Preparing for University Calculus

Prepared by the APICS Committee
on Mathematics and Statistics

Edited by Robert Dawson

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Disclaimer: This booklet is intended to give prospective students an idea of what a typical introductory calculus course at a university in the Atlantic region is like. Readers should note, however, that there are differences between calculus courses at different universities, and even within universities.

There is much important material in high school mathematics courses that is not reviewed in this booklet - to name only a few topics, probability, linear algebra,and coordinate geometry of conic sections. Omission of a topic from this booklet is not intended as an indication that it is of lesser importance than other topics.

This booklet is not intended as a textbook or achievement test. Moreover, we do not advocate any particular method of teaching the material contained herein.

Please send all error reports and suggestions for improvement to **robert.dawson@stmarys.ca**, or write to

CALCULUS BOOKLET
Dept. of Mathematics and Computing Science
Saint Mary's University
Halifax, Nova Scotia B3H 3C3

 For details see page 9.

1 About First-Year Calculus

1.1 What math will I need in university?

The answer to this depends on what you are taking, and you should check with the university or universities that you are thinking of attending. As a general rule, however, if you are planning to get a degree in science, you will probably need one or more courses in calculus (some universities permit programming or statistics instead in some cases). If you are taking courses in the social sciences or life sciences, you will probably need one or more courses in statistics; and degrees in physics, engineering, astronomy, mathematics, or computing science (and some other subjects) will require additional mathematics courses beyond the first year level.

Some departments (other than mathematics) may teach their own courses in calculus, linear algebra, or statistics. In this case, you may not need to take the standard first-year courses offered by the mathematics department.

1.2 What is calculus?

Calculus is a branch of mathematics that deals with rates of change. Its roots go back as far as Ancient Greece and China, but calculus as we know it today began with Newton and Leibnitz in the 17th century. Today it is used extensively in many areas of science.

Basic ideas of calculus include the idea of *limit*, *derivative*, and *integral*. The derivative of a function is its instantaneous rate of change, with respect to something else. Thus, the derivative of **height** (with respect to position) is **slope**; the derivative of **position** (with respect to time) is **velocity**; and the derivative of **velocity** (with respect to time) is **acceleration**.

The integral of a function can be thought of as the area under its graph, or as a sort of total over time. Thus, the integral of **slope** is (up to a constant) **height**; the integral of **velocity** is, up to a constant, **position**; and the integral of **acceleration** (with respect to time) is **velocity**. As you may have guessed, integrals and derivatives are related, and are in a sense opposites.

Now, many functions (though not all) can be represented by algebraic expressions. For instance, the area of a circle is related to its radius by the formula $A = \pi r^2$; and the distance that a body falls in a time t, starting at rest, is given by $x = 1/2at^2$. Given such an expression, calculus allows us to find expressions for the integral and derivative of the function, when they exist.

1.3 Why is calculus important?

In the sciences, many processes involving change, or related variables, are studied. If these variables are linked in a way that involves chance, and significant random variation, statistics is one of the main tools used to study the connections. But, in cases where a deterministic model is at least a good approximation, calculus is a powerful tool to study the ways in which the variables interact. Situations involving rates of change over time, or rates of change from place to place, are particularly important examples.

Physics, astronomy, mathematics, and engineering make particularly heavy use of calculus; it is difficult to see how any of those disciplines could exist in anything like its modern form without calculus. However, biology, chemistry, economics, computing science, and other sciences use calculus too. Many faculties of science therefore require a calculus course from all their students; in other cases you may be able to choose between, say, calculus, statistics, and computer programming.

It should be understood that there is more to mathematics than calculus. Linear algebra, probability, geometry, and combinatorics are just some of the other branches of mathematics, introduced in schools, that are important at the unversity level. And problem-solving skills, which cut across all branches of mathematics, allow the mathematics to be applied to other subjects.

1.4 What background do I need?

You should have taken Grade 12 precalculus mathematics, or an equivalent course, and understood the material. The second part of this booklet gives some examples of the sort of thing that you should be able to do.

You do *not* need a high school calculus course; if you have taken one, you should not assume that you can skip classes, or not study, during the first part of the course. University calculus goes deeper than most courses in high schools do, even if they appear to cover the same material. If you have been lucky enough to have a calculus course in high school, consider it as a preview of the upcoming feature that you are now about to see! And remember to keep using your algebra and other mathematical skills, to keep them sharp.

1.5 What is taking university calculus like?

You will probably find that university calculus is faster-paced than your high-school courses. At most universities the lecture sections will be bigger — possibly over a hundred students — and the professor will not be able to slow the class to the pace of the slowest students. You — and you alone — will be responsible for handing work in on time, and for being present for classes, tests, and exams.

This is not a course that can be passed by just memorizing everything; you have to understand it. This won't happen instantly, and the instructor cannot make it happen; you have to do that yourself, and be an active participant in the course. If you make this effort consistently through the term, you will find that it pays off.

The course material consists of a rather small number of big ideas, and a moderate number of formulae you will need to know, not hundreds of short cuts and special rules. A common mistake, especially with word problems, is trying to learn one "plug-in" rule for each different kind of problem that you might encounter. Don't do that; instead, **try to understand the underlying patterns.**

Here are the stages by which you will learn and master a new idea in calculus.

- First, you should **read ahead** in the textbook, so that you have at least a rough idea of what the instructor is going to say. This will help you follow the lecture and make note-taking easier.
- In the lecture, **the instructor will introduce the new idea,** give some examples, and perhaps explain how it fits in with other ideas or give some problem-solving tips.
- At this point, a few students think the process is over till the final exam and that if they don't know everything then, it will be the instructor's fault. Not so; the ball is back in your court! **Work on problems**, do the assignments (including additional study problems if you need to, whether assigned or not), and make sure that you understand what you are doing. It's better to work for an hour or so several times a week than in one killer session. **This is the heart of the learning process; the lecture is just to help you get started.**
- If you don't understand something, decide what it is that you don't understand and **go for help.** Go prepared to tell the instructor (or friend, tutor, learning

center person, etc.) what it is that you don't understand. ("Everything" is not helpful.) On the other hand, do not go along demanding that the instructor tell you how to answer Question 6.11 and refusing to listen to any explanation of why it's done that way.

- Finally, **your knowledge will be tested** on quizzes, midterms, or final examinations. The exams will probably make up the majority of your grade. If you have handed in copied, half-understood assignments, you will lose more marks at exam time than you gained in the short term. But if you kept up with the lectures and assignments, making sure you understood each bit before going on, you will probably do well.

If you find yourself falling behind, you will have to catch up. Don't panic - catching up is not impossible. There are various resources to help.

- **Yourself.** If you are falling behind and not putting plenty of time in studying - say five or six hours outside class per week *for each course* - the solution may be as near as your desk and textbook.

- **Your textbook** contains hundreds of worked examples, and thousands of problems. Usually, about half of the problems have answers in the back of the book; and you may be able to buy a study guide that shows the working of those problems in more detail. You can also get other books such as "Schaum's Outline of Calculus" containing more worked problems.

- Your instructor will have **office hours** during which you can go for help. Try to figure out ahead of time *what* you need help with; you will get more out of the visit.

- **Discuss** your problem with your classmates. This is not the same thing as copying their assignments, of course! University penalties for cheating are severe, and can include expulsion.

- You might want to hire a **tutor.** Do not try to get the tutor to do your assignments for you; you will not be able to bring the tutor into the exam with you! Get the tutor to make sure you *understand* the material.

- Your university may have a **math learning center**, group tutoring sessions, or other resources that you can use. Take advantage of them!

- Your university or student union may well have **workshops** on effective studying, effective note-taking, exam nerves, etc. There will also be counselling available for other problems that might interfere with your study.

1.6 HELP! I have to write a placement test!

Many Atlantic-area universities have found that incoming students are not uniformly well-prepared for calculus. Many students are very well prepared, and ready to start the usual calculus curriculum; but some are not. As a result, placement tests are common, and at some universities, if you do not pass the placement test, you will be required to take a remedial course covering important material from the high school curriculum **before you can take calculus.** At other universities you may just be advised to take the remedial course, or allowed into a slow-moving section that takes two semesters to cover the first semester's material.

This is to stop students from starting calculus without adequate preparation and then falling behind and failing. Experience has shown that most students who start calculus without adequate preparation do fail.

