^2}\,dx$:

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dy\,dx \\ &= \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,d\theta\,dr = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r\,dr \\ &= 2\pi \int_0^{\infty} e^{-u}/2\,du = \pi,\end{aligned}$$

which gives

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{1}$$

Lord Kelvin is said to have once told a class [**3**, p. 1139] that a mathematician was one to whom (1) was as obvious as "twice two is four" was to them. Probably few mathematicians would accept this compliment without at least a slight blush!

Nonetheless, Kelvin's words must certainly have added to the determination of many beginning mathematicians to understand this ingenious calculation thoroughly.

One question that must have occurred to many over the years is: What else can I do with it? The surprising answer to this natural question is: Absolutely nothing! Specifically, we have:

Theorem. *Let* $r(x, y) = \sqrt{x^2 + y^2}$. *Any Riemann-integrable function* f *on* $(-\infty, \infty)$ *such that* $f(x)f(y) = g(r(x, y))$ *for some* g *is of the form* $f(x) = ke^{ax^2}$.

The solution of this functional equation is not new (see, for instance, Aczél [**1**, sec. 2.2]). Its application to (1) is quite possibly not new either, but does not appear to be well known. In any case, the proof given here is elementary and a good example of a technique, sometimes known as "hill climbing," that can be used on other problems, both practical and recreational.

Proof of theorem. First, letting $y = 0$, we obtain $g(x) = f(0)f(x)$. Then for positive u and v we have

$$f(\sqrt{u})f(\sqrt{v}) = g(\sqrt{u+v}) = f(0)f(\sqrt{u+v}). \tag{2}$$

Setting $u = v = x/\sqrt{2}$ in (2), we get

$$f(x/\sqrt{2})^2 = f(x)f(0). \tag{3}$$

If $f(0) > 0$ we have $f(x) \geq 0$ for all x in $\mathbf{R}^+$. Similarly, $f(0) < 0$ implies that $f(x) \leq 0$ for all x in $\mathbf{R}^+$. If $f(0) = 0$, then f is identically 0.

Suppose that $f(0) \neq 0$ but that $f(a) = 0$ for some $a > 0$. By (3) we have $f(a/\sqrt{2}) = f(a/2) = \cdots = 0$, ensuring that there are arbitrarily small positive a for which $f(a) = 0$. Moreover, for any $x > a$, setting $u = x^2 - a^2$ and $v = a^2$ in (2) gives $f(x)f(0) = f(\sqrt{x^2 - a^2})f(a) = 0$, so $f(x) = 0$. We conclude that, if $f(a) = 0$ for any a whatsoever, f is identically 0.

The function f is thus always 0, always strictly positive, or always strictly negative on $\mathbf{R}^+$. In the first case, the result follows immediately, and the third case reduces to the second upon replacing f with $-f$. It thus suffices to consider the case in which $f(x) > 0$ for all x in $\mathbf{R}^+$.

Let $h(x) = \log(f(\sqrt{x}))$ for $x > 0$. Then (2) yields

$$h(x) + h(y) = h(0) + h(x + y).$$

This is essentially Cauchy's functional equation, the solution of which is well known. (For further discussion of this and related functional equations, the reader is referred to [**1**] or [**2**].) By induction, the restriction of h to any set of the form $\mathbf{N}\alpha$ with α a positive real number (hence also to any set of the form $\mathbf{Q}^+\alpha$) must have the form $h(x\alpha) = [h(\alpha) - h(0)]x + h(0)$. It follows that the restriction of f to any such set has the form $f(x) = \exp(ax^2 + b)$. As $f(x)f(0) = f(|x|) = f(-x)f(0)$ and $f(0) > 0$ by assumption, it follows that $f(x) = f(-x)$ for all x. Thus, $f(x) = \exp(ax^2 + b)$ on all of $\mathbf{Q}\alpha$.

Finally, we extend this to $\mathbf{R}$. Suppose to the contrary that there exist α and α' such that f restricts to $\exp(ax^2 + b)$ on $\mathbf{Q}\alpha$ and to $\exp(a'x^2 + b')$ on $\mathbf{Q}\alpha'$, where either $a' \neq a$ or $b' \neq b$. There are then at most two points, the solutions of $(a' - a)x^2 = (b - b')$, for which $\exp(ax^2 + b) = \exp(a'x^2 + b')$. For any interval $[c, d]$ avoiding such points, there exists $\epsilon > 0$ such that every subinterval of $[c, d]$ contains x in $\mathbf{Q}\alpha$

and x' in $\mathbf{Q}\alpha'$ with $|f(x) - f(x')| > \epsilon$. We conclude that upper and lower Riemann sums over this interval must differ by at least $\epsilon(d - c)$, with the consequence that $\overline{\int}_c^d f(x)dx \neq \underline{\int}_c^d f(x)\,dx$. . Therefore, such an f cannot be Riemann integrable over $[c, d]$, much less over $(-\infty, \infty)$. ■

Corollary. *The only nondegenerate definite integrals that can be evaluated by the process used in (1) are those of the form*

$$\int_{-\infty}^{\infty} ke^{ax^2}\,dx,$$

where k and a are constants and $a < 0$*.*

Proof. The only domains that are both "square" and "round"—that is, both of the form $\{(x, y) : a \le x \le b,\ a \le y \le b\}$ and of the form $\{(x, y) : x^2 + y^2 \le r\}$—are the empty set, the singleton $\{(0, 0)\}$, and $\mathbf{R}^2$. Moreover, $\int_{-\infty}^{\infty} ke^{ax^2}\,dx$ converges if and only if $a < 0$. ■

Attempts to generalize Poisson's method in other ways (for instance, by multiplying more copies of the original definite integral to obtain a triple or quadruple integral that can be evaluated using spherical polar coordinates) do not seem any more fruitful. It is surprising that so elegant a trick should have one, and only one, application; and it is gratifying that that one application should be one so vitally important to mathematics and statistics.

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Solution to Puzzle on Page 211

$$\binom{4.4}{4} \Big/ \sqrt{.\bar{4}}$$

School pupils may well object to counting the number of ways in which four objects may be chosen from $4\frac{2}{5}$ objects, but university students should be able to recognise that

$$\binom{4.4}{4} \Big/ \sqrt{.\bar{4}} = \frac{\Gamma(5.4)}{\Gamma(5)\Gamma(1.4)} \Big/ \frac{2}{3} = \cdots = \frac{22 \times 17 \times 12 \times 7}{10^4} = 3.1416.$$