Errata and Suggestions for Improvement
in the
Mathematical Modeling Textbook Series

by

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On June 5, 2002, David Hamilton contacted Donna Gorman, Mathematics Consultant for the Nova Scotia Department of Education, to report an error he had noticed in the text *Mathematical Modeling* 2. After some discussion of the issues, David offered to provide a written report, describing his concerns about Chapter 5. David was aware that faculty at Saint Mary's University and UNB, Fredericton, have also been recording errors in the *Mathematical Modeling* series, and discussions between the three campuses have led to compilation of this report.

The authors of this report have considerable experience reading technical papers, yet we all found these texts difficult to read.

1 Background to the Common Atlantic Mathematics Curriculum

In 1989, the National Council of Teachers of Mathematics (NCTM, a large North American group) produced a document presenting standards for mathematics curriculum and evaluation, for kindergarten through high school graduation. (The document is available at `http://standards.nctm.org/Previous/CurrEvStds/`). The introduction to the Standards lists "mathematical expectations for new employees in industry," including:

- the ability to see the applicability of mathematical ideas to common and complex problems;
- preparation for open problem situations, since most real problems are not well formulated;

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- belief in the utility and value of mathematics.

Of course, employers value these skills, and educators should be aware of such expectations. Unfortunately, the new Atlantic Curriculum interprets the above to say: “All mathematics taught in the schools should be immediately applicable.”

The NCTM Standards also make several references to the importance of “technology” in mathematics education. The Texas Instruments company has based a brilliant marketing campaign on these references, convincing K-12 educators across the continent that graphing calculators are essential for understanding mathematics.

Today’s news media are full of statistics, and it is not surprising that these twin incentives (immediate relevance and technology) have led to an overemphasis of statistics in the school curricula. Students should certainly be encouraged to watch news reports and read newspapers critically. (Does the report explain how data were gathered? Does it give the response rate for a survey? Does it quantify the doubt associated with numeric estimates?) Indeed, newspaper and magazine articles provide an ideal scenario in which educators can make connections between language and analytical skills.

If some of our students are eventually to become practicing statisticians, then they need to understand basic mathematics. The writers of this curriculum were not prepared to say: “Trust us. If you learn this today, you’ll thank us later on.” Multiplication tables may not look useful to a child in Grade 4 or 5, but the teacher knows that to do (for instance) arithmetic with fractions those multiplication facts will be essential. Of course, many “experts” in grade school education say that, with the advent of calculators, there is no need to learn arithmetic with fractions. Such statements demonstrate a basic ignorance of mathematics beyond grade school. Imagine trying to learn the rules of algebra if you don’t know the rules for arithmetic with fractions. And so it goes on.

The theory of statistical inference is based on sophisticated mathematics. If students are to be shown the “black boxes” of statistical gimmicks available on a graphing calculator, then their teachers must be very much aware of when such boxes are appropriate and when they are inappropriate. Teachers must stress the importance of describing possible errors associated with conclusions based on data and, as with any subject, teachers must understand statistics at a level well beyond that which they expect of their students. The statistical concepts introduced in the series *Mathematical Modeling* are more difficult than the authors realize.

2 The *Mathematical Modeling* Series

In this document, our criticisms fall, roughly, into three categories:

1. Misinformation.
   (a) Errors in fact. Such errors occur primarily (but not exclusively) in the sections about statistics. This topic is treated much less competently than algebra, trigonometry, or geometry.
   (b) Errors in explanation. These errors are usually the result of over-brief and over-simplified discussion of difficult topics.
   (c) Errors in emphasis. For example, suggestions that trial-and-error is as good as an algebraic approach which can be adapted to handle more complicated problems; or basing algorithms / mathematical rules on observed patterns, in situations where students could understand an algebraic proof.

   Sometimes material is presented in illogical order. For example, students are expected to grasp the function $y = 2^x$ before they have seen expressions such as $2^{\frac{1}{3}}$.

   Some sections take up a lot of student and teacher time while providing little new information.

2. Poorly constructed questions and problems.

   In many cases, introductions to supposed real-world problems lack information which would help students to understand the situation, thus causing unnecessary confusion. In other problems, the "input" (numbers, functions, etc.) are unrealistic. (These examples are pedagogically bad, because they damage the development of the students' number sense.) In some applications of mathematics to other disciplines, technical terms are used incorrectly, and in some cases "facts" are just plain wrong.

3. Constructive criticisms which should be addressed when revising the texts. In some cases, these are suggestions for more lucid presentation of material. In others, they are examples of poor wording which one of us could not ignore.

Teachers need clear guidelines as to which results *must* be presented in class, and which are more "sideways enrichment." Teachers should be advised that, if time does not allow investigations and discoveries, then specified key material must be presented in more traditional ways.

We do *not* claim to have found all errors in the *Mathematical Modeling* series. However, we present evidence that extensive revisions are needed in all four texts.

A few minor points about conventions:

Grouping of digits. It is true that the "pure" Systéme Internationale mandates the use of spaces, rather than commas [or periods], to separate groups of three digits. However, this usage is by no means universal among scientists in North America, and is more or less unknown among non-scientists. It might be better to compromise with common usage, as the book already does with time units (kilometres per hour, cents per minute: both excluded by strict SI usage). Perhaps a sidebar feature would be appropriate, explaining the French, English, and SI conventions along with a warning that, when presented with a problem, students should check to see which convention is being used. The feature should mention also that year numbers are *never* broken.

Spacing between constants and trig functions Throughout the book, in expressions such as $3\cos x$, the space between the numerical constant and the function is omitted. This is nonstandard and should be corrected.

Parenthesizing function arguments such as $\sin(x)$**.** This is not done as often as it might be. It is pedagogically valuable as it discourages students from thinking of $\sin x$ as a product (and that $\sin x/x = \sin$). It also avoids monstrosities such as "$y = \sin 282\ 239x$" (Book 2, page 123).

Principal values While there is merit in the convention of using capitalized forms to distinguish the principal value *function* from the inverse *relation*, it should be made clear that this convention is not universally accepted and that students may expect to meet other conventions.

Real numbers, range. The use of "R" for both of these sets is unfortunate, especially as they are likely to be used together. The authors should use bold or "blackboard bold" for the real numbers, and consider "Range" or "Ran(f)" for the range.

3 Mathematical Modeling 1

In many places, key results are "buried" inside investigations and problems, with no clear summary of those key ideas. Notable examples are the illogical introduction to quadratics in Focus L of Chapter 3 (page 133) and the derivation of the formula for the sum of the interior angles of a polygon (Focus question 1, page 265). Many investigations and problems encourage students to notice patterns and draw conclusions, but opportunities are missed to show the difference between noticing a pattern (which may or not generalize) and actually proving a result. For example, students know enough algebra to actually prove the results summarized at the bottom of page 289 (effect of rescaling on surface areas and volumes), for all the figures considered in the chapter.

Organization of material in Chapters 3 and 4 is confusing. A possible reorganization would place straight lines in Chapter 3: both those straight lines arising from mathematical formulae and those forced upon scatterplots of data, with discussion of issues related to each. Chapter 3 should also include an introduction to functions. Looking ahead to Chapter 7, the new Chapter 3 should also introduce graphs of linear inequalities, and introduce systems of linear equations in two unknowns. (See the graphs on pages 112 and 118. The obvious question is not asked of the graph on page 118.) At this point, students should be made aware of the connection with Chapter 2: that systems of linear equations can be written using matrices (with some indication that they will see this topic again).

The new Chapter 4 would then introduce quadratics, with an introduction to one-to-one functions. Then, an extra chapter would introduce a few more functions. This extra chapter should include functions of the form $y = x^{\frac{1}{2}}$, $y = x^{\frac{2}{3}}$, etc, with the obvious introduction to the notion of an inverse function (after all, calculators have inverse buttons). Square roots are used in Chapter 5, and cube-roots in Chapter 6, so such functions should appear earlier. In its present form, the text assumes that students understand only non-negative integer exponents. So there is a logical inconsistency in Chapter 4 where functions of the form $y = 2^x$ are introduced.

Chapter 5 is correct, though the wording is often clumsy.

Chapter 6 is well-written, but time-consuming. Teachers need advice as to possible paths through the chapter, so as to make reasonable progress in the allocated time.

Chapter 7 starts off too slowly, then rushes the good stuff.

Many pedagogical experts believe that some students are more likely to understand how to factor quadratic expressions if they have access to algebra tiles. If so then, by all means, allow them to use the tiles *for a time*. But do not lose sight of the goal: ability to factor quadratics should eventually be a routine skill in each student's mathematical tool box. And problems requiring interpretation of diagrams of such manipulatives are counter-productive. (Diagrams are abstractions, and the manipulatives were introduced to help those kids who prefer the concrete.)

Last, but by no means least: serious errors occur in statistical topics (Chapters 1, 3, 4).

3.1 Chapter 1. Data Management

This chapter provides an extremely terse introduction to big statistical ideas. The authors demonstrate confusion about several concepts, including basic data types, outliers, precision, and fitting lines to data. By far the worst errors are those associated with fuzzy, incorrect references to the Central Limit Theorem.

This chapter should introduce scientific notation. Applications throughout all four texts should occasionally use scientific notation (in appropriate contexts), and remind students to answer with reasonable implied precision (significant digits).