The test is often multiple-choice so that it can be graded as quickly as possible. It will generally be closed-book and written without a calculator (see the next section). Because success in first-year calculus requires a fairly high level of preparation, and because the test covers some very elementary (but important) material, the pass mark may be higher than 50%. You will probably have to write the test at or before the beginning of term; find out ahead of time what the rules are at the university you plan to attend.

Some sample questions for one such placement test may be found at `http://conway.math.unb.ca/Placement/`. Some harder questions, to help you see if you're *really* ready, are at `http://www.math.unb.ca/ready/exercise.html`.

1.7 HELP! They want to take my calculator away!

In some first-year calculus courses, calculators are not permitted. To be exact, they do not permit them in the pretest, tests, or exams; you can certainly have one in the lecture if you want, or use one for your homework. In other introductory calculus courses, calculators or computers are used intensively to explore the behaviour of functions. You should find out the policy at the university you plan to attend!

- If your calculus course has a no-calculator policy, the questions will be designed so that a calculator is not needed. It would be unreasonable for the professor to ask you to compute $(1.42433x + 2.4577)^3$ without a calculator. But (s)he won't. You might be asked to compute $(x+2)^3$; the calculator will not be needed here.

- In a situation where calculators are not permitted, any expression that does not have a well-known simplification may always be left as it is. Thus, for instance, while you are expected to know that $\sqrt{25} = 5$, and that $\log_{10} 1000 = 3$, you may always leave (eg) $\sqrt{17}$ or $\ln(1000)$ in those forms. Now, that is even easier than using the calculator! Do note, however, that you would generally be expected to simplify (eg) $\sqrt{17}/\sqrt{17}$ to 1, should such an expression arise! The use of algebra (as opposed to arithmetic) to simplify expressions will be common in your course. Conventions as to what should and should not be simplified will be made clear during the course.

- Even if your calculus course does allow calculators, you will often be required to give, as an answer, an algebraic expression involving integers. For instance, you might be asked: "Give the answer as an expression of the form $\sqrt{b}/c$ where b and c are integers." It is far more important, for the purposes of calculus, to know that $\sin(\pi/4) = \sqrt{2}/2$, **and why**, than to know that it is approximately equal to 0.707.

When you get into courses where there is a reason for test questions to involve significant amounts of arithmetic, you will be allowed — and encouraged — to use a calculator, perhaps even a computer. In the meantime, if you are not allowed to use a calculator in tests, you can be fairly sure that the arithmetic will be kept simple.

1.8 More Information

You can get further information about requirements and regulations at specific universities from their admissions offices, calendars, and Web sites.

2 Ordering information

Getting a copy free

Please note, this is the truly cool way to get this booklet (and saves us time).

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3 The Math You'll Need

In this section, we present over 175 questions that cover the absolute minimum of mathematical skills that you will need for university calculus. We give worked answers for a few; the answers to the rest are in the back of the booklet.

There are important topics in high school mathematics that are not covered here, and there may even be some things you'll need for calculus that we have missed. A good student should be able to answer significantly more challenging problems than these. However, if you understand all of this material, you will be reasonably well prepared for calculus.

3.1 Arithmetic

You should be able to do basic arithmetic without a calculator, including operations on fractions, negative numbers, and decimals. You should be able to compute simple powers and roots. This material, which is from the elementary and junior high school curriculum, is fundamental for everything that follows. Also, many arithmetic problems are really "algebra with numbers".

Examples:

1. Find $\frac{1}{3}+\frac{2}{5}$.

   Solution: To add (or subtract) fractions, you must first find a common denominator:
   $$\frac{1}{3}+\frac{2}{5}=\frac{1\cdot 5}{3\cdot 5}+\frac{2\cdot 3}{5\cdot 3}=\frac{5}{15}+\frac{6}{15}=\frac{11}{15}$$

2. Convert 1.25 to a fraction in lowest terms. **Solution:** To convert a decimal to a fraction, write as a fraction over a power of 10, then reduce.
   $$1.25=\frac{125}{100}=\frac{5\cdot 25}{4\cdot 25}=\frac{5}{4}$$

   (HINT: Memorizing some common equivalences between decimals and fractions, such as $1/2 = 0.5$, $1/3 = 0.333\ldots$, etc., could save you some time here!)

Problems: (answers on page 40)

1. Find $\frac{2}{3} + \frac{3}{5}$
2. Find $\frac{3}{4} - \frac{1}{5}$
3. Find $\frac{1}{3} \times \frac{6}{5}$ in lowest terms.
4. Convert 3.125 to a fraction in lowest terms.
5. Convert 37/25 to a decimal.
6. Convert 17/10 to a decimal.
7. Find $0.0001 \times 0.01/0.001$
8. Find 1.23×0.1
9. Find $1.005 + 9.995$
10. Find $1 - (-2)$
11. Find $\sqrt{64}$
12. Find $20^3/20^5$ as a decimal.

3.2 Basic Algebra

As observed above, basic algebra is closely related to arithmetic, and many of its rules are familiar as "rules of arithmetic". You should be aware that $a \times b$ may be written as ab.

You should know the basic rules for addition, subtraction, multiplication, division, and exponents, and be aware of the operations such as division by 0 and taking the square root of a negative number that cannot be done within the real number system.

You should know how to solve a simple equation, simplify an algebraic expression, and evaluate an expression by plugging values into it. All of these will be very important when you take calculus.

Examples:

1. Simplify $\frac{a^7b^3}{a^4b^4}$.
   Solution: By basic rules of exponents
   $$\frac{a^7b^3}{a^4b^4} = a^{7-4}b^{3-4} = a^3b^{-1} = \frac{a^3}{b}$$
2. Solve $ax + 4 = 2x - a$ for x when $a = 5$.
   Solution: Substituting $a = 5$ and adding, subtracting and dividing like quantities on both sides of the equation we reduce to an equation which is its own solution:
   $$\begin{aligned} 5x + 4 &= 2x - 5 \\ 5x + 4 - 2x &= 2x - 5 - 2x \\ 3x + 4 - 4 &= -5 - 4 \\ 3x\left(\frac{1}{3}\right) &= -9\left(\frac{1}{3}\right) \\ x &= -3 \end{aligned}$$
   so $x = -3$.

Problems: (answers on page 40)

1. Simplify $(a^4/a^2)^2$
2. Simplify $abc - acb + acd - ace$, factoring if possible .
3. Simplify $\frac{\frac{y}{x} + x}{\frac{2}{x}}$.
4. Simplify $(a^2 + a^3 + a^4)/a$, factoring if possible.
5. Solve $3x + 2 = x + 2$ for x.
6. If $x = 5$ and $y = 7$, find $x^2 - xy + 3y$.
7. Simplify $\frac{a^2 - (-a^2)}{a^2}$
8. If $a = 5$ and $b = 2$, find $a - b^2 + 2ab$
9. Solve $x^5 + 2x + 3 = x^5 - x$.
10. If $x = 3$ and $xy = 1$, find y.
11. Simplify $(a^2/a^{-2})(b^{-2}/b^2)$.
12. Solve $a + x + 2 = a - x + 4$.

3.3 Inequalities and absolute values

You should be able to solve simple inequalities and perform algebraic operations with them. In particular, you should know which operations reverse inequalities and which ones preserve them. You should understand interval notation, including open, closed, and half-open intervals, and intervals with limits at ∞. You should know how to compute an absolute value, and to do simple algebra using the absolute value function.

Examples:

1. Solve $\dfrac{7-2x}{3} \leq 4$.

   Solution: $\dfrac{7-2x}{3} \leq 4$ implies $7-2x \leq 12$ and $-2x \leq 5$.
   Thus $x \geq -\dfrac{5}{2}$, or $x \in [-\frac{5}{2}, \infty)$.

2. Solve $x^2 + 3x - 10 \leq 0$.

   Solution:

   Factoring the left side, $(x+5)(x-2) \leq 0$. The corresponding equation $(x+5)(x-2) = 0$ has solutions -5 and 2 which divide the real line into three intervals: $(-\infty, -5)$, $(-5, 2)$, $(2, \infty)$.

   On each of these intervals we determine the sign of the factors as follows:

Interval	$x+5$	$x-2$	$(x+5)(x-2)$
$(-\infty, -5)$	$-$	$-$	$+$
$(-5, 2)$	$+$	$-$	$-$
$(2, \infty)$	$+$	$+$	$+$

   From the table we find that $x^2 + 3x - 10 \leq 0$ on the interval $x \in [-5, 2]$.

