GRACEFULLY LABELED TREES FROM SKOLEM SEQUENCES

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Abstract

In this paper it will be shown that the existence of a Skolem sequence of order n implies the existence of a graceful tree on $2n$. Moreover, we will show two particular constructions of graceful trees from a Skolem sequence construction of Th. Skolem.

1 Introduction

All graphs considered will be finite with no loops or multiple edges. For any graph G, its vertex set and edge set will be denoted V_G and E_G respectively. Any graph theoretical definitions not included in this paper can be found in West [8].

Definition 1.1 *Given a graph, G, a graceful labeling of G is an injective mapping $\alpha : V_G \longrightarrow \mathbb{Z}_{|E_G|}$ such that the associated mapping $\beta : E_G :\longrightarrow \{1, \ldots, |E_G|\}$ defined by $\beta(\{u, v\}) = |\alpha(u) - \alpha(v)|$ is bijective.*

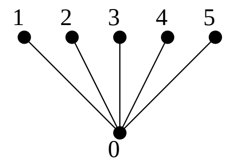

Figure 1: A Graceful Labeling of a Tree.

The notion of a graceful labeling was introduced by Rosa [6]. Much work has been done in this field, however there remain many open questions. The famous conjecture that all trees are graceful is attributed to Ringel [5], where most accomplishments have been made by showing particular classes of trees to be graceful. For a survey of result on graceful graphs see Gallian [4].

Definition 1.2 *A Skolem sequence of order n is a sequence $(s_0, s_1, \ldots, s_{2n-1})$, consisting of the elements $\{1, \ldots, n\}$, such that $\forall k \in \{1, 2, \ldots, n\}$, there are exactly two subscripts $i(k), j(k)$ for which $s_{i(k)} = s_{j(k)} = k$. These subscripts also satisfy the equation $|i(k) - j(k)| = k$.*

As an example, consider (6,1,1,2,8,2,6,7,3,4,5,3,8,4,7,5), which is a Skolem sequence of order 8. It should be noted that a Skolem sequence of order n is equivalent to a partition, P of $\mathbb{Z}_{2n}$ into n subsets of size 2, such that for $\{a_i, b_i\} \in P$

$$\bigcup_{i=1}^{n} \{|a_i - b_i|\} = \{1, \ldots, n\}.$$

The idea of a Skolem sequence was created by Th. Skolem [7], who found that the necessary and sufficient conditions for the existence of a Skolem sequence of order n to be that $n \equiv 0$, 1 (mod4). His proof of sufficiency resulted in specific constructions of such sequences. We will use these to construct specific families of graceful trees.

Given a Skolem sequence of order n, $S = (s_0, s_1, \ldots, s_{2n-1})$, we will consider a correlation between sequences an labelled trees, where the i^{th} position corresponds to the vertex labeled i. As one considers the Skolem pair $\{a_i, b_i\}$, $1 \leq i \leq n$, it is evident that these represent edges having value $\{1, 2, \ldots, n\}$, where the edge of value i is incident with the vertices labeled a_i and b_i. These edges constitute the perfect matching of the tree.

To create graceful trees from these Skolem sequences requires using these edges and adding edges of value $\{n+1, n+2, \ldots, 2n-1\}$ so as not to create cycles.

2 Specific Construction of Graceful Trees from Skolem Sequences

Given the existence theorem, we will now constructively create graceful trees from Skolem sequences. Recall that the necessary and sufficient condition for the existence of a Skolem sequence of order m is that $m \equiv 0, 1 (mod 4)$.

2.1 Skolem sequences of order $4k$

Given a Skolem sequence of order $m = 4k$ we are able to generate an infinite class of graceful trees. Consider the following construction of Skolem pairs as $4k$ of the edges of a tree on $8k$ vertices:

Construction 1:

$\{8k-2, 8k-1\}$	
$\{6k-r-2, 6k+r-1\}$	$1 \leq r \leq k-1$
$\{k-r-1, 3k+r-2\}$	$1 \leq r \leq k-1$
$\{k-1, 5k-2\}$	
$\{2k-r-1, 2k+r-1\}$	$1 \leq r \leq k-1$
$\{4k-2, 6k-2\}$	
$\{5k-r-2, 7k+r-2\}$	$1 \leq r \leq k-1$
$\{2k-1, 6k-1\}$	

This generates $4k$ disjoint edges with values $1, 2, \ldots, 4k$. To create a graceful tree we must now add in the edges of value $4k+1, \ldots, 8k-1$, while being careful not to create cycles. The additional edges given below will create a infinite class of graceful trees:

Construction 1 (cont.):

$\{0, 8k-1\}$	
$\{1, 8k-1\}$	
$\{0, 6k+r\}$	$0 \leq r \leq 2k-3$
$\{2k+r, 8k-1\}$	$0 \leq r \leq 2k-5$
$\{5k-2, k-3\}$	
$\{6k-2, 2k-4\}$	
$\{6k-1, 2k-4\}$	

For $k \geq 5$ this generates a graceful labeling of the $2-star$ on $4k-3$ vertices onto which is attached four additional graphs

- A P_1 at the root vertex.
- A $2-star$ on $4k-3$ vertices by its root vertex to a vertex, v_1, of degree 2.
- A P_4 by its central vertex to a leaf, l_1, not adjacent to v_1.

- A P_2 by its end vertex to a leaf, $l_2 \neq l_1$, which is also not adjacent to v_1.

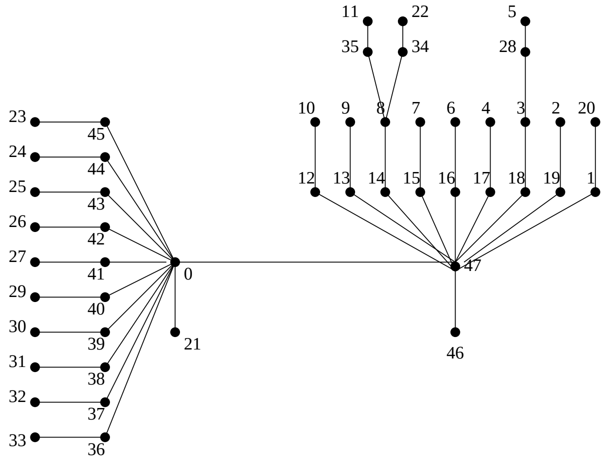

Figure 2: The graceful tree obtained from Construction 1, where $k = 6$.

This construction also generates graceful trees for $k < 5$, however the P_2 and the P_4 do not attach in the above described positions.

2.2 Skolem sequences of order $4k + 1$

Given a Skolem sequence of order $m = 4k+1$ we are also able to generate an infinite class of graceful trees. Consider the following construction of Skolem pairs as $4k + 1$ of the edges of a tree on $8k + 2$ vertices:

Construction 2:

$$\begin{array}{ll}
\{8k-2, 8k-1\} & \\
\{6k-r-2, 6k+r-1\} & 1 \leq r \leq k-1 \\
\{k-r-1, 3k+r-2\} & 1 \leq r \leq k-1 \\
\{k-1, 5k-2\} & \\
\{4k-2, 8k-1\} & \\
\{2k-r-1, 2k+r-1\} & 1 \leq r \leq k-1 \\
\{5k-r-2, 7k+r-2\} & 1 \leq r \leq k-1 \\
\{2k-1, 6k-1\} &
\end{array}$$

This generates $4k + 2$ disjoint edges with values $1, 2, \dots, 4k + 1$. To construct a graceful tree we must now add in the edges of value $4k+2, \ldots, 8k+1$, while being careful not to create cycles. The additional edges given below will create a infinite class of graceful trees:

Construction 2 (cont.):

$\{0, 8k+1\}$	
$\{1, 8k+1\}$	
$\{0, 8k-1\}$	
$\{0, 8k-2\}$	
$\{0, 8k-3-r\}$	$0 \leq r \leq 2k-5$
$\{2k+r, 8k+1\}$	$0 \leq r \leq 2k-5$
$\{2k-4, 6k-1\}$	
$\{2k-4, 6k\}$	
$\{2k-4, 6k+1\}$	
$\{k-4, 5k-2\}$	

For $k \geq 6$ this generates a graceful labeling of the $2-star$ on $4k-3$ vertices onto which is attached four additional graphs

- A P_1 at the root vertex.
- A $2-star$ on $4k-3$ vertices by its root vertex to a vertex, v_1, of degree 2.
- A $2-star$ on 7 vertices by its root vertex to a leaf, l_1, not adjacent to v_1.
- A P_2 by its end vertex to a leaf, $l_2 \neq l_1$, which is also not adjacent to v_1.

This construction also generates graceful trees for $k < 6$, however the P_2 and the 2-star on 7 vertices do not attach in the above described positions.

3 Gracefully Labelled Trees from Skolem Sequences

Recent work has been done by Abrham[1] which uses Skolem sequences to find graceful labelings of 2-regular graphs, and vice-versa. We will restrict ourselves to the use of Skolem sequences to find graceful labelings of trees.