Page 4. Controlled variable. Definitions in the text and sidebar are mutually contradictory. This one paragraph tries to summarize big ideas which need pages, with several concrete examples. Perhaps the word "independently" (second paragraph) is referring to the notion of "blocking?"

Page 9. Focus Question 2. Tools for measuring. "Why is it important to use the same tool to measure length and width?" If I wanted to measure the length of a guitar string, for example, then I would not use the same instrument that I would use to measure its width.

Page 9. Focus Question 4. Precision. " . . . the precision of a calculated measurement should be considered the same as the least precise measurement." This is an oversimplification. For instance, 102m + 0.5m is accurate to 3 significant figures, not 1. The sidebar on page 11 explains this better, but is still an oversimplification.

Page 12. Problem 14. Precision. "Environment Canada stated yesterday that the temperature was 21.225° C." The statement is nonsense. Environment Canada would never make such a statement. (For the record, we checked on this. But think about it.)

Page 13. Problem 21. Precision. Contrary to information given in the Teacher's Resource (page 17), 7503 mm is not approximately equal to 75 cm, but wrong by a factor of 10. A typical calculator-user mistake?

The Teacher's Resource says "The finest scale is in millimetres so students will probably have most confidence in 7503 mm." If the intent was to point out an anticipated student error, then further elaboration is required.

With reference to numbers appearing in the problem, 75.38 cm is the same level of precision as 750.3 mm and *more* precise in absolute terms than 7503 mm. The implication that a measurement in millimetres is more precise than one in centimetres, even when the cm measurement has one more decimal place is silly.

Page 14. A note about the mode. The mean and median describe location for continuous (measurement) data. Sets of continuous data often have no mode: we have to "round off' to see a mode (as in the height measurements on page 33). The mode is used as a measure of location for discrete data (including non-numeric categorical data).

These are important issues. University-aged students often find it difficult to distinguish between discrete and continuous data. They also have trouble identifying nominal versus ordinal data. An earlier course in this curriculum discuss different types of data, but the obvious "connections" have not been made in this text.

Page 15, Question 1(a). 100 **repetitions of an experiment.**

The answer in the Teacher's Resource (p. 20) suggests confusion between data as time series and behaviour of cumulative means: "It is likely that after some initial variations the measurements will 'cluster' around some value."

Page 15. Focus E. Outliers. "An outlier may be accurate but it is not typical." This is terse to the point of being vague. The authors were probably trying to make the following points.

Some sets of data contain a few observations which are much smaller (or larger) than all the rest. There are various "rules of thumb" for declaring an observation to be an "outlier." Some outliers are mistakes in the data (data entry errors, equipment failure errors, technician recording errors). Other outliers are *not* mistakes; they are recorded accurately and are very unusual. An example of an outlier that is not a mistake would be the height of the tallest person you have ever seen: your eyes did not deceive you; that person is unusual.

Page 16. Problem C. "Which is best?" should always have a description of the intended use. This problem is meaningless otherwise.

Pages 18-21. Boxplots. Since the concept of an outlier has been introduced, why not use full boxplots (which indicate outliers)? What the book currently uses is the "skeletal boxplot." The standard among statisticians and experimenters is the "full boxplot," as invented by the late John Tukey, perhaps the greatest authority on exploratory data analysis. This is not much more complicated and gives a standardized way of flagging outliers that is understood worldwide, as well as a quick and easy way to see the basic shape of the data set.

The box (extending from first to third quartiles) and the median line are drawn exactly as in the skeletal boxplot. The width of that box (technically called the "interquartile range," but "box width" will do in this context) is a nice descriptor of the variability displayed by the set of data. It is easier to calculate than the standard deviation, and easier to think about. A small box width means low variability; a large box width says high variability.

Tukey's insight, widely adopted by scientists, is that outliers are conveniently flagged as observations more than 1.5 box widths above the upper end or below the lower end of the box. We thus pencil in two lines 1.5 box widths above and below the box; these, called the"inner fences," are working lines that do not appear on the final plot. The whiskers extend to the most extreme data within these fences, not to the fences themselves. Every datum beyond the inner fences is an outlier, and is indicated by a dot.

Many researchers go further and draw a second pair of fences 1.5 box widths outside the inner fences (3 box widths beyond the box), and use a ring or bigger dot to indicate really unusual data that are even outside the outer fences.

Points which are isolated in this way by a boxplot are called "outliers" (or "extreme outliers," for the really unusual ones). In particular, in approximately normally distributed data, way fewer than 5% of observations are classified as outliers.

Boxplots have two main uses:

1. Statisticians use them as a way to take a quick first look at large sets of data, looking for possible recording errors, unusual shapes, etc.

2. When plotted on the same scale, they provide quick informative comparisons of two or more sets of data. (eg male and female heights).

The boxplots on page 21 would give quick visual comparisons of relative bowling abilities *if* the same scale had been used for all five players. This demonstration misses the point of boxplots.

Page 24, Procedure E and Page 25, Problem 27 (Description of data) Here, students are asked to "Rate how your set of data is distributed using a 0 to 10 scale." This scale is *not* ordinal, though the use of numbers implies a natural ordering. A data set could be "moderately spread out" (Category 5) whether or not it had outliers (Category 7), and a data set could be "really spread out a lot" (Category 10) with no outliers at all.

In a chapter intended to introduce students to descriptions of data (in their many forms), the authors have used an inappropriate ordinal scale.

Since this is an "investigation," students must decide what constitutes "moderate" or "a lot of" spread. These are difficult questions for amateurs to answer.

Page 24. Question 21. Description of data. "Explain why the range has no influence on your rating in step E." Outliers are associated with a large range. Generally, range increases with standard deviation. So, what is the intent of this question?

Page 24, Question 23. Effect of removing outliers. This is a confusing treatment of outliers. The largest/smallest observation in a data set need not be an outlier. In this problem, "distribution" surely means "shape of histogram." Technically, we describe the basic shape of the histogram in the same way, with or without the odd outlier and, technically, one very large/small outlier can cause a huge change in any one of mean, standard deviation, range.

Good students and teachers will realize that this is a huge question, and wonder what answer was expected, while weaker teachers will probably give confusing answers.

Page 29. Table to calculate standard deviation. A big idea in statistics is that of "residual," the difference between an observation and the predicted value under some prescribed model. (In this example, the predicted value is the average, 10.4). The calculation of standard deviation is a student's first exposure to residuals, even if the term is not used. One important feature of a residual is its sign, positive if the observed value is greater than the model value, negative otherwise. The signs in column two of this table are precisely the opposite of what they should be. While the calculation of the standard deviation is correct, graphical description (using dots on a number line) would make much more sense to students if positive numbers in the second column corresponded to observations that were greater than the observed average.

Page 31. Question 7. Quality control. Surely the goal of the manufacturer is to minimize some measure of the number of bubbles per jar. Choice of an appropriate measure is difficult: mean? median? 90th percentile? A secondary consideration would be consistent performance.

Better students will be confused by the emphasis of this problem.

Page 37. First line. Outlier. This definition flags 5% of data as "outliers." The usual algorithm used in construction of boxplots would label about 0.7% of data from a large sample of Gaussian data as outliers (2% of a sample of size 30, 3% of a sample of size 20). Are the authors aware of these issues?

Page 37. Sidebar. Normal distribution. "A normal distribution occurs when the data size is very large and selected randomly." This blatant error, which occurred more frequently in earlier drafts of the text, suggests that the authors do not understand the Central Limit Theorem.

Page 38. Question 10(a) : Inference based on sample estimates of mean and standard deviation. The problem here is with imprecise language. We don't *know* any of the individual measurements. The problem should be changed so as to ask: "Given the information, what would you guess to be an unusually large (or small) sleeping time? What would you consider large, but not unusually so? Where would you 'draw a line' between unusual and not unusual?"

This is just one of many examples in which the authors illustrate the NCTM statement that "most real problems are not well formulated."

Page 39. Problem 14. Time series data. "In what year was there an unusual number of films spotlighted?" The teachers' manual (page 47) claims that there are no such years. This data should be treated as a time series, with an increasing trend in all three measured variables. To appreciate these ideas requires maturity well beyond that of high school students.

This section of the Teacher's Resource repeatedly writes:

"outlier = [mean] + 2[SD] and outlier = [mean] − 2[SD]."

Small wonder so many of our students abuse the symbol "=."

Page 40 ff. Line of best fit. How is this line to be determined? Later, on page 42 (F), the reader is directed to use the eye-and-ruler approach, which is fine.

Page 41. Investigation 5. Fitting a line to data. Step B creates the impression that raw data must be combined before drawing a scatter plot. There is some attempt to point out that each student's line will be different, but no attempt to point out that, on another day, any one student's line will be a bit different than his/her previous line. (In fact, with the eye-and-ruler technique, a student could well fit different lines to identical scatter plots.) There is not sufficient emphasis placed on quantifying doubt, margin of error, whatever you want to call it. Teaching statistics without these ideas is dangerous, if not unethical.

Graphing calculators actually give numbers that help to quantify the doubt, but this is difficult material for students and teachers in grade 10.