3. Solve $|4x+5| > 9$.

   Solution: $|4x+5| > 9$ is equivalent to
   $4x+5 > 9$ or $4x+5 < -9$
   $4x > 4$ or $4x < -14$
   $x > 1$ or $x < -\frac{7}{2}$;
   which can be written as $x \in (-\infty, -\frac{7}{2}) \cup (1, \infty)$.

Problems: (answers on page 40)

1. If $y > x$, $x \geq w$, and $w < z$, which of the following must hold?

   (a) $y > w$, (b) $y < w$, (c) $y > z$, (d) $y < z$, (e) none of these.

2. For what values of x does $(x-2)^2 \geq 0$?

3. If $a > b$, can we conclude that:

   (a) $a^2 > b^2$ always; (b) $a^2 > b$ always; (c) $a^2 > b^2$ if $b > 0$;
   (d) $a^2 \geq b^2$ always; (e) none of these are true.

4. For what values of x is $1/(1+x) > -1$?

5. For what values of y is $y^2 > 0$?

6. For what values of y is $y^2 \geq 2$?

7. For what values of x is $|x-3| \leq 1$?

8. Which of the following intervals contains the point 0?
   (a)$(-\infty, 0)$ (b)$(-1, 1)$ (c)$(0, \infty)$
   (d)all of a,b,c (e) a and c only

9. Which of the following intervals contains the point 0?
   (a)$(-\infty, 0]$ (b)$[-1, 1]$ (c)$[0, \infty)$
   (d)all of a,b,c (e) a and c only

10. Find $|3 - |3-6||$

11. Simplify $x^2 - 2|x^2|$.

12. Solve $-1 < 2x - 5 < 7$.

13. Solve $x^2 - 3x + 2 > 0$.

14. Solve $|2x-3| \leq 5$.

15. Solve the equation $\frac{|x+3|}{|2x+1|} = 1$.

3.4 Functions

You should understand the concept of "function" and "inverse function" and know how to compute the composition of two or more functions. You should be able to determine the range and domain of a simple function. This will be important for understanding the Chain Rule, various methods of integration, and limits.

Examples:

1. Find the domain of the functions defined by:
   (a) $f(x) = \frac{1}{x^2-1}$ (b) $g(x) = \sqrt{4-x}$

   Solution

   (a) The domain of $f(x) = \frac{1}{x^2-1}$ is the set of all values of x such that $x^2-1 \neq 0$; that is $x \neq \pm 1$ or $x \in \{(-\infty, -1) \cup (-1, 1) \cup (1, \infty)\}$

   (b) The domain of $g(x) = \sqrt{4-x}$ is the set of all values of x such that $4 - x \geq 0$; that is $x \leq 4$ or $x \in (-\infty, 4]$.

2. If $f(x) = 3x + 5$, find $\dfrac{f(x+h) - f(x)}{h}$

   Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[3(x+h)+5]-[3x+5]}{h} = \frac{3x+3h+5-3x-5}{h} \\ &= \frac{3h}{h} = 3\end{aligned}$$

3. If $f(x) = \sqrt{x-1}$ and $g(x) = x^2$, find:
   (a) $f(g(x))$ (b) $g(f(x))$

   Solution

   (a) $f(g(x)) = f(x^2) = \sqrt{x^2-1}$; this has domain $(-\infty, -1] \cup [1, \infty)$.

   (b) $g(f(x) = g(\sqrt{x-1}) = (\sqrt{x-1})^2$
   This is $x - 1$ on the domain $[1, \infty)$ but is undefined elsewhere.

Problems: (answers on page 40)

1. If $f(x) = 1/x$, find the largest possible domain (within the real numbers) on which f could be defined.
2. If $f(x) = 1/(x+1)$, find the domain of f.
3. If $g(x) = x^2 + 2$, find $g(1+x)$
4. If $f(x) = x^2$, find the range of f.
5. Find the domain and range of f(x) $= \dfrac{|x|}{x}$
6. If $f(x) = \dfrac{1}{x+1}$, find (a) $f(x^2)$; (b) $f(-2)$; (c)$f(\sqrt{x})$.
7. If $f(x) = 1/x$ and $g(x) = x^2$, find $f(g(a))$.
8. If $f(x) = 1/x$ and $g(x) = x^2$, find $g(f(2))$.
9. If $f(x) = 1/(x+1)$, find $f(f(x))$
10. If $f(x) = x^{-3}$, find the inverse function.
11. Which of the following functions is its own inverse:
    (a) $f(x) = x^2$, (b) $g(x) = -x$, (c)$h(x) = x$
    (d) each of f,g and h (e) g and h only
12. Which of the following functions has an inverse function?
    (a) $f(x) = x^3$, (b) $g(x) = x^5$, (c)$h(x) = x$
    (d) each of f,g and h (e) g and h only
13. Finds the inverse of $f(x) = \dfrac{x+1}{2x+1}$.

3.5 Polynomials

You should know how to add, subtract, multiply, divide, and factor polynomials. You should know special forms such as the difference of powers. You should understand the connection between roots and factorizations, and be able to solve a quadratic equation using the quadratic formula. You should be able to work with a polynomial function of something nontrivial: eg, factor $\sin(x)^2 + 2\sin(x) + 1$.

Examples:

1. Expand: $(x^2 - 1)(x^2 + 3)$

   Solution:
$$\begin{aligned} (x^2-1)(x^2+3) &= x^2(x^2+3) - 1(x^2+3) \\ &= x^2 \times x^2 + x^2 \times 3 - 1 \times x^2 - 1 \times 3 \\ &= x^4 + 3x^2 - x^2 - 3 \\ &= x^4 + 2x^2 - 3 \end{aligned}$$

2. Expand: $(x - 1)(x + 1)(x^2 + 1)$

   Solution:
$$\begin{aligned} (x-1)(x+1)(x^2+1) &= (x-1)(x+1)\ \ (x^2+1) \\ &= (x(x+1) - 1(x+1))\ \ (x^2+1) \\ &= (x^2 + x - x - 1)(x^2+1) \\ &= (x^2-1)(x^2+1) \\ &= x^2(x^2+1) - 1(x^2+1) \\ &= x^4 + x^2 - x^2 - 1 \\ &= x^4 - 1 \end{aligned}$$

3. Solve: $y^2 - 7y + 12 = 0$

   Solution: This problem can be solved in one of two ways: either by factoring the quadratic, or by using the quadratic formula. We observe that
$$\begin{aligned} y^2 - 7y + 12 &= 0 \qquad \text{through factoring becomes} \\ (y-4)(y-3) &= 0 \end{aligned}$$
   which implies that the zeroes are $y = 3$ and $y = 4$.

Problems: (answers on page 40)

1. Solve: $x^2 - 5x + 3 = 0$.
2. Factorize $2x^2 + 5x + 2$.
3. Put $x^2 + 2x + 2$ in completed square form, and graph it.
4. Simplify: $(x^2 + 3)(x^2 + a) - x^4$.
5. If we divide $x^4 + x^3 + x^2 + x + 1$ by $x + 1$, what is the remainder?
6. Expand: $(x + 2)^4$.
7. Expand and simplify: $\dfrac{(x+1)^3 - (x-1)^3}{x}$
8. Expand and simplify: $\dfrac{(x+1)^3 + (x-1)^3}{x}$
9. Expand: $(x^2 - 1)(x^2 - x + 1)$
10. How many real solutions does $x^4 - 5x^2 + 6 = 0$ have?
11. Factorize: $x^4 - 13x^2 + 36 = 0$

3.6 Algebra with Fractions

You should be able to simplify a fractional expression, convert a "stacked" fractional expression into a simple one, put fractional expressions over a common denominator, and perform a partial fraction expansion. These skills will be useful in finding various derivatives, simplifying derivatives and integrals, and in particular for the "partial fractions" technique of integration.