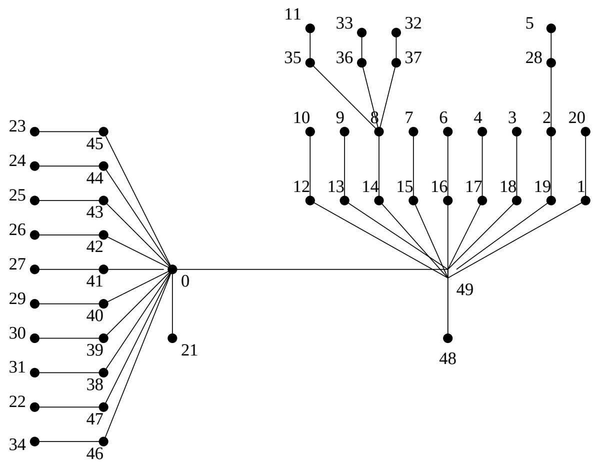

Figure 3: The graceful tree obtained from Construction 2, where $k = 6$.

Before we consider the existence theorem we will need to introduce some new definitions and notation, which will be particularly useful for our technique of edge addition.

Definition 3.1 *Consider a Skolem seqeunce of order* n, $S = (s_0, s_1, \ldots, s_{2n-1})$.

- *Its core,* C_S*, is the central two positions. Namely* $n-1$ *and* n.
- *Given a vertex labeled* $i \notin C_S$*, the maximal connection of* i *will be the addition of the edge* $\{0, i\}$*, if* $i \geq n$*, or* $\{i, 2n-1\}$ *otherwise.*
- *Given a position* i *its primary follower,* $f_p(i) = j : s_{2n-1-j} = s_i, i \neq 2n-1-j$*. In general, the primary followers of* i *are elements of the form* ${f_p}^k(i)$*, where* ${f_p}^k(i) = f_p({f_p}^{k-1}(i))$
- *Given a position* i *its secondary follower,* $f_s(i) = j : s_j = s_{2n-1-i}, j \neq 2n-1-i$*. The secondary followers of* i *are of the same form as its primary followers, where* f_p*, is replaced by* f_s*.*
- *The act of 'tracing' involves choosing a vertex and maximally connecting its successive primary and/or secondary followers until the original vertex is repeated. A vertex used in determining such primary or secondary followers is considered to have been traced.*

Theorem 3.2 *The existence a Skolem sequence of order* n *implies the existence of a graceful tree on* $2n$ *vertices.*

Proof. Consider a Skolem sequence, $S = (s_0, s_1, \dots, s_{2n-1})$, and consider the correspondence in which the i^{th} position corresponds to the vertex labeled i.

For all pairs $\{a_i, b_i\}$, where $b_i - a_i = i$, add the edge $\{a_i, b_i\}$. This gives a perfect matching on the vertices labeled $\{1, \dots, n\}$. We must add edges of value $\{n+1, \dots, 2n-1\}$ on these vertices without creating any cycles.

Now perform the following procedures:

1. Choose an untraced position, $i \in C_S$. Maximally connect the vertex currently adjacent to it, providing it is not in C_S and then trace the secondary followers of the new vertex, until one of the following occurs.

   (a) Position $2n-1-i$ is reached, in which case do not connect it.

   (b) Position 0 or $2n-1$ is reached. Assume WLOG it is 0. Maximally connect 0, then trace the primary followers of $2n-1$ until position i is reached. Do not maximally connect it, for it is in the core.

   There are no more untraced elements left in the core so proceed as follows.

2. Choose a remaining untraced position, i. It is not in the core so maximally connect the vertex adjacent to it and then trace the secondary followers until one of the following occurs

   (a) It is position $2n-1-i$, in which case it is not in the core so maximally connect it and stop.

   (b) It is 0 or $2n-1$. Assume WLOG it is 0. Maximally connect 0 then remove the untraced edge adjacent to $2n-1$ which occured in the perfect matching (This is the reason for the at most 2 vertices excluded from the perfect matching). Then trace the remaining secondary followers of 0 until position $2n-1-i$ is reached. Maximally connect it then stop.

   If there are any remaining untraced positions repeat this procedure.

Since all positions have been traced and primary and secondary follwers have always been maximally connected, all of the edge values in $\{n, \dots, 2n-1\}$ have been added. Since no vertex can be traced more than once using primary and secondary followers no cycles have been created.