Page 45. Extension 1. Controlled dependent variables. This is a confusing problem. Average scores are provided for students at a music festival, broken down by age group and type of instrument. Both age and instrument are independent variables. Presumably, we are interested in two questions:

Does average score vary with instrument?

Does average score vary with age group?

Somehow, we want to answer each of these questions, *allowing for* the effect of the other factor, a classic two-way analysis of variance problem, way beyond high school.

But what is expected here? Any sensible student will say: "I don't have 'control' of anything. Kids register for a music festival; some are younger than others; some play piano and others play strings, etc." A sensible student would want to know how many children contributed to each of the nine reported averages.

Page 51. Point 1.5. Central Limit Theorem. What is a "large distribution?" The statement "This distribution results when very large amounts of data are collected randomly about a characteristic of the population" is *incorrect*. See the above comment about the sidebar on page 37.

3.2 Chapter 2. Networks and Matrices

Over the last few decades, mathematics professors across the continent have been invited to talk at teacher in-service functions. Many of us have tried to stretch the teachers' minds, introducing thought provoking topics beyond the standard curriculum. The intent was not to suggest that such material should be added to the school curriculum, and certainly not that it be added at the expense of more basic material. Networks and graphs have been a popular topic at such in-service events.

Page 58. Sidebar. Complete. This is not the standard meaning of "complete." The correct word is "Eulerian."

Page 59. Sidebar. Euler circuit. This is not what an Euler circuit is, either. An Eulerian graph *has* an Euler circuit.

Page 69. Question 12(a). Paths in a directed graph. The assumption that the routes are acyclic is implicit in the answers given in the teachers' manual and is contrary to the spirit of the matrix formalism.

Page 74 ff. Matrix multiplication. Matrices are used in many fields. Their entries may be positive or negative, and certainly need not be integers. A few problems requiring multiplication of matrices with fractional entries would be appropriate here, *and* a valuable opportunity to review basic arithmetic facts before the algebra of subsequent chapters.

A simple application of matrices, which convinces many students that matrices any indeed be useful, is this. Given a spread sheet of numbers (rows labeled by student names, columns labeled by test scores), show how to calculate each student's term mark (possibly a weighted average) as a problem requiring the multiplication of two matrices. etc.

3.3 Chapter 3. Functions, Relations, Equations, and Predictions

Most of the comments listed below refer to sloppy use of language, which is unforgivable in a textbook. Some issues are discussed more thoroughly in the errata for Chapter 4. As in other chapters, a few practical examples are overused. By the end of Chapter 3, students will be thoroughly fed up with problems about internet connection fees.

There are two major conceptual errors in this chapter. The first concerns the idea of average cost (per hour, say) versus marginal cost (for one additional hour). Consider the example used to introduce Section 3.3 (page 111). The wording is sloppy. It would be better to describe the fee scales as follows: "Company A charges a monthly fee of \$20.00, *plus* an additional fee of \$2.00 per hour connection time (or part thereof)." Similarly for Company C. The connection-time fee describes the *marginal* cost of each additional hour connected to the web. If I use Company A and connect for a total of 3 hours over the next month, then my bill will be \$26.00, which *averages* to \$8.67 per hour. If, during the following month, I connect for many more hours (let's say 10 hours altogether), then my bill will be \$40.00, which averages to \$4.00 per hour of internet connection. This important point is lost on the authors of Chapter 3, and has led to serious errors. See also: Page 119. Question 3 (average hourly rate depends on hours of use); Page 120, Question 7(a) should read "How much does it cost to print each *additional* copy of the school newspaper? What is the average cost per copy if 1000 copies are printed? 2000?"; Page 123, Question 19; Page 127, Question 28.

The opening lines of Section 3.3 (page 111) ask the all-important question about the two internet companies: "For what number of hours of Internet use are the costs the same?" This is an important question because the answer (20 hours) describes a critical number of hours of monthly internet use: below 20 hours one should use Company C; above 20 hours one should use Company A. We cannot find discussion of this crucial idea anywhere in the chapter.

The other major conceptual error is with the unusual (and inaccurate) introduction to quadratic functions. These errors occur in Focus L and Focus N, and are not corrected elsewhere in the text.

Page 133. Focus L. Julia's light-bulb idea is illogical. *If* I know that a function is of the form $f(x) = ax^2 + bx + c$ and if I also know that $f(1) = 0$ and $f(3) = 0$, then I can certainly prove that $f(x) = a(x-1)(x-3)$. *However*, there are many *other* types of function g for which $g(1) = g(3) = 0$.

For example, the graph of $y = \sin x$ (with x measured in degrees) crosses the x-axis at $0, 180, 360, \ldots$, but multiplying $(x-0)(x-180)(x-360)\cdots$ gives no insight into the relationship between x and y.

Page 137. Focus N. The crucial idea behind using factoring to find the x-intercepts of a quadratic function is this:

> If two real numbers w and z satisfy the equation $wz = 0$, then at least one of them is zero.

We cannot find this point mentioned, let alone stressed, in Chapter 3.

Other problems with Chapter 3:

Page 97, C; page 99, 10(a); page 100, 13(e). "Are the data continuous or discrete?" We understand why this question was asked on pages 97 and 99, but does the reader? The discussion on page 97 should state the obvious, that to join points would imply the possibility of parts of cubes. In the same vein, 10(a) should simply ask: "Does it make sense to join the points?" But, on page 100, the reader does not know the story behind the points on the graphs. So how can the question be answered?

Discussion of discrete versus continuous data is sadly missing in Chapter 1, confusing in Chapter 3.

Page 97, Question 5. (b) "Is a graph the best way to do this?" If the graph is drawn at all carefully then (since the answer to 5(a) will be a positive integer) the graph is as good as any other method.

5(c) "How confident are you that your answer is correct?" In Chapter 4, little attention is paid to quantifying doubt. In a situation where there is nothing random about the underlying model, the authors ask about doubt.

See also page 131. Question 2(c) and the next point.

Page 99. Sidebar. "When you use a graph to find a value, you can only estimate the number of cubes." The authors are confusing two kinds of scatter plots

- those that arise from observations which vary randomly about an underlying model;

- those that arise from plotting points (x, y) where y is a function of x.

Such confusion is more blatant in Chapter 4. In this particular problem, any tidy graph will lead to correct answers, since students understand that the answer will be a positive integer.

Page 100, Question 14. Unrealistic data. If Frau could find a way to measure gas consumption at 1-hour intervals, then the data would *not* fit perfectly on a straight line.

Page 101. Sidebar. The definition is provided for an "equation" in one variable, x. Students have already been deriving equations involving two variables.

Page 102. Question 19(a). "Check the accuracy of your estimate." How?

Page 109. Focus Question 10. "Explain why constructing a table of values and / or using a graph would not be good strategies for getting an exact solution for this problem." Yes, it is best to resort to algebra when the numbers get complicated. Curriculum documents do not stress this point.

Page 109. Instructions for Questions 11–13. Each equation involves *two* variables, not one.

Pages 112, 117, 119. Intercepts. The sidebar on page 112 asks a question about the y-intercept, which is defined on page 117. The more general notion of intercept does not appear until page 119.

Page 122. Focus H. Realism. This seems to be a "friendly loan" with no interest. Perhaps the student borrowed from a parent. There should be more detail in the introductory story.

Page 123. Question 17(a). "Write an equation to represent the relationship." The authors probably intended to say: "Write an equation to represent the relationship between the number of families who came to the food bank and the total weight of potatoes given away."

Page 123. Question 18. Grammar. Paragraph should read: "In 1996, . . . Water levels and rate of flow *were* tracked . . . "

The note in the sidebar admits that 18(a) belongs in Chapter 4.

Page 123, Question 20. Extrapolation. Students are asked to fit a straight line to data which surely requires a different model. They are then asked to extrapolate way beyond the range of observed data.

Page 125. Question 23(a). This problem requires ideas from Chapter 7. No hint is given that this is a challenge problem. It is under the heading "Check your understanding."

Page 125. Question 25. Wording. The intent was: "For *weekly* sales of ..." Writers of textbooks should bear in mind that they are setting an example of (what should be) logical thinking and precise use of language.

Page 126. Question 26. Wording. Better to say: "Refer to the problem described at the beginning of this Investigation."

Page 127. Question 28. Wording. In order to do this problem, we need to be told that Craig is renting from a store that charges an annual membership fee. (Not a common practice these days.)

Page 130. Investigation 7. The pattern is difficult to follow: three of the four squares have black in the top left position. For an obvious pattern (add a row on the bottom and on the right) reverse the colours in the second square.

This investigation introduces quadratic functions. Questions should move from the familiar (straight line equation in H) to the unfamiliar (A–G).

Page 131. Question 1(a). Grammar. "How much area *would be* covered?"

Page 132. Sidebar. "Why does the π key give you a more accurate answer than solving by hand?" Does "by hand" refer to the approximation 22/7 ?

Page 133. Focus L. "Describe the shape of the graph using the word symmetry." The graph (six points) has no axis of symmetry.

Page 134. Sidebar. quadratic. Are students any wiser after reading all these words?

Page 139. Sidebar. Challenge yourself. The phrase "an expression" should read "the expression $x^2 + 6x + 10$."