Examples:

1. Simplify $\dfrac{(x-1)^2+4x}{x+1}$.

   Solution: $\dfrac{(x-1)^2+4x}{x+1} = \dfrac{x^2-2x+1+4x}{x+1} = \dfrac{x^2+2x+1}{x+1} = x+1.$

2. Simplify $\dfrac{\frac{x-1}{x+2} - \frac{x-2}{x+1}}{x}$.

   Solution: $\dfrac{\frac{x-1}{x+2} - \frac{x-2}{x+1}}{x} = \dfrac{\frac{(x-1)(x+1) - (x-2)(x+2)}{(x+1)(x+2)}}{x} = \dfrac{(x^2-1)-(x^2-4)}{x(x+1)(x+2)} = \dfrac{3}{x(x+1)(x+2)}.$

3. Expand $\dfrac{x^2+1}{x^2-1}$ in partial fractions.

   Solution: $\dfrac{x^2+1}{x^2-1} = 1 + \dfrac{2}{x^2-1} = 1 + \dfrac{2}{(x+1)(x-1)}.$

   We want to write this as $1+\frac{A}{x+1}+\frac{B}{x-1}$ for suitable A, B. Putting the fractions over a common denominator, we have $A(x-1)+B(x+1) = 0x+2$; so $A=-1$, $B=1$, and

   $$\frac{x^2+1}{x^2-1} = 1 - \frac{1}{x+1} + \frac{1}{x-1}\ .$$

   .

Problems: (answers on page 40)

1. Simplify $\dfrac{1}{x+1} - \dfrac{1}{x-1}$.
2. Write $x + 1 + 1/x + 1/x^2$ in the form $P(x)/Q(x)$ where P,Q are polynomials.
3. Simplify $\dfrac{x+1}{x+3} - \dfrac{x}{x+2}$. Is this equal to 0 for any x?
4. Simplify $\dfrac{1/x}{1/x^2}$.
5. Simplify $\dfrac{\frac{1}{x} - \frac{1}{x+h}}{h}$.
6. Solve $\dfrac{x+1}{x-2} - \dfrac{x}{x-1} = 0$.
7. Simplify $\dfrac{1/x \ + \ 1/y}{1/x \ - \ 1/y}$.
8. Simplify $\dfrac{x}{x+1} - \dfrac{1}{x(x+1)}$.
9. Expand $\dfrac{1}{x^2 + x - 6}$ in partial fractions.
10. Expand $\dfrac{x^3 + 4x^2 + 4x + 1}{x^2 + 2x}$ in partial fractions.
11. Expand $\dfrac{1}{x^3 - x}$ in partial fractions.

3.7 Rationalizing numerators or denominators

You should know how to eliminate square (and other) roots from the numerator or denominator of a fraction by multiplying both the numerator and denominator by an appropriate expression. This technique will be important in finding the derivatives of certain expressions involving roots.

Examples:

1. Rationalize the denominator of $\dfrac{1}{\sqrt{a}}$

   Solution: By multiplying both numerator and denominator by $\sqrt{a}$ we obtain

   $$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}}\frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

2. Rationalize the denominator of $\dfrac{1+\sqrt{x}}{2+\sqrt{x}}$

   Solution: Recall, when we multiply $(a+b)(a-b)$ we obtain a difference of squares a^2-b^2. So, if the denominator of a rational expression contains a constant added to a square root term, we can eliminate the root term by multiplying by the difference of the constant and the root term. In this case if we were to multiply both the numerator and the denominator by $2-\sqrt{x}$ we will obtain the desired result.

   $$\begin{aligned} \frac{1+\sqrt{x}}{2+\sqrt{x}} &= \frac{(1+\sqrt{x})}{(2+\sqrt{x})}\frac{(2-\sqrt{x})}{(2-\sqrt{x})} \\ &= \frac{2-\sqrt{x}+2\sqrt{x}-x}{4-2\sqrt{x}+2\sqrt{x}-x} \\ &= \frac{2+\sqrt{x}-x}{4-x} \end{aligned}$$

3. Rationalize the numerator of $\dfrac{1+\sqrt{x}}{2+\sqrt{x}}$

   Solution: This time we multiply numerator and denominator by the conjugate of the numerator:

   $$\frac{1+\sqrt{x}}{2+\sqrt{x}} = \frac{(1+\sqrt{x})}{(2+\sqrt{x})}\frac{(1-\sqrt{x})}{(1-\sqrt{x})} = \frac{1-x}{2-\sqrt{x}-x}$$

Problems: (answers on page 40)

1. Rationalize the denominator of $\dfrac{x+1}{\sqrt{x}}$
2. Rationalize the denominator of $\dfrac{a}{\sqrt{a}\sqrt[3]{b}}$
3. Rationalize the denominator of $\dfrac{1}{\sqrt{x}+\sqrt{y}}$
4. Rationalize the denominator of $\dfrac{a-\sqrt{x}}{b-\sqrt{x}}$
5. Rationalize the denominator of $\dfrac{1}{1+x^{1/3}+x^{2/3}}$
6. Rationalize the numerator of $\dfrac{\sqrt{x}+\sqrt{y}}{x}$
7. Rationalize the numerator of $\dfrac{a-\sqrt{x}}{b-\sqrt{x}}$
8. Rationalize the numerator of $\dfrac{\sqrt{x+1}-\sqrt{x}}{x}$
9. Rationalize the numerator of $\dfrac{\sqrt[3]{y+h}-\sqrt[3]{y}}{h}$

3.8 Linear Graphs

You should be able to graph linear functions and inequalities, determine the slope and intercept of a line from its equation and vice versa, determine where two lines meet, use the negative-reciprocal rule for orthogonal lines, and find the distance between two points.

Many of these ideas will be conceptually important in calculus, which deals a lot with slopes, tangent lines, secant lines, etc.

Examples:

1. Find the point of intersection of the lines $x = 2$ and $x + y = 5$.

   Solution: This does not require a graph to be drawn. Remember that the line (or curve) corresponding to an equation is the set of points (x, y) for which the equation is true; and the intersection of two lines or curves is the set of pairs (x, y) for which both equations are true.

   So we need to find a pair (x, y) such that $x = 2$ and $x + y = 5$. The first equation gives us the value of x immediately; plugging that into the other equation we get $2 + y = 5$, $y = 3$, so the answer is $(x, y) = (2, 3)$.

2. Find the slope of the line through the points $(-5, 0)$ and $(3, 2)$.

   Solution: The "rise" is $2 - 0 = 2$ and the "run" is $3 - (-5) = 8$.
   Slope = rise/run = $2/8 = 1/4$.

3. Find the equation of the line through $(5, 0)$ and orthogonal to $x + 2y = 3$.

   Solution: First we find the slope of the given line $x + 2y = 3$. To find the slope we isolate y in the equation, rewriting it first as $2y = 3 - x$ and then as $y = \frac{3}{2} - \frac{1}{2}x$. Thus the given line has slope $m = -\frac{1}{2}$.

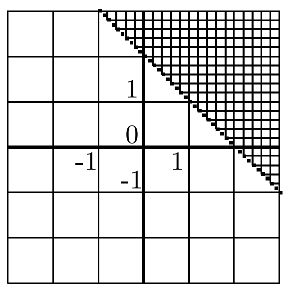
   The line we are looking for is orthogonal to the given line, so its slope is the negative reciprocal of the first line's slope. $-1/(-\frac{1}{2}) = 2$, so we are looking for a line with slope 2, hence with eqation $y = 2x + b$.

   Every point on that line satisfies that equation, so $0 = 2 \times 5 + b$. Solving, we find that $b = -10$, and our equation is

   $$y = 2x - 10 \ .$$

Problems: (answers on page 40)

1. Find the slope of the line $2y = x - 2$
2. Give the equation of a line with slope 3, through $(0, 0)$.
3. If a line has slope $3/2$ and passes through $(2, 2)$, give its equation.
4. Find the distance from the point $(0, 3)$ to the point $(3, 0)$
5. Find the slope of the line through the points $(0, 3)$ and $(3, 0)$.
6. Find the point of intersection of the lines $y = 5 - x$ and $y = 2x + 2$
7. Find the equation of the line with slope $-1/2$ and y-intercept 3
8. Find the slope of a line orthogonal to the line $y = 3x + 5$.
9. Find the equation of the line through $(3, 2)$ and $(1, 0)$
10. Which inequality is this the graph of?