The trace that passed through the core left one edge unconnected (by the consistent use of the primary and secondary followers) except for the unique trace which contained 0.

If this trace originated from the core it finished by entering the core and maximally connecting the vertex adjacent to it. Hence there are no components not containing 0 and no edge values left unused.

On the other hand if this trace had originated from outside the core it would have ended outside the core. In this case the edge adjacent to $2n-1$ which had been deleted will also have to be added. Fortunately, this also means that the trace which originated from the core did not contain 0, so it left one edge not connected to the component containing 0. Hence there is exactly one component remaining to achieve exactly one edge value, which is less than n, so there is no difficulty in adding in this edge.

The resulting graph is a tree on $2n$ vertices. □

Corollary 3.3 *The existence of a Skolem sequence of order n implies the existence of a graceful tree on $2n$ vertices which exhibits a perfect matching on either $2n$ or $2n-2$ vertices.*

Proof. Recall that the edges $\{a_i, b_i\}$ determine a perfect matching on the vertices $\{1, \ldots, n\}$. The alogrithmic addition of edges removes at most one edge from this perfect matching, giving a perfect matching on either $2n$ or $2n+2$ vertices. □

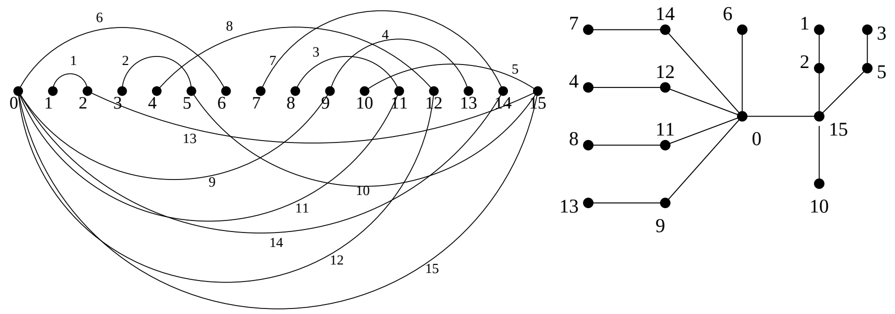

Figure 4: Obtaining a graceful tree with a perfect matching on $2n$ vertices

However it is not the case that any graceful tree on $2n$ vertices, where $n \equiv 0{,}1 \pmod 4$, gives yield to a Skolem sequence of order n. It may not be the case that the perfect matching involves the edges of value $\{1, \ldots, n\}$.

In the case where there a perfect matching on $2n$ vertices the tree obtained is always a 2-star with the following attached (let the root of this 2-star be r).

- A P_1 at the root vertex.
- A $2-star$ by its root vertex to a vertex, v_1, adjacent to r.

The number of vertices in each 2-star is dependant on the Skolem sequence chosen. However if the number of vertices in each is $2p+1$ and $2q+1$ then $p+q = n-1$.

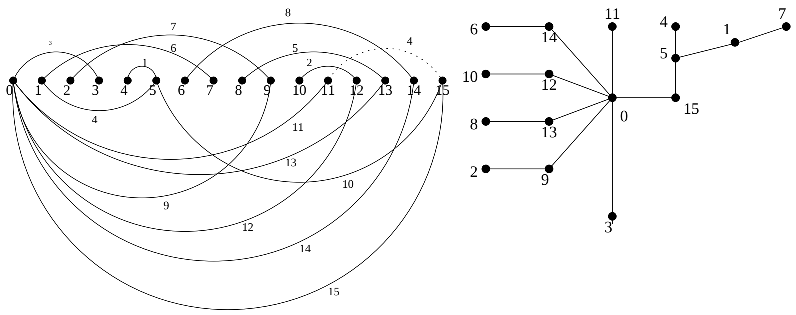

Figure 5: Obtaining a graceful tree with a perfect matching on $2n-2$ vertices

In the case where there is a perfect matching on $2n-2$ vertices the tree obtained is always a 2-star with the following attached (let teh root of this 2-star be r).

- A $2-star$ by its root vertex to a vertex, v_1, adjacent to r.
- A P_1 at v_1.

As well, a P_2 will be attached to some vertex of this graph other than v_1 or r. The number of vertices in each 2-star is dependant on the Skolem sequence chosen. However if the number of vertices in each is $2p+1$ and $2q+1$ then $p+q=n-2$.

Neither of these types of graphs have previously been shown to be graceful. However it not possible to say that this is a new family of graceful trees because it is unclear whether all such values of p and q can be obtained.

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