Page 140. Question 26. Grammar. Problem with tenses: David "plans" to do something that he is already doing.

Page 140. Chapter Project. For clarity, (a) should start with a sentence such as "Sketch two or three dog pen designs, carefully indicating lengths for each side of the pen."

Questions about maximum area are relevant only after constraints have been placed on the problem (e.g. Seana can afford to pay for only so much fencing).

Page 142. Question 4(b). Wording. This problem should read: " ... These four people each told four others *who had not yet heard the rumour*, and so on."

Page 144. Question 11(g). It is good to see a problem for which the answer is not an integer, but this problem is beyond the scope of "mental math."

Page 146. Continuing Case Study 2. The instructions are incomplete. If there is to be an "equation," then more than one number must be requested. Students should be asked to describe independent and dependent variables.

Page 152. Questions 3 and 4. "Assume a partial ride is charged a part of the \$0.60." Is the term "partial ride" meant to mean "a part of a minute"? The question might be better rephrased as:

"The cost of a taxi is \$2.00 to start and \$0.60 per minute; fractions of a minute are charged for proportionately, so that (for instance) a ride of 1 minute and 40 seconds would cost \$3.00."

Page 153. Question 8. "...pairs of numbers whose average always equals 2." The word "always" is redundant; either the average of a pair of numbers equals 2 or it does not.

3.4 Chapter 4. Modeling Functional Relationships

This chapter considers two very different sorts of graphs: those that arise from plotting points (x, y) which satisfy some kind of mathematical equation (e.g. problem 8(b), page 168, relates the volume of a cylinder to its radius), and those that arise from plotting empirical data. With the first type of graph, there is nothing random, no "doubt." There is all kinds of doubt (also called "error") when lines are fitted to empirical data. There is no discussion of the inherent differences between these two sorts of graphs and, consequently, the material is confusing and disjointed.

Both situations (mathematical equation, empirical data) can lead to plots which are not the graphs of functions. The text seems to imply that non-functions are associated primarily with empirical data, omitting discussion of classic mathematical equations whose graphs are not the graphs of functions. The text omits explanation that least squares models force a functional relationship to a scatter plot.

The haste to introduce data-driven examples has confused introduction of the basic mathematical concept of function. For the purpose of this text, a function f is a *rule* which associates a unique real number $f(x)$ with each real number x in its domain. We often write $y = f(x)$ and describe the function f as a set of ordered pairs (x, y), graphing the set of points on coordinate axes. When we use the word "function," the rule should be clearly stated in the form $f(x) = \ldots$ or $y = \ldots$.

Some teachers believe that students can perform the exercise:

sketch the graph of all points (x, y) which satisfy the <u>equation</u> $-(y + 3) = x^2$

more readily than they can perform the exercise:

sketch the graph of the <u>function</u> $y = -3 - x^2$,

even though the graphs are identical. The concept of "function" is so important that we should never be sloppy with its use. In particular, it is appropriate to use the word "function" for the second task, but *not* for the first. Throughout Chapter 4, and in sections of Book 3, the word "function" is used for the first task. The list below does not include all such sloppy uses of the all-important word "function."

Page 176. A–D. (Sloppy use of "function.") What is the advantage in writing $3y = x^2$ rather than the more usual $y = \frac{1}{3}x^2$? With the latter notation, one can say (see D): "I can see a vertical 'stretch' of one third." There is no need to take a reciprocal.

Page 177. Question 9(a). The equation $2\frac{1}{2}y = x^2$ displays confusing, non-standard notation.

Pages 177 ff. There is some advantage to writing an equation such as $y = \frac{1}{3}(x-4)^2 + 5$ in the form $y - 5 = \frac{1}{3}(x-4)^2$ until students understand the geometry, but functions defined by equations of the form $y = \ldots$ are fundamental in mathematics, so the goal should be understanding of functions in the classic "completed square" form.

Page 192, sidebar. The"Line of Best fit" is still undefined, though the suggestion seems to be made that there is a unique such line that "allows everyone to get a consistent equation."

Page 192. Median-median line. Why confuse the issue by having points on the vertical lines in the figure? There is an error in the first median-median point. This is not the median-median line used by other authors, and it is easy to construct examples where the line proposed here gives silly answers.

Pages 196. Correlation coefficient. There is nothing wrong with introducing the word "correlation" and using it when describing the relative strengths of trends in two different scatter plots. We see no advantage to presenting it as a number that can be obtained by pushing buttons on a calculator, with no indication of how it might be calculated.

Page 196. Step D. Fitted Line. This step of the procedure asks for the "linear-regression equation" . . . yet another kind of line to fit to data.

Page 198, Sidebar. Fitted Line. By this point, students have seen several ways to fit a straight line to a scatter plot of points. They must be wondering: "Which line should I fit?" Rather than answer this pertinent question, the second note in the margin asks students: "Do you think that the least squares line is the more accurate line?" There are theoretical arguments for using least squares in many situations, but such arguments are beyond the scope of this course.

What makes a line "accurate"? It is easy to construct examples where the least squares line does not give an accurate description of any pattern in the data. The word "accurate" is used frequently in these texts, and its intended meaning is often unclear.

Page 199. Investigation 6. Fitting curves to data. Several curriculum documents encourage students to plot a set of points and then search their graphing calculators until they find a curve that fits the scatter. Such practice is called "data snooping," and is *not* the way to do statistics: one tends to find results which cannot be replicated. This investigation teaches students bad habits.

3.5 Chapter 5. How Far? How Tall? How Steep?

This chapter is technically correct. Most of the comments below refer to inaccuracies in wording, and lack of connection to other chapters.

Page 213. Second set of triangles. The phrase "not a dilation" is confusing. Suggestion: "A dilation, followed by a rotation."

Page 215. Sidebar. There is room in the sidebar to review a few facts which are used in this chapter: a straight angle is 180°; the sum of the angles in a triangle is 180°; formula for area of a triangle.

Page 217.8. Focus C. The statement that "compass bearings are based on magnetic north" is not true in general, and the addition of the sidebar note (page 217) "this focus assumes that you are working with magnetic north" does not make the first statement correct. These two statements add nothing to the focus section, and should be removed. A statement —perhaps in the "Did You Know?" on page 216—to the effect that *orienteers* use bearings based on magnetic north would be useful.

Page 218. Question 16(a). " . . . after they have followed their instructions." It is not clear from the context that Jane and William were following instructions.

Page 219. Introduction to Chapter Project. "with isosceles triangle bases." The "bases" are actually at the ends of the roof shown. Perhaps it would be better to say: "with isosceles triangle roof supports."

"The roof can be different sizes." Poor grammar.

Page 222. Question (b). "Would an instrument that is 55.0 cm long fit inside the case?" The answer depends upon the *width* of the instrument (especially near the ends).

Page 223. Sidebar. Proof. Where in the text is "proof" defined or explained?

Page 222–223. The symbols A and B have different meanings on these two pages.

Page 226. Introduction to square roots. The preceding chapter spent considerable time on quadratic functions, yet there is no reference to the graph of $y = x^2$ when the "principal square root" is introduced—nor elsewhere in the chapter.

With a carefully drawn graph it is easy to demonstrate that, for example, $\sqrt{9+4} \neq \sqrt{9} + \sqrt{4}$.

"The square root of 16 is 4 and −4." Grammatical issues aside, this statement is false.

It is appropriate to say: "There are two solutions to the equation $x^2 = 16$, 4 and −4 . . . ."

Page 227. Sidebar. "Consecutive sides of a square are perpendicular." Most mathematicians would say "adjacent" rather than "consecutive."

Page 230. Problem 7. This problem is illogical. For example, $\sqrt{72}$ could be the length of the longest side of a flower bed similar to any one of the three given triangles: dilations of 6, $\sqrt{14.4}$, $\sqrt{7.2}$, going clockwise from top right. (The authors of these texts are disturbingly unaware of non-integers.)

Page 231. Questions 12, 13. There is room here for one harder problem, with the smaller square placed in the *middle* of the big square, rather than in a corner.

Page 133 ff. Investigation 4. This investigation should at least refer back to Investigation 1, which is remarkably similar.

Page 235. Figure. This simple figure should have appeared much earlier in the chapter.

Page 236. Sidebar. Definitions. "$\tan X$—a constant value based on the ratio of the side opposite to angle X to the side adjacent to angle X in a triangle." If tan were constant, then all values in the last row of the table on page 234 would be the same (and we wouldn't need tan buttons on calculators).

Of course, the authors are trying to say that the size of angle X is all that matters; that, no matter what their side lengths, *all* right triangles with angle X will yield the same value for $\tan X$. Wording of definitions must be clear and precise.

Page 236. Sidebar diagram. This diagram suggests that $\tan 45°$ is some value greater than 6.

Page 237. Question 16. This problem introduces the classic $30° - 60° - 90°$ triangle. This triangle and the classic isosceles right triangle should be featured in a sidebar of the chapter.

Page 238. Trig table. It would be appropriate to include a note about interpolation, for angle measures that are not whole numbers.

Page 242. Question 8. The story should clearly state that the first pole is directly in front of Jesse.