(a) $y > 2$ (b) $x > y + 2$ (c) $x < y + 2$ (d) $x > 2$ (e) $x + y > 2$

11. Which of these lines is parallel to $2y = 6x - 1$?

(a) $2y = 5x - 1$ (b) $y = 6x + 1$ (c) $y = 3x + 1$
(a) $2y = 3x + 2$ (a) $2y = -6x + 1$

12. Which of these lines is orthogonal to $y = 3x - 1$?

(a) $y = -3x - 1$ (b) $y = -x/3 + 1$ (c) $y = x/3 + 1$
(d) $3x = y + 2$ (e) $y = 1 - 3x$

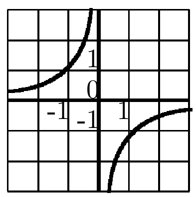
3.9 Graphs

You should be able to graph polynomials and rational functions, showing features such as zeroes, y intercept, horizontal, vertical, and slant asymptotes, and points of discontinuity. You should also be able to read significant features from a graph.

A graph - for our purposes - should be drawn by determining the main features and joining them together with smooth curves. It should not be drawn by plotting five or six points and joining them with straight lines! In your calculus course, you will learn to extend these graphing skills by adding other features, such as maxima, minima, and points of inflection.

Examples:

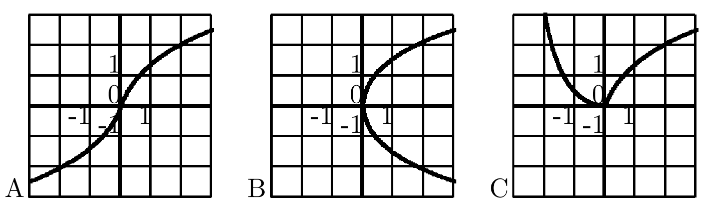
1. Which equation best fits the given curve?



(a) $y = x^2$
(b) $y = 1/x$
(c) $y = \sin(x)$
(d) $x = -1/y$
(e) $x = y^2$

Solution: As the graph has a vertical asymptote (here, $x = 0$), we can rule out (a),(c), and (e). Of the remaining two options, (b) has a graph consisting of points in the first and third quadrants (x and y are either both positive or both negative), while the equation (d) is satisfied by points such as $(1, -1)$ that are on the given curve. So we choose (d), which does indeed have all the right features.

2. Which of the following is the graph of a function $y = f(x)$?

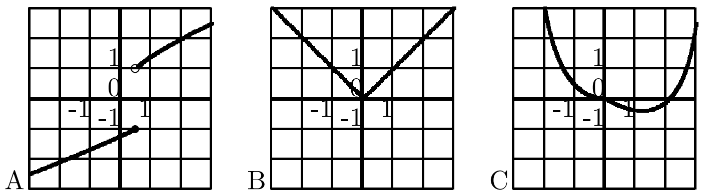


(a) A only
(b) B only
(c) C only
(d) all three
(e) A and B
(f) A and C

Solution: (f). Graph B does not have a unique y value for every x, and the others do. The smoothness of graph B, and the fact that it appears to be a parabola, are not relevant. (It *is* the graph of an equation $x = f(y)$.)

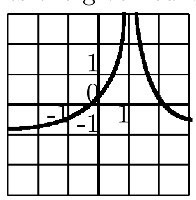
Problems: (answers on page 40)

1. Which of the following is the graph of a function $y = f(x)$?



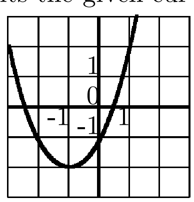
(a) A only
(b) B only
(c) C only
(d) all three
(e) B and C
(f) A and C

2. Which equation best fits the given curve?



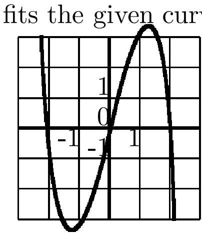
(a) $y = 1/(x-1)^2 - 1$
(b) $y = 1 - 1/x$
(c) $y = 1/(x+1)^2 - 1$
(d) $y = 1/x^2$
(e) $y = 1/(x+1)^2 + 1$

3. Which equation best fits the given curve?



(a) $y = x^2 - 2$
(b) $y = (x+1)^2$
(c) $y = x^2 + 2x - 1$
(d) $y = x^2 - 2x - 1$
(e) $y = x^2 + 2x - 2$

4. Which equation best fits the given curve?



(a) $y = x^3$
(b) $y = -x^3$
(c) $y = x^3 + 4x$
(d) $y = x^3 - 4x$
(e) $y = 4x - x^3$

3.10 Exponents and roots

You should know the basic identities for exponents and roots, and be able to use them to solve equations and derive other identities. In particular, you should be able to convert a reciprocal to a negative power, or a root to a fractional power. These identities will be extremely important for differentiation and integration, because they let us use one rule to differentiate and integrate many apparently different expressions. For instance, we use the same rule to differentiate x^n, $\sqrt[k]{x}$, and $1/x$.

Examples:

1. Find $4^{18}/4^{16}$.

   Solution: $4^{18}/4^{16} = 4^{18-16} = 4^2 = 16$.

2. Simplify $\sqrt{a^{7/2}a^{5/2}a^{3/2}a^{1/2}}$.

   Solution: First we simplify the expression under the square root sign:

   $$a^{7/2}a^{5/2}a^{3/2}a^{1/2} = a^{7/2+5/2+3/2+1/2} = a^8 \ .$$

   So

   $$\sqrt{a^{7/2}a^{5/2}a^{3/2}a^{1/2}} = \sqrt{a^8} = a^4 \ .$$

3. Solve: $x^{-2} = 100$.

   Solution: Taking the reciprocal of both sides, $x^2 = 1/100$. Taking the square root of both sides, $x = 1/10$ or 0.1.

4. If $a^4 = 9$, find a^{-2}

   Solution: If $a^4 = 9$, $a^2 = 3$ and $a^{-2} = 1/3$.

Problems: (answers on page 40)

1. Simplify $\dfrac{\sqrt{x} \times x^2}{1/x^2}$

2. If $a^{3/2}a^{-2}\sqrt[3]{a} = a^b$, what is b?

3. Solve: $4^x 2^x = 64$.

4. Solve: $4^0 + 2^x = 9$.

5. Find $\left(\dfrac{1}{\sqrt{16}}\right)^{1/2}$

6. If $a^4 = b^8$, for real numbers a, b, which of the following must hold?
   (a) $a = b^2$ (b) $a = -b^2$ (c) $a = b^{-2}$
   (d) $a = b^2$ or $a = -b^2$ (e) none of these must hold.

7. $\sqrt{a^{16}b^{16}} =$ (a) $(ab)^2$ (b) $(ab)^4$ (c) $(ab)^8$ (d) $(ab)^{16}$
   (e) none of these.

8. If $a^b = (\sqrt[3]{a})^{12}$, what is b?

9. If $a^3 = b$, and $b^{1/6} = 10$, find a.

10. If $\sqrt[b]{a} = a^2$, find b.

3.11 Logarithms

Logarithms are extremely important in many of the sciences, and it is important to be able to differentiate and integrate expressions using them. To do that, you have to be able to manipulate logarithms algebraically.

You should know the definition of logarithms to various bases, their relation to powers and roots, and the change-of-base formula $\log_a(b)\log_b(c) = \log_a(c)$. While logarithms appear tricky at first, there are actually not very many things to learn about them.

Examples:

1. Find $\log_5(125)$.
   Solution: We recognize 125 as the cube of 5, (or, in a pinch determine that fact by trial and error.) $5^3 = 125$, so $\log_5(125) = 3$.

2. Simplify $\log_2(3)\log_3(4)$
   Solution: Using the change-of-base rule, $\log_2(3)\log_3(4) = \log_2(4)$. Because $4 = 2^2$ we have $\log_2(4) = 2$.

3. What is the decimal logarithm of $10\sqrt{10}$?
   Solution: $10\sqrt{10} = 10^{3/2}$, so $\log_{10}(10\sqrt{10}) = 3/2$.

4. If $\log_a(b^2) = 3$, find $\log_b(a^2)$
   Solution: If $\log_a(b^2) = 3$, then $\log_a(b) = 3/2$. By the change-of-base rule, $\log_b(a) = 2/3$ and so $\log_b(a^2) = 4/3$.

5. (Without a calculator or notes!) The natural logarithm of 10 is between:
   (a) 0 and 1 (b) 1 and 2 (c) 2 and 4 (d) 4 and 6 (e) 6 and 10
   Solution: It would be useful here to know $\ln(10) = 2.3025\ldots$, but you do not need to. If you just know that e, the base of the natural logs, is between 2 and 3, you can rule out (a) and (b) as too small (because $e^2 < 3^2 < 10$) and (d) and (e) as being too big (because $e^4 > 2^4 > 10$).

   (The actual value of e is $2.71828\ldots$)

Problems: (answers on page 40)

1. Find $\log_{10}(0.001)$.
2. Find $\log_2(64) + \log_3(9)$
3. If $3^x = 7$, which is true?
   (a) $7 = \log_x(3)$ (b) $x = \log_7(3)$ (c) $3 = \log_x(7)$ (d) $7 = \log_3(x)$
   (e) $x = \log_3(7)$ (f) $3 = \log_7(x)$
4. $\log_3(5)\log_5(7)\log_7(9) =$
   (a)1 (b)2 (c)3 (d)4 (e) none of these
5. If $\log_a(25) = 4$, what is a?
   (a)$1/5$ (b)$\sqrt{5}$ (c)5 (d)$\log_2(5)$ (e) none of these
6. Find $\log_{100}(1,000,000)$
7. Find $\log_{100}(10)$
8. If $\log_{10}(2) \approx 0.301$, which of these is closest to $\log_{10}(2000)$?
   (a) 0.6 (b) 3 (c) 6 (d) 30 (e) 60 (f)300
9. If $\log_{10}(2) \approx 0.301$, which of these is closest to $\log_{10}(8)$?
   (a) 0.3 (b) 0.6 (c) 0.9 (d) 1.2 (e) 2.4 (f)6
10. If $\log_{10}(2) \approx 0.301$, which of these is closest to $\log_2(10)$?
    (a) 0.5 (b) 1 (c) 3 (d) 5 (e) 20 (f)50
11. Find $2^{\log_2(17)}$
12. Find $4^{\log_2(3)}$
13. If $\log_3(10) = K$, then $\log_9(10) =$
    (a)$2K$ (b)$K/3$ (c)$K/2$ (d)K^2 (e)$3K$ (f)$\sqrt{K}$

3.12 Geometry and basic trigonometry

Calculus does not use very advanced geometry, but you should be thoroughly familiar with similar triangles, Pythagoras' theorem, and parallel lines; and, from analytic geometry, the midpoint and distance formulae, and the "negative reciprocal" rule. Trigonometry is important in various branches of science, but especially in mathematics, physics, and engineering.

You should be able to convert between degrees and radians ($180° = \pi$ radians). You should know the definitions of the trig functions, and be able to use them to find sides and angles of triangles. You should know and be able to use the sine and cosine laws for triangles.

Most angles do not have trig functions that are easy to give as exact expressions rather than decimal approximations (and you will not be expected to do so), but you should know the trig functions of a few common angles, such as 0°, 30°, 45°, 60°,and 90°. You should also know how to find the trig functions of angles outside the range $[0°, 90°]$ in terms of trig functions of angles in that range.

You should also be familiar with the inverse trigonometric functions. Note that although (for instance) $\sin^2(x)$ means $(\sin(x))^2$, and $\sin(x)^{-1}$ means $1/\sin(x)$, which is $\csc(x)$, $\sin^{-1}(x)$ means $\arcsin(x)$.

Examples:

1. If the hypotenuse of a right triangle is 10, and one leg has length 8, how long is the other leg?
   Solution: Let the length of the unknown leg be x. By Pythagoras' theorem, $x^2 + 8^2 = 10^2$, so $x^2 = 100 - 64 = 36$ and $x = 6$.

2. Find the midpoint of the segment from the point $(2, 3)$ to $(8, -3)$.
   Solution: By the midpoint formula, the coordinates of the midpoint are $\left(\frac{2+8}{2}, \frac{3+-3}{2}\right)$ which simplifies to $(5, 0)$.

3. If two sides of a triangle have length 1 and 2, and the angle between them is 45°, what is the length x of the remaining side?
   Solution: By the cosine law, $x^2 = 1^2 + 2^2 - 2(1)(2)\cos(45°) = 5 - 4(\sqrt{2}/2) = 5 - 2\sqrt{2}$; and therefore $x = \sqrt{5 - 2\sqrt{2}}$.
   We cannot simplify this further so we leave it in this form.

Problems: (answers on page 40)

1. If a triangle has sides 15, 20, and 25 units long, what is the measure in degrees of its largest angle?

2. If a rectangle has edge lengths 10 and 15, how long is its diagonal?

   (a) 25 (b)$\sqrt{25}$ (c)$\sqrt{125}$ (d)$\sqrt{225}$ (e)$\sqrt{325}$

3. Which of these triples could *not* be the edge lengths of a right triangle?

   (a)$(3, 4, 5)$ (b)$(12, 16, 20)$ (c)$(2, 2, 3)$ (d) $(12, 13, 5)$ (e) $(\sqrt{2}, \sqrt{2}, 2)$

4. In the plane, what are the coordinates of the midpoint of the line segment whose ends are $(10, 6)$ and $(4, 10)$?

   (a) $(2, 3)$ (b) $(3, 2)$ (c) $(5, 5)$ (d) $(7, 8)$ (e) none of these are correct.

5. What is the distance between the points whose coordinates are $(-2, 3)$ and $(1, -1)$?

   (a) 5 (b) 7 (c) 9/2 (d) $\sqrt{7}$ (e) $\sqrt{5}$

6. Find the cosine of 225°

7. Find the arctangent of 1, in degrees

8. Find the cotangent of 30°

9. How many radians is 75°?

10. How many degrees is 3/2 radians? (a)$2\pi/3$ (b)180 (c)$270/\pi$ (d)$\pi/120$ (e)$3\pi/2$

11. How many angles θ between 0° and 360° have $\sin(\theta) = 1/2$?

    (a)0 (b)1 (c)2 (d)3 (e)4

12. Find $\tan(\pi/6)$

13. The value of $\arcsin(0.5)$ is: (a) 0° (b) 30°
    (c) 45° (d) 60° (e) 90°

14. If $\cos(\theta) = 3/5$, then $\sin(\theta)$ can be

    (a) 4/5 only (b)5/4 only (c) 5/4 or $-5/4$
    (d) 4/5 or $-4/5$ (e) $-5/4$ only

3.13 Trigonometric identities

There are many identities that are satisfied by the trigonometric functions. They are important in calculus because they are used to reduce the number of rules you have to learn. Note that $\cos^2(\alpha)$ means $(\cos(\alpha))^2$.

The identities for one angle can all be derived from the definitions of the six functions, via the relations $\tan(\alpha) = \sin(\alpha)/\cos(\alpha)$, $\cot(\alpha) = \cos(\alpha)/\sin(\alpha)$, $\sec(\alpha) = 1/\cos(\alpha)$ $\csc(\alpha) = 1/\sin(\alpha)$ and from the identities $\sin^2(\alpha)+\cos^2(\alpha) = 1$, $\sec^2(\alpha) - \tan^2(\alpha) = 1$, and $\csc^2(\alpha) - \cot^2(\alpha) = 1$.

There are also various identities for trigonometric functions of sums and differences of angles, and for double and half angles. These are aso useful in calculus.

Examples:

1. $\sin^2(\theta)\sec(\theta)\csc(\theta) =$
   (a) $\sin(\theta)$ (b) $\cos(\theta)$ (c) $\tan(\theta)$ (d) $\sec(\theta)$ (e) $\csc(\theta)$ (f) $\cot(\theta)$

   Solution:

$$\begin{aligned}\sin^2(\theta)\sec(\theta)\csc(\theta) &= (\sin(\theta)\sec(\theta))\ (\sin(\theta)\csc(\theta))\\ &= (\sin(\theta)(1/\cos(\theta)))\ (\sin(\theta)(1/\sin(\theta)))\\ &= (\tan(\theta))\end{aligned}$$

   so the answer is (c).

2. If $\sin(35^\circ) = \cos(\beta)$, and $\beta \in [0^\circ, 90^\circ]$, find β.
   Solution: $\sin(\alpha) = \cos(90^\circ - \alpha)$ for all α; so here $\beta = 55^\circ$.

3. If $\cos(\alpha) = 0.3$, find $\cos(2\alpha)$.
   Solution: We do not need to know α for this! One form of the "double angle formula" for cosines is

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1$$

   from which we see that $\cos(2\alpha) = 2(0.3^2) - 1 = -0.82$.