Page 243. Question 11(a) The hexagon in the diagram appears to be "equilateral;" this means only that the edges are the same length. An equilateral hexagon need not be convex, or have opposite edges parallel. In order to make progress on the problem, students need more information. If they are to assume that the hexagon is also "regular," then the story should clearly say so.

3.6 Chapter 6. The Geometry of Packaging

This Chapter is well written, and uses techniques learned in the preceding chapter. There is a lot of packaging for a relatively small amount of geometry. As in other chapters, teachers need to be told which material is crucial for understanding of subsequent sections of the chapter, and which material can be skipped.

Answers that are irrational numbers must not be given as integers or terminating decimals. Question 6, page 27 (and Teacher's Resource page 321) provide an example; we have not recorded all instances of this mistake in the texts.

Page 257. Shipping costs are a major consideration of package designers.

Page 159. Second set of figures. All symbols appearing on formulae should also appear on the diagrams: w, d, r are missing.

Page 265 ff. Focus B. "Regular polygonal prisms have rotational symmetry equal to the number of sides of the polygon." This is sloppy usage. The rotational symmetry (within the plane) of a polygon is not a number; it is a *group* whose *order* is the number of sides. These concepts should both be rigorously defined. If necessary, it is permissible to forget the group operation and consider the symmetries as a *set*.

Moreover, the rotational symmetry group of a regular polygonal *prism* is generally twice as large as that of the corresponding polygon. It includes rotations, about axes parallel to the top and bottom polygonal faces, that interchange the top and bottom face. (In the special case of a cube it is still larger, including rotations that carry the top and bottom to sides.)

The very next sentence is, at best, vague (but probably misleading): "This makes it easier to design a lid and packaging for the container." Rotational symmetry has nothing to do with the ease of design. In fact, for many purposes, a rectangle (rotational symmetry group order 2) is easiest. Cylinders permit easier manufacture and the use of threaded lids.

The sidebar defines rotational symmetry as "the property of a shape where … ." Incorrect grammar.

Page 265. Question 1. Results from this problem are so fundamental that they should be summarized somewhere in the chapter. (They are needed on the next page.)

Page 267. Question 7 and Teacher's Resource page 321. Since the number of sides must be an integer, the continuous graph shown in the sidebar of the Teacher's Resource is inappropriate.

Page 268, Question 9(c). Page 277, Question 4. The term "limiting shape" is sloppy. (One should say "...as the number of sides goes to ∞... .") The term "maximum shape" is simply wrong.

Page 269. Question 12. Unrealistic numbers. Rectangular lumps of fudge (as illustrated) of any reasonable size will not come close to packing such a box (base not square, barely 3" across, half that in height). In particular, the implied precision of "3.8 cm" is unrealistic.

Page 268. Question 10(d). "Research the best shape for an eavestrough. Why do you think it is the best shape?" A far more reasonable problem would be: "What shapes are used for the cross-section of eavestroughs? Why do you think these shapes are used?"

Page 275. Economy Rate. Students should be taught that economy rates have dimension, equal to length (measured in cm on page 275), and are not a dimensionless quantity as suggested.

Page 279. Cube-root. How many parents will rush out to try and buy a calculator with a cube-root button? See also pages 280, 281: "Use a calculator to find out." See our introductory comments on the organization of *Mathematical Modeling* 1.

Page 279. Question 9. The intent is not clear here. Perhaps the following was intended (in the spirit of Chapter 1): "You are handed a cube which has (although you are not told this) volume 100cm^3. How would you measure the edge length of such an object?"

Perhaps this meaning was intended: "You are handed a cube, and told that its volume is 100cm^3. Knowing the volume, how would you calculate the edge length of such an object?"

Page 282. Question 1. Volume of a snow house. Most Canadian children would wonder what was expected here: the walls of an igloo are thick; the *usable* inside volume would be much less than that calculated from the overall diameter.

Page 282. Question 2. Realism. Most Canadian students know that a snow house with square base and flat roof would be impossible to build.

Page 285. Example 1. Unnecessary confusion. Since 3 appears as a dimension on each of these boxes, the set of inequalities is unnecessarily confusing. The ratios 2: 3: 5 and 1: 1.5: 2.5 would be better.

Page 289. Focus G. Missing proof. We find a (rare) summary of facts "discovered" in preceding problems. These particular facts can be proven: a worthwhile algebraic exercise.

Page 290. First paragraph. Overly brief explanation. It is good to see a summary, but it is too terse. Address cylinders and prisms separately, and then state the obvious.

Page 293. Question 1(b). Order of presentation. We hope that teachers are reading ahead and can tell their students that the next few pages will show them how to do this problem.

Page 292. Sidebar. "Think about . . . . " The statement that a triangular mailing tube is more rigid than a cylindrical tube is simply wrong. A triangle is "most rigid" as a *framework*, with rigid struts joining pivoting corners. A triangular mailing tube made of somewhat flexible material is more subject to crushing than a cylinder, as students can readily demonstrate. Note that most bones are round.

Page 292. Sidebar and figures. The perpendicular bisector, median, angle bisector, and altitude are all lines. The sidebar describes the first as a line and illustrates it as a ray; describes and illustrates the second and fourth as segments; and describes the third as a segment and illustrates it as a ray.

Page 293 ff. Investigation 8. Simple **proofs** of the four line coincidence theorems exist. In keeping with both the *NCTM Standards, "Reasoning and Proof,"* and mathematical tradition, these could be introduced (for the strongest students, at least) after their exploration. The fact that a proof exists would reassure those students who (due to clumsiness or cheap compasses) failed to achieve concurrence.

It would be nice to have stronger students explore the altitudes of an *obtuse* triangle, and discover that these intersect even though they may hit their bases outside the triangle! On the same theme, they can determine whether the other three points can ever be outside the triangle.

A rediscovery of the Euler line (" Do any three of the four points you've found appear to lie on a straight line? Check with your classmates—have they found the same thing?") would not be out of the question, though the proof is definitely too hard.

Page 294. Question H. Since a triangle is sometimes defined as a figure consisting of three line segments, it should be made clear that the center of gravity of the *interior* of the triangle is intended.

The only way that the Teacher's Resource suggests of obtaining the answer is a physical experiment. It is true that we do not have the axiomatic background at this point to work rigorously with this concept. One could at least make it plausible by showing that every segment $XY||BC$ with $A-X-B$ and $A-Y-C$ must balance on the median from A to BC (using similar triangles); it is then reasonable to suppose that the whole triangle does.

It is interesting to note that the barycenter is also the center of gravity of the three vertices, but *not* in general of the edge skeleton.

3.7 Chapter 7. Linear Programming

The key idea in linear programming is this: if the feasible region is bounded by straight line segments in the xy plane, then any objective function of the form $f(x, y) = ax + by + c$ will be minimized / maximized at a vertex of the feasible region. Chapter 7 spans pages 305 to 335. The term vertex is defined on page 333, and the fact that the vertices are all that matter is first alluded to in the worked example on pages 332–333. (It is hoped that students "discovered" it earlier, but there is no formal reference to this key fact.) Heather and her problem are introduced on page 306, but the reader does not see a graph of the feasible region until page 317. Progress is too slow for those students who have mastered the preceding six chapters. All students will be frustrated by the relatively small emphasis placed on description of a formal algorithm. The practice problems on pages 334–335 are appropriate. Teachers *must* schedule time so that students work these problems.

Valuable opportunities have been missed in this chapter: there are obvious connections with material in Chapters 2 and 3. See our introductory comments on the organization of *Mathematical Modeling* 1.

Page 306. Too vague. The introduction to Heather's problem is too vague, and will confuse readers. It begs the question: "For how long can Heather work?"

Students will quickly realize that there are many constraints on such a problem. The introduction to constraints moves too slowly.

Page 307. Focus A. Setting examples. Think about the example being set here. No time is allocated for leisure reading, or for help with chores around the home. Question 5 could be interpreted to suggest that school students could work at a job for five hours per day, seven days per week. Question 6(b) could be interpreted to suggest that regular exercise is not necessary.

Page 308. Question 7. Photo in sidebar. Realism Most individually-owned vegetable farms are much smaller than 2000 ha. *If* Sue owns such a large farm, then she will not be tilling by hand as shown in the photograph.

Page 312. Sidebar. Definition of inequality. Once again, an attempt to be terse leads to a misleading statement. It is best to define strict inequalities *first*. For example, the inequality $x < 12$ describes a set of real numbers: all the numbers to the left of 12 on a number line. Then $x \leq 12$ describes a set that has one additional point, etc. If this is the students' introduction to inequality symbols, then the topic deserves more attention. See our introductory comments on the organization of *Mathematical Modeling* 1.

Page 312. Procedure B. Bullet 2. In order to avoid student frustration, add the extra instruction: "with the *same scale* as that used in Procedure A."

Page 313. Note in Sidebar. The statement "All units must be the same," is clearly incorrect. For Heather's problem, (as is clearly stated in the note on page 312) the variable x represents the number of chair bundles cut, and the variable y represents the number of couch bundles cut (as is clearly stated in the note on page 312). One might respond that both variables describe " number of bundles," but look at some more problems. For example, in problem 1 on page 334, the units are "pillow" and "cushion."

Page 314. C. Realism Surely hockey stick manufacturers make fewer goalie sticks than regular sticks?