Problems: (answers on page 40)

1. $\sin(2x) =$
   (a) $2\sin(x)$ (b) $\sin(x) + \cos(x)$ (c) $\cos(x)\sin(x)$
   (d) $2\cos(x) - 2\sin(x)$ (e) $2\cos(x)\sin(x)$

2. $\cot(\alpha)\cos(\alpha)\sin(\alpha) =$
   (a) $\cos(\alpha)\sin(\alpha)$ (b) $\sin^2(\alpha)$ (c) $\cos^2(\alpha)$ (d) $\cos(\alpha)$ (e) $\sin(\alpha)$

3. $\tan(\alpha)\cot(\alpha) =$
   (a) $\sin(\alpha)$ (b)$\cos(\alpha)$ (c) $\sin^2(\alpha)$ (d)$\cos^2(\alpha)$ (e) 1

4. If $\cos(\theta) = \sqrt(2)/2$, then $\tan(\theta) =$
   (a) 2 (b) 1 (c) 0 (d) 1/2 (e) ± 1

5. If $\tan(\theta) = 3$, then $\cot(\theta) =$
   (a) 3 (b) 1/3 (c) 0 (d) -3 (e) $-1/3$

6. If $\sin(\gamma) = 1/3$, $\sin(-\gamma) =$
   (a) 2/3 (b) 1/3 (c) $-1/3$ (d) $-2/3$ (e) none of these

7. If $\cos(\gamma) = 1/3$, $\cos(-\gamma) =$
   (a) 2/3 (b) 1/3 (c) $-1/3$ (d) $-2/3$ (e) none of these

8. If $\cos(\alpha) = 3/5$, find $\cos(2\alpha)$

9. How many angles θ between 0° and 360° have $\sin(\theta) = \cos(\theta)$?

   (a)0 (b)1 (c)2 (d)3 (e)4

10. How many angles θ between 0° and 360° have $\sin(\theta) = \sec(\theta)$?

    (a)0 (b)1 (c)2 (d)3 (e)4

3.14 Problem solving

Solving applied problems ("word problems") using calculus uses many of the same skills as solving applied problems using algebra. In each case, you have to pick out the important numerical quantities — known or unknown — from the problem, and determine the relations between them. These yield a set of equations that must be solved to yield the desired quantity. You may also need to know certain quantities and relations that are *not* given in the problem.

Do **not** try to learn "the formula for each type of problem." A common mistake is to think that there is one formula for "the problem with the lawnmowers" and another for "the problem with the antifreeze". There are many different problems involving time to complete tasks, and many different problems involving mixtures. Instead, learn basic relations and heuristics, such as that when two agents (people, taps, etc) cooperate on a task, the *amount of work done* can be added and the *time taken* cannot.

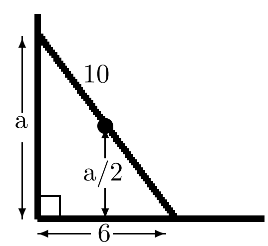
Examples:

1. A tap fills a 100-liter tank in 2 hours, if the tank starts empty and the drain is closed. If the drain is open, and the tap is off, the tank drains at a constant rate; starting full, it would empty in 3 hours. If the tank starts half-full at noon, when is it completely full?

   Answer: We combine the effects of the tap and drain by adding or subtracting work done per hour. The tap adds 100 ℓ in 2 hours, or $100/2\ \ell/\text{hr}$. The drain removes 100 ℓ in 3 hours, or $100/3\ \ell$. We need to ad 50ℓ to the tank; this takes $50\ell/(100/6\ \ell/\text{hr}) = 3\text{hr}$, so the tank is full at 3:00 PM.

2. If a $10m$ ladder leans against a vertical wall, with the foot of the ladder $6m$ from the wall, how high above the ground is the midpoint of the ladder?

   Answer:**Draw a diagram and label it.** Because the wall is vertical, and we assume the ground to be horizontal, the wall and ground make a right angle. Therefore the wall, ground, and ladder make a right triangle (with the ladder as hypotenuse) and Pythagoras' theorem applies. $a^2 + 6^2 = 10^2$, so $a = \sqrt{(100-36)} = 8$, and the height of the midpoint is $a/2$ or 4.



Problems: (answers on page 40)

1. If the area of a triangle is 100 cm^2 and its base is equal to twice its height, what is its base? (a) 5 cm (b) 10 cm (c) 20 cm (d) 50 cm (e) 200 cm
2. If the perimeter of a rectangle is 28 meters, and the diagonal is 10 meters, what is the area?
3. If the perimeter of a rectangle is 26 meters, and the area is 30 square meters, what is the diagonal?
4. A right triangle has area 30 square meters and perimeter 30 meters. What is its hypotenuse?
5. If a goat can eat the grass on a lawn in 1 day and a sheep can eat the grass on the same lawn in 2 days, how long would it take two goats and a sheep working together? Assume the grass is not growing.
   (a) 2/5 of a day (b) 3/8 of a day (c) 1 day (d) 2 days (e) 4days
6. Joe started day trading with a certain amount of money. On the first day he doubled his money, and on the second day he lost 2/3 of what he had at the end of the first day. At the end of the second day he had \$ 100. How much did he start with?
   (a) \$50 (b) \$75 (c) \$80 (d) \$125 (e) \$133.33 (f)\$150
7. If a puppy weighs a third as much as a dog, and together they weigh 12 kg., how much does the puppy weigh?
8. A driver is driving 100 km. She drives the first 50 km at 50 km/hr and the second 50 km at 150 km/hr. How long does the trip take? (a) 30minutes (b) 50 minutes (c) 60 minutes (d) 80 minutes (e) 90 minutes
9. The towns of Ayton and Beaton are 60 km apart. At noon, Alice leaves Ayton and drives to Beaton at 80 km/hr, and Bob leaves Beaton and drives (on his tractor) to Ayton at 40 km/hr. At what time do they meet?
10. The towns of Ayton and Beaton are 60 km apart. At noon, Alice leaves Ayton and drives to Beaton at 90 km/hr. If Bob rides from Ayton to Beaton on his bicycle at 15 km/hr, when must he leave Ayton to get to Beaton at the same time that Alice does?

3.14159265358979... Harder Problems

The material earlier in this booklet represents the minimum that you need to know before you begin university calculus. We hope that it won't be *all* that you know! You should also know something about linear algebra, geometry, statistics, and other areas of mathematics; you should have experience applying mathematics in other subjects; and you should be able to write clear explanations of what you know, and solve problems that require a certain amount of lateral thinking. In this section, we present some problems intended to challenge the stronger student. Such students may also want to look for problems in publications such as Crux Mathematicorum, or compete in mathematics contests.

Answers to these questions are *not* given.

1. Find all solutions to $x^7 + 4x^5 + x^3 - 6x = 0$.

2. Explain why it is impossible for $x^{11} - 7x^7 + x^3 - 4x = 0$ to have an even number of real solutions. Find a general statement about polynomials that this is a general case of.

3. You want to know if $ax^8 + bx^6 + cx^4 + dx^2 + e$ has an odd or even number of real solutions, where a, b, c, d, e are real numbers (not all 0) that you don't know. You are allowed to "buy a letter" and find out its value for $10 per letter . Explain how to find the answer for minimum cost.

4. Explain why $123abc567$ cannot be a perfect square (where a, b, c are unknown digits), without using a calculator or doing any lengthy calculations.

5. Given that $2^{10} = 1024$, show how to estimate the decimal logarithm of 2; and find the first digit and number of digits of 2^{100}.

6. Explain why we cannot define $0/0$ to be 1 without changing other rules of mathematics.

7. Find the area of a regular octagon, if its edge length is 1.

8. (a) If we put the biggest possible circle inside a square of radius 1, and the biggest possible square into that circle, what would its area be?

   (b) If we put the biggest possible sphere inside a cube of radius 1, and the biggest possible cube into that sphere, what would its volume be?

9. Which is larger, 100^{200} or 200^{100}? Explain your answer.

10. For $n = 1, 2, 3, \ldots$ we define $n! = n \times (n-1) \times \cdots 3 \times 2 \times 1$. (This is pronounced "$n$ factorial" and is important in combinatorics, probability, and other areas of math.).
(a) Find $n!$ for $n = 3, 4, \ldots, 8$
(b) How many 0's does 100! end in?

11. By using Pythagoras' theorem and elementary geometry, find $\sin(30°)$, $\sin(45°)$, and $\sin(60°)$.

12. Determine $\sin(15°)$ and $\sin(75°)$.

13. Starting with the sum-of-angles formulae for sines and for cosines, derive the difference-of-angles formulae, the double-angle formulae, the half-angle formulae, and the formulae for $\sin(a)\cos(b)$, $\sin(a)\sin(b)$, and $\cos(a)\cos(b)$.

14. A rhombus has the edge length equal to its short diagonal. Find the area in terms of the length of the *long* diagonal.

15. A river runs straight east and west. A horse and rider are 4km north of the river. The rider could get to camp by riding 2km south and then 2km west, but the horse needs to get to the river to drink, too. What is the shortest distance that the horse and rider must travel to get first to the river and then to the camp?

16. The decimal expansion of 1/3, 0.333..., repeats with period 1; and that of 1/7, 0.142857142857... repeats with period 6. Find fractions whose decimal expansions repeat with periods 2, 3 and 4. Can you find the *smallest* denominators that work?

17. Show (without calculus) that for any positive a, b, we always have

$$\frac{a+b}{2} \geq \sqrt{ab}$$

.

Answers to problems:

Section 3.1 (1) $\frac{19}{15}$ (2) $\frac{11}{20}$ (3) $\frac{2}{5}$ (4) $\frac{25}{8}$ (5) 1.48 (6) 1.7 (7) 0.001 (8)0.123 (9) 11.0 (10) 3 (11) 8 (12) 0.0025

Section 3.2 (1) a^4 (2) $ac(d-e)$ (3) $\frac{y-x^2}{2}$ (4) $a(1+a+a^2)$ (5) $x=0$ (6) 11 (7) 2 (8) 21 (9) $x=-1$ (10) $y=1/3$ (11) a^4/b^4 or $(a/b)^4$ (12) $x=1$

Section 3.3 (1) a (2) all x (or $-\infty,\infty)$) (3) c (4) $(-\infty,-2)\cup(-1,\infty)$ (5) $y\neq 0$ (6) $(-\infty,-\sqrt{2}]\cup[\sqrt{2}.\infty)$ (7) $[2,4]$ (8) b (9) d (10) 0 (11) $-x^2$ (12) $x\in(2,6)$ (13)$x\in(-\infty,1)\cup(2,\infty)$ (14)$x\in[-1,4]$ (15)$x=-\frac{4}{3},2$

Section 3.4 (1) $x\neq 0$ or $(\infty,0)\cup(0,\infty)$ (2) $x\neq -1$ or $(\infty,-1)\cup(-1,\infty)$ (3) x^2+2x+3 (4) $[0,\infty)$ (5) Domain: $\neq 0$. Range: $\{-1,1\}$ or ± 1 (6) (a) $\frac{1}{x^2+1}$ (b) -1 (c) $\frac{1}{\sqrt{x}+1}$ (7) a^{-2} (8) 1/4 (9) $\frac{x+1}{x+2}$ (10)$x^{-1/3}$ (11) e (12) d (13)$\frac{1-x}{2x-1}$

Section 3.5 (1) $x=(5\pm\sqrt{13})/2$ (2) $(2x+1)(x+2)$ (3) $(x+1)^2+1$ (4) $(3+a)x^2+3a$ (5) 1 (6) $x^4+8x^3+24x^2+32x+16$ (7) $(6x^2+2)/(x)$ (8) $2x^2+6$ (9) x^4-x^3+x-1 (10) 4 (11)$(x-3)(x+3)(x-2)(x+2)$

Section 3.6 (1) $\frac{-2}{x^2-1}$ (2) $\frac{x^3+x^2+x+1}{x}$ (3) $\frac{2}{(x+3)(x+2)}$;no. (4) x (5) $\frac{1}{x(x+h}$ (6) $x=1/2$ (7) $\frac{y+x}{y-x}$ or $1+\frac{2x}{y-x}$ (8) $\frac{x-1}{x}$ (9) $\frac{-1/5}{x+3}+\frac{1/5}{x-2}$ (10) $x+2-\frac{1/2}{x}+\frac{1/2}{x+2}$ (11) $\frac{1/2}{x+1}-\frac{1}{x}+\frac{1/2}{x-1}$

Section 3.7 (1) $\frac{x\sqrt{x}+\sqrt{x}}{x}$ (2) $\frac{\sqrt{a}\,\sqrt[3]{b^2}}{b}$ (3)$\frac{\sqrt{x}-\sqrt{y}}{x-y}$ (4)$\frac{ab+(a-b)\sqrt{x}-x}{b^2-x}$ (5)$\frac{x^{1/3}-1}{x-1}$ (6)$\frac{x-y}{x(\sqrt{x}-\sqrt{y}}$ (7)$\frac{a^2-x}{ab+(b-a)\sqrt{x}-x}$ (8)$\frac{1}{x(\sqrt{x+1}+\sqrt{x}}$ (9)$\frac{1}{(y+h)^{2/3}+(y+h)^{1/3}h^{1/3}+h^{2/3}}$

Section 3.8 (1) 1/2 (2) $y=3x$ (3) $y=\frac{3}{2}x-1$ (4) $3\sqrt{2}$ or $\sqrt{18}$ (5) -1 (6) $(1,4)$ (7) $y=3-x/2$, or $y=-\frac{1}{2}x+3$ (8) $-1/3$ (9) $y=x-1$ (10) e (11) c (12) b

Section 3.9 (1) d (2) a (3) d (4) e

Section 3.10 (1) $x^{9/2}$ (2) $-1/6$ (3) $x=2$ (4) $x=3$ (5) 1/2 (6) d (7) c (8) 4 (9) 100 (10) 1/2

Section 3.11 (1) -3 (2) 8 (3) e (4) b (5) b (6) 3 (7) 1/2 (8) b (9) c (10)c (11) 17 (12) 9 (13) c

Section 3.12 (1) 90° (2) e (3) c (4) d (5) a (6) $-1/\sqrt(2)$ or $-0.707\ldots$ (7) 45° (8) $\sqrt{3}$ (9) $75\pi/180$ or $5\pi/12$ (10) c (11) c (12) $1/\sqrt(3)$ (13) b (14) d

Section 3.13 (1) e (2) c (3) e (4) e (5) b (6) c (7) b (8) $-7/25$ (9) c (10) a

Section 3.14 (1) c (2) 24m^2 (3) $\sqrt{109}$ m (4) 13 m (5) a (6) f (7) 3 kg (8) d (9) 12:30 PM (10) 8:40 AM

Some useful facts:

Algebraic identities	
$a^2 - b^2 = (a+b)(a-b)$	$(a+b)^2 = a^2 + 2ab + b^2$
$a^3 - b^3 = (a^2 + ab + b^2)(a-b)$	$a^3 + b^3 = (a^2 - ab + b^2)(a+b)$
Exponential identities	
$a^x b^x = (ab)^x$	$a^x a^y = a^{x+y}$
$(a^b)^c = a^{bc}$	$a^{-b} = 1/a^b = (1/a)^b$
$\sqrt[a]{b^a} = \left(\sqrt[a]{b}\right)^a = b$	$\sqrt[a]{b} = b^{1/a}$
$a^{\log_a(b)} = b$	$\ln(a) = \log_e(a), \quad e = 2.71828\ldots$
$\log_a(b) + \log_a(c) = \log_a(bc)$	$\log_a(b) - \log_a(c) = \log_a(b/c)$
$\log_a(b)\log_b(c) = \log_a(c)$	$\log_a(1/b) = -\log_a(b)$
Trigonometric facts	
Let a triangle have sides of length a, b, c opposite angles A, B, C. Then	
Pythagoras' Theorem: If C is right, $a^2 + b^2 = c^2$.	
Sine Law: $\sin(A)/a = \sin(B)/b = \sin(C)/c$	
Cosine Law: $c^2 = a^2 + b^2 - 2ab\cos(C)$	
Trigonometric identities	
$\tan(\alpha) = \sin(\alpha)/\cos(\alpha)$	$\cot(\alpha) = \cos(\alpha)/\sin(\alpha)$
$\cot(\alpha) = 1/\tan(\alpha)$	
$\sec(\alpha) = 1/\cos(\alpha)$	$\csc(\alpha) = 1/\sin(\alpha)$
$\sin^2(\alpha) + \cos^2(\alpha) = 1$	
$\sec^2(\alpha) - \tan^2(\alpha) = 1$	$\csc^2(\alpha) - \cot^2(\alpha) = 1$
$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$	
$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$	
$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$	
$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$	
$\tan(A+B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$	$\tan(A-B) = \dfrac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$

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