Page 315. Question 14. Confidence rating. Discussion of "confidence level" is appropriate when making inferences based on random samples. The teacher should be continually try to gauge students' grasp of material. But this question is inappropriate and will cause confusion. See also question 3(c) on page 320.

Page 316. Question 16. Such simple inequalities belong earlier in the chapter?

Page 316. Sidebar. Photograph. There should be a caption to say who is in the photograph.

Page 316. Question 17(b) This problem should read: "Draw a set of coordinate axes, labeled s and p, and mark in all points that satisfy the constraints." (Note that the appropriate graph will show discrete points.)

Page 317. Focus C. There is a huge jump here. The discussion between Heather and her brother assumes they understand that the answer to question 9 on page 313 leads to graphs of the form shown on page 317. The introduction to Focus C should clearly make the connection.

Page 318. Discussion of graphs. The solid line / broken line notations are *conventions*, not facts to be "discovered." (Even the issue of which side of the line to shade is a convention. If we shade in the other sides of the lines, then the feasible region shows up clearly as the only unshaded area.)

At this point, it would be appropriate to remind students about the graphs of equations of straight lines. That is, make the connection with Chapter 3.

Page 318. Question 20(d) The word "minimum" should be "maximum."

Page 318. Sidebar. Challenge. Grammar. Tenses.

Page 319. Sidebar. The definition "Objective function—a function that allows you to find the maximum or minimum values using the given constraints," is badly worded to the point that it is incorrect.

The objective function is the function which one wants to minimize or maximize. (It is usually a function of several variables, and those variables are usually required to satisfy constraints.)

4 Mathematical Modeling 2

Most of the chapters of this book are reasonably accurate, although there are many small mistakes. One structural change that ought to be made involves the introduction of radian measure. This is done at present at the end of Chapter 4, well after sections in which radians should have been used.

In Chapter 3, and in Section 4.1, angles are measured exclusively in degrees. This is unfortunate, as for most calculations involving periodic functions and phenomena the radian is the natural unit (because of its pre-eminent position in calculus), with the cycle ($= 360^\circ$) probably the runner-up. Serious consideration should be given to moving what is now Section 4.3 to the beginning of Chapter 3, and thereafter stating problems involving physics in radians, and those involving music, etc, in cycles.

The convention that radians are not labeled while all other units are must be emphasized. Conversion between units should be practised wherever possible, often as a rider ("What is this in degrees?"). Explicit discussion of which unit is appropriate for a given problem should be encouraged, and students should develop this skill themselves. (In section 4.2, the *unit-independence* of trigonometric identities should be stressed; the book's practice of stating "x is in degrees" in such a problem is to be deprecated.) A black-line master for a radian protractor would be a nice touch.

In Chapter 6, the emphasis on degree measure is appropriate, although some exposure to trigonometry done using radians would not hurt. The emphasis on degree measure in Chapter 4 of Book 3, and on radian measure in Chapter 4 of Book 4, are appropriate.

In contrast with the other chapters, Chapter 5 of this book is extremely poorly written, to the point where it should probably never be used in the classroom. This is not hyperbole; just about every topic covered involves fundamental misconceptions. Details are given below.

4.1 Chapter 1. Investigating Equations in 3-Space

The chapter title is a misnomer since, in fact, the dimensions of the spaces involved run from 2 to 5. The mathematics is mostly correct and appropriate, except in Section 1.5.

The selection of problems shows little imagination. One theme—problems involving alternative schemes of payment—is stressed to the near-exclusion of others. For a textbook series that claims to stress modeling and problem solving, this situation is a step backwards from the days of bathtubs, lawnmowers, and antifreeze. For students to learn to transfer knowledge and skills, a much wider range of problems should be given.

There is no discussion of the doubt associated with predictions based on data.

In Section 1.5 there is a recurrent theme of overfitting, which is particularly dangerous with polynomial models. Even when there are more degrees of freedom than parameters to be fitted, the Teacher's Resource recommends that students abandon data (see discussion of Questions 6(a) and 7). What was Chapter 4 of Book 1 all about?

Question 6 (Section 1.5) encourages extrapolation far beyond the range of the data, on an overfitted model (something that any statistician would avoid). Nonetheless, the Teacher's Resource does not seem to suggest that there is a problem.

4.2 Chapter 2. Mathematics—Check it Out!

The concept—independent study and research—underlying this chapter is excellent. Unfortunately, the quality of the suggested material is variable: good ideas are mixed with half-baked ones and a few outright errors. The Teacher's Resource seems, in places, to describe a completely different version of the chapter.

Page 76. "The cardinal number for a set with an infinite amount of numbers is called 'aleph-null'." This is not true; it is the cardinal number for a set with a *countable* infinity of numbers. The first of the three questions is trivial: two sets with the cardinal number 4 can also have the same number of elements. The second question is meaningless unless some other system of enumeration (such as ordinal numbers) is used. What the authors probably have in mind is the fact that an infinite set (such as the natural numbers) can be in 1–1 correspondence with a proper subset of itself (such as the even numbers). It is a worthy goal that students understand that the real numbers form a "dense" set, that each tiny little dot on a number line corresponds to a real number, that integers and fractions are just the "tip of an iceberg." But discussion of cardinality of infinite subsets of the real numbers is too ambitious.

Page 79, Wallis's product, parts c,d. The implication is apparently that the first calculation should be carried out to the number of terms shown before the ellipsis, and the second to five more numbers in both numerator and denominator. This should be made explicit.

By the way, Wallis's product is more normally written as

$$2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots$$

which makes its convergence trivial to check, as even partial products provide upper bounds while odd partial products provide lower bounds.

Page 80. It might be good to mention that the sacrifice of the hundred oxen is probably a myth; Proclus mentions a single ox.

Page 82. "Why won't Alice get to twenty?" It is hard to see what this question means (and the Teacher's Resource diverges from the textbook at this point). Following the apparent pattern, Alice will reach 20 when she attempts to compute four times thirteen.

"How might you apply the numbers in the Fibonacci sequence to the keys of the piano?" The fact that there are five black keys in an octave, and eight white keys (if you include the tonic at both ends), is pure coincidence, and there is no relation to the Fibonacci numbers. This is not mathematics, but numerology. (Note also that (for instance) the chromatic scale of C♯ contains six black keys and seven white.)

4.3 Chapter 3. Sinusoidal Functions

This is a solid chapter, mathematically. The section on music is interesting, (though it might be good to note at some point that the human ear does not detect phase, and get the students to produce several *different* shapes that an A major chord might have.)

There is, unfortunately, the same overdependence on one type of problem—here, the Ferris wheel. There are also some examples of sloppy problem composition, with physically absurd numbers pulled out of the air. This completely spoils the effect of "trying to make the problems relevant."

Page 90. Example. The only way to reconcile the description with the graph is if the graph shows *cumulative* sales while the sales that "gradually increased until a more exciting game was introduced" are non-cumulative. However, the use of the word "total" in the sentence and not in the graph suggests the opposite.

Page 93. Question 8. This is not a Ferris wheel, this is a centrifuge. The rotation described is so fast that the seats would be upside-down at the top of the ride. (Easy check: in the 1 sec after reaching the top, the car descends 6m. An object in free fall would descend 5m.)

What is the point of "relevant" examples if numbers are going to be pulled out of thin air without any understanding of what is going on?

Page 117. Sidebar. The main reason that you have to bring special converters to run North American appliances in the UK is that the voltage is 220VAC. The difference in frequency is of secondary importance, and most converters do not correct for it. (This causes motors to run slightly slow and hot; most electronic or heating devices are unaffected.)

Page 123. Table. As given (without degree symbols), these functions define ultrasonic notes a few octaves above the limits of human hearing. Unwieldy though they may be, the correct equations are $y = \sin(158400x)^\circ$, etc. Numerical angles without an indicated unit are *always* in radians.

Page 123. (c)–(e). This is a lovely idea, but does not go far enough. Students should be informed that the human ear does not detect phase, and should graph the same chords several times, with different phase shifts between the constituent wave forms. It would be a fascinating exercise to try to describe the visual difference between all versions of the C major chord on one hand and all versions of the example discord on the other—or just to learn to distinguish them by eye. (A hint: all the versions of the C major chord will be periodic, with the same period.)

The use of a piano or keyboard in conjunction with this (ideally, a synthesizer with a pure sine-wave tone) should be encouraged.

Page 126. Example. Are the authors aware that a pendulum's length and period are interdependent? A pendulum with length l must have period (in the small-arc approximation) $p = 2\pi\sqrt{(l/g)}$ seconds. Plugging in the given value for p, we conclude that $l = g/\pi^2 \approx 1$ metre. This is not only unrealistic for a trapeze, but makes the indicated 10m amplitude shown in the graph impossible.

Page 145 Question 34 Any angle in degrees must be marked as such. The function $y = 3\sin(180x)$, if x is in seconds, represents an oscillation of around 29 Hz.

Page 146. Question 38. The Earth [technically, the center of gravity of the Earth-Moon pair, but this is within the Earth] is not at the center of the Moon's orbital ellipse (as shown in the diagram), but at one focus. [Historical note—the same mistake was made on the British one-pound note commemorating Newton.]

Page 149. "Did you know?" Trigonometric identities are still used to find *exact* values of trigonometric functions. Decimal approximations were usually found using methods based on power series (and the Calculus of Finite Differences that will be introduced in Book 3, Chapter 2).

Page 155. Questions 13, 15. It is completely irrelevant that x is in degrees; stating this will just confuse students. Q15a, of course, is in terms of θ anyway.

4.4 Chapter 4. Trigonometric Equations

This chapter is fairly well written, and covers important traditional material. One flaw is the treatment of decimal approximations in section 4.1; students should *never* see statements such as "$\sin 105° = 0.9659$." This can be written correctly as "$\sin 105° \approx 0.9659$" or as "$\sin 105° = 0.9659\ldots$." The "equals" sign should be used in mathematics only to represent the exact equality of two expressions.

Here and throughout the series, the typesetter should leave a gap between numbers and the expressions that follow them, e.g. "4 tan x" rather than "$4\tan x$."

While the emphasis on exact solutions is praiseworthy, it is not clear what goal is served by avoiding precise decimal approximations in favour of graphical solutions. The inverse trig function keys are introduced on page 141; this could be expanded, perhaps *in combination with* graphical solutions.

4.5 Chapter 5. Statistics

This chapter tries to cover much of the material presented in a first year university statistics course. Given the time allotted and the relative immaturity of high school students, this scope is unrealistic. There are numerous blatant errors and inaccuracies in the text, particularly concerning confidence intervals and hypothesis tests.

In many places, wordings of definitions and conclusions are confusing or incorrect. Perhaps the intent was to sound "friendly," but the choice of words betrays incomplete understanding by the authors, and can only perplex the students.

There are so many problems with Chapter 5 that we have chosen to organize our comments into categories, with examples, rather than as a page-by-page list of errors. We fear that many of these comments are "above the heads" of the authors of the textbooks. They are *serious* criticisms of a chapter that will do more harm than good.

The extent of errors in Chapter 5 underlines a crucial point: the material is too difficult for high school students.

Small populations. Investigation 5, on page 184, suggests that students take a sample of size 80 from a "population" of 100 squares. When sample size is comparable to population size, all sorts of technical problems arise: read about the "finite population correction." An introduction to the problem of estimating a population proportion should avoid such issues, while at the same time avoiding any technical errors. Investigation 5 contains technical errors.

Boxplots. were introduced in *Mathematical Modeling*, Chapter 1, as a means to compare sets of measurement data. That is their strength. Leave them there. In Chapter 5, important issues are confused.

Confidence intervals. Confidence intervals are introduced for the binomial probability p. We know of no other text which introduces the idea of confidence intervals using the binomial, because of the difficulty of getting exact answers. Far more usual is to use the normal distribution with known variance. The idea of inverting box plots of binomial distributions to get confidence intervals is cumbersome and inaccurate, because only a coarse grid of values for p can be used. The central idea that a confidence interval is a random interval with a pre-specified probability of containing the true p can be explained using this approach, but is *not* done effectively in Chapter 5. The idea is subtle and often gives our second year university students difficulty.

Simulation is used to study the sampling distribution of the sample proportion, so it is surprising that simulation is not used to demonstrate that roughly 95% of 95% confidence intervals actually contain the true p.

Interpretation of a confidence interval. The conclusions at the bottom of page 195 and top of page 196 are wrong. Hanna cannot say this. She may say that she is 90% confident that the *population proportion* is between .25 and .6. But probability statements about *samples* of size 20 can only be made with precise knowledge of the population proportion. The reader who may think these comments are "nit-picking" does not understand enough statistics to evaluate the merit of this chapter. This error comes at the crucial point where the concept of confidence interval is introduced.

In several instances the authors seem to assign a probability that the population proportion lies inside a confidence interval. This is inappropriate, the true value is either inside it or not. Probabilities can only be attached to the *method* of constructing the interval, not to a particular interval. (Listen to the radio: " ... using a method known to be accurate 19 times out of 20 ... ")

On page 196, line 5, we read "Hanna can use a point estimate with the box plots to write a *probable* range of values for the population proportion." Much better would be to use the word "plausible" or to say "Hanna can use a point estimate with the box plots to write a range of values for the population proportion within which she is 90% confident the population proportion lies."

In problem 24, page 199, we cannot give a probability that Angi is wrong: the interval either contains the true proportion or it does not. Angi has used a procedure known to be accurate 18 times out of 20 (under all the usual assumptions). We understand that these technical issues are difficult to grasp, but a text book should be a model of clear thinking, and precise mathematical statements.

Problem 9, page 213 reads "...how *confidently* can Fred order a shutdown of the production line if his 100-can sample has ... ?" Problem 10 is similar. Fred's confidence is in his interval, not in how he feels when he shuts down production. This kind of decision problem is much better handled using the methods of hypothesis testing. While there is a direct correspondence between confidence intervals and hypothesis tests with two-sided alternatives, Fred would probably use a one-sided alternative hypothesis here, because he is not interested in the defective rate being less than 2%. The practical issues associated with this problem are beyond the experience of high school students, and the statistical issues are (clearly) beyond the experience of the authors of this text book.

Questionnaire data. In Sections 5.1 and 5.2 the issue of non-response is not addressed.

Statisticians sometimes perform variations of the best-case/worse-case calculations displayed on page 208 in order to demonstrate the effect of non-response on opinion poll data. It is *not* an appropriate calculation for a setting with random selection.

Binomial trials. Binomial trials are discussed primarily in the context of Yes/No survey questions. However, surveys technically do not satisfy the requirements of binomial experiments. Apart from the requirement that there are only two possible outcomes of each trial, there are two further requirements not discussed in the book. First the probability of a Yes outcome must remain the same for each trial and, secondly, the trials must be statistically independent. Flipping a coin, rolling a die, or spinning a wheel would satisfy these conditions, but taking a survey where the same person will never be chosen twice (i.e. without replacement) technically does not. Of course for small samples from very large populations the binomial approximation is good, but failure to address this issue is sloppy and bound to confuse students. There is no discussion of the difference between sampling with replacement and sampling without replacement. This issue is vital if the binomial distribution is to be used. Problem 15 on page 197 cannot be done using techniques presented in this course unless sampling with replacement is used.

Population parameters and their sample estimates. Page 213 has a box in the margin which states that $\sqrt{p(1-p)/n}$ is the standard deviation of the sampling distribution, where p is the *sample* proportion. Since p is a mere estimate of the population proportion, this formula gives an *estimate* of the standard deviation of the sampling distribution. The formula would have to be evaluated at the (unknown) *population proportion* to find the standard deviation of the sampling distribution. The answer could be quite different numerically, and is certainly different conceptually.

Focus M, starting on page 217, repeats this mistake. The margin of error was defined on page 213 as $2\sqrt{p(1-p)/n}$ where p is the *sample* proportion. The statement "Any sample proportion has a 95% probability of being located a distance from the population proportion that is no more than the margin of error" is therefore false for the reason given above. The probability is *approximately* 0.95 of a sample proportion being within $2\sqrt{p_T(1-p_T)/n}$ of the *population* proportion p_T. The difference between the sample and population proportions is important.

Interpretation of simulation results. Focus Question 8 on page 212 probably intended to show the standard deviation of the sample proportion rather than the count. (They are different.) Although it is not clear, we assume the results shown are from simulation of many samples of size 100 or 1000, for otherwise we would not be talking about a sampling distribution. (The 3 digit accuracy indicates that at least 1000 samples were taken.) Part (b) cannot be done because confidence intervals can only be calculated for particular samples, not using results from simulations of several samples. This means that part (c) cannot be done either.

Why is there so much emphasis on simulations and spinners? "Real" examples make more sense: dice-rolling, mock lotteries, even (that old standby) balls in urns.

Hypothesis testing. Hypothesis testing is introduced using the chi-square goodness of fit test. This is the only book we have seen which does this. Much simpler would be a sign test or a test for a binomial probability equaling a particular value (which uses binomial probabilities) or a test for the mean of a normal distribution with the variance known. The distribution of the test statistic is exact for these situations whereas the chi-square distribution is only an approximation to the distribution of the goodness of fit statistic. An advantage of the usual tests is that the null *and* alternative hypotheses can be easily stated in terms of the parameter of the distribution. Little or no mention of the alternative hypothesis is given in this chapter, and the null hypothesis is rarely if ever stated in terms of probabilities, as it should be.

There are aspects of the approach to testing used in this section which are not standard. Most texts include an alternative hypothesis when discussing testing. In many situations, the form of the alternative affects the conclusion one makes. Inclusion of an alternative allows discussion of type I and type II errors, and how failure to reject the null does *not* imply its acceptance. Most scientific literature now reports test results using the P-value; very few adhere to the decision approach described here.

Focus M, starting on page 217, is seriously flawed. In the first place, the phrase "statistical significance" should be used in conjunction with a hypothesis test. Such tests have not been introduced at this point in the text. Rejecting a null hypothesis at the α level is equivalent to stating that the result is statistically significant at the α level.

In problem 24, page 218, a margin of error is given for the old sales (41%). This implies that the old sales were determined with a sample. After the advertising campaign, sales increased to 46%. We are not told whether this was determined using a survey or not. If so, then there are two sources of variability here, one from each survey, and any assessment of the population difference must take this into account, using a two sample Z test for binomial p's or a χ-square test for 2×2 contingency tables. If not (and a census was taken), one could argue that 41% (the random quantity) is significantly lower than 46%, but this is not quite the same as saying that 46% is significantly higher than 41%. The same applies to problems 29 and 30 on page 219.

In problem 27, page 218, once again we have two groups. Is the proportion for the control (surely the *percentage* is shown?) based on a sample or not? If based on a sample, then our margin of error would include both sources of randomness. If not, then this is a very difficult problem (remember: estimated margin of error is calculated from the observed sample proportion).

Problem 24(b), page 218, asks "Would you classify the increase in revenue as a significant amount of money?" The amount may seem quite substantial for the reader, but to a millionaire, it could be trifling. The everyday and statistical usage of the word are colliding here. An important point to be made in any treatment of hypothesis testing is the difference between statistical and practical significance.

On page 222, paragraph 3, the sales manager and Anselem choose the null hypothesis after looking at the data. Such data snooping is not permitted. The usual probability statements are invalidated when the null is chosen in this way. (The same is done in Example 1 of page 224.)

Chi-squared tests. We believe the intent of the chapter was to introduce the chi-square goodness of fit test, where observed counts in various categories are compared to expected counts. In this test, the null hypothesis test specifies the probabilities of the various outcomes, and these outcomes add to 1. The data results from independent multinomial trials, where each outcome can be classified into one of several disjoint categories. Example 1 page 224 is a valid use of the test, as are problems 17 and 18 on page 225.

Another correct use of the test occurs when the counts in the categories arise through a Poisson counting process. Fairly advanced statistical theory tells us that the goodness of fit test can be used when

(i) the total over all cells carries no information about differences among the mean rates in each cell;

(ii) the total over all cells also has a Poisson distribution;

(iii) conditional on this total, the cells have a multinomial distribution.

If students are not told about this logic, they will be puzzled as to how the expected counts are generated, but they are clearly not ready for this logic. Example 2, page 224 is of this type, *if* we are willing to ignore the randomness in the number of shots in the previous year. Problem 38, pages 229-230, is an example of this situation: numbers of customers are reported under good weather and bad. There is no fixed number of trials in advance here; the two counts will be random. (The data should *not* be converted to averages here, but kept as a total. It was only by great good luck, or by rounding, that the numbers are integers.) If raw data had been kept, then a simple two-sample test to compare two means (or two medians) would be more appropriate.

There are numerous examples in the text where the goodness of fit test is used when it should not be. When comparisons are made between survey results in two years, or in males and females, or in different shifts, the correct analysis uses a test for homogeneity or independence in a *contingency table*. For example, the analysis presented for Example 5, page 245, "loses" 2 of the 4 cells from the table in the margin. The correct analysis produces expected counts for all 4 cells in the table. The method of getting expected counts is more complicated (row sum times column sum divided by overall sum), and the rule for degrees of freedom is also complicated (number of rows -1 times number of columns -1). The analysis is incorrect; the correct value for the chi-squared test statistic is 3.0.

In questions 7, 8 and 9 of page 221, results from previous surveys are used to provide expected counts for new surveys. This ignores the randomness in the results of the previous surveys, which is substantial because the sample sizes are small. The correct analysis also uses the 2×2 contingency table test. Case Study 7, page 237 is also like this, but the issue is confused by the fact that we are told to use 28 as the expected count. Problems 14 and 15 on page 248 are also of this type. In all these cases the contingency table test should be used.

The scenario in Focus O, page 222, is really one of comparing two binomial probabilities. Is the proportion of new customers the same for the two parts of town; i.e., is

$p_{TW} = p_{TB}$? If we knew the population size for both parts of town, a chi-square test for a 2×2 contingency table or a Z test for two binomial proportions would be appropriate (and it would be possible to use box plots or the normal curve in problem 10).

Is there a place in this chapter where the subject of degrees of freedom is discussed, how to calculate degrees of freedom, how to use degrees of freedom?

The writers of this chapter do not understand the difference between the two basic types of chi-square tests described above. In particular, the authors cannot see the fundamental difference between the table in the margin of page 223 and that in the margin of page 224.

One indication of the level of difficulty of chi-square tests is that the writers of Chapter 5 use them incorrectly several times in this section.

4.6 Chapter 6. Trigonometry and its Applications

This chapter, dealing with the solution of triangles, is well-written, useful, informative, and accurate. The mention of surveying practice is a good touch. It would be nice to have a larger selection of exercises and examples.

The suggestion in Example 1 on page 268 that air-sea rescue workers take the Bermuda Triangle myth seriously and use it in rescue planning is unfortunate.

5 Mathematical Modeling 3

Overall, this book is more solid than Books 1 and 2. There are some praiseworthy attempts to find real-world applications, though often spoiled by artificiality and sometimes by carelessness in making up the problems. There are a few serious errors.

5.1 Chapter 1. Quadratics

This is traditional material, handled in a fairly traditional way, and without major errors. The use of the calculus of finite differences to analyze sequences is an excellent idea and might usefully be extended.

Page 34. Question 27. The assumption is clearly made that the edge of the lake is linear (over 36 km). This should be stated explicitly.

Page 41. Upper diagram. The diagram, said to "illustrate" the relation

$$\text{distance} = \text{speed} \times \text{time}$$

is content-free. Diagrams are wonderful aids to learning where they illustrate something, but this does not. A rectangle with its interior and two edges properly labeled would have *some* illustrative value.

Page 49. Question 29. "An *imaginary* or *complex number* can be ... " In normal usage, a sentence using italics in this way is a definition.

Used this way in a definition, "or" should mean that the two terms are synonymous. But here they are not. Compare the confusion in Book 4, page 63, sidebar.

Page 49, "Did You Know?" As in Book 4, the attempts to describe the phasor formalism are mangled beyond usefulness. This is a pity, because it is a beautiful and useful example well within the abilities of the best students. It would be better to write: "For instance, in electronics, capacitance and inductance in filters, tuners, and other alternating-current circuits can be represented as resistances, using complex numbers."

Page 50, Question 33; Page 55, Question 51 and note in Sidebar.

Question 51 asks: "Explain why functions of the form $y = (x - q)^2$ have one zero and their corresponding quadratic equations have two equal real roots." But, on page 41 a root of a quadratic equation is defined as "a value ... that makes the equation true." By this definition it is meaningless to talk about "two equal real roots" without noting that this is a *convention* which, like the term "double root" must be defined by reference to the factorization of the polynomial. The lack of discussion about conventions has caused unnecessary confusion.

A possible addition to the text would be:

> "If a quadratic equation can be written as $(x - q)^2 = 0$, so that the root $x = q$ appears twice in the factorization, it is said to have a *double root* or *two equal roots*."

This comment should be placed on or near Page 41, certainly before Page 50.

5.2 Chapter 2. Rate of Change

This chapter explores some of the basic ideas of differential calculus from a graphical and numerical viewpoint.

Page 92. Question 22. The use of a quadratic function here is exceptionally artificial, especially around New Year's Day. (An appropriate model would use trigonometric functions, to capture yearly cycling.)

Page 92. Question 23. Why is the formula not given as $A = m^{2/3}$? Given that way, the reason why the formula works is obvious: $A \sim h^2$, $m \sim h^3$. (This should be familiar from Book 1, Section 6.5) Using a decimal approximation hides this—a standard problem with over-reliance on "technology."

Page 93. Question 28. The marginal cost formula

$$C = 1000 + 10n + 0.05n^2$$

is completely unrealistic. One would normally expect economies of scale, not diseconomies. (The authors of Book 1 were unaware of the term "marginal cost"—see notes on Chapter 3 of Book 1.)

5.3 Chapter 3. Exponential Growth

This chapter involves classical material and is, for the most part, written well.

Page 124. Bulleted list. Density of bacteria in bulk ground beef is repeatedly given in terms of bacteria per square centimetre; but bulk ground beef does not have a well-defined surface area.

Page 126. part (f). An otherwise good exercise spoiled by an absurd supposition. See also Question 27 on page 183.

5.4 Chapter 4. Going 'Round in Circles: Circle Geometry

There is little new material in this chapter, but few errors. The "sports complex" theme is perhaps overdone. It would have been nice to see more problems in which the use of a circle was not just an aesthetic whim. Such examples certainly exist. The use of proofs (including the traditional 2-column format) is praiseworthy.

Page 209. Question 11. If these "5cm $\times$ 15cm beams" are supposed to be the standard 2×6 dimensional lumber, it is worth noting that the Imperial dimensions are still used (and are nominal only).

Page 227. Question 15 and Teacher's Resource. Firstly, "latitude" and "longitude" have been confused. There is no such thing as "121° N," nor "34° W latitude."

Secondly, a degree of latitude (or a degree of longitude at the equator) is about 110 km (recall $10,000$km $\approx 1/4$ circumference of the Earth (the original definition) $= 90°$). The two towns are therefore around 1000 km apart, and cannot realistically share a campsite.

Thirdly, over distances such as this, the curvature of the earth may matter, depending what precision is needed. The Teacher's Resource categorically claims that "it does not matter," a dangerous statement.

Page 243. Question 46. We cannot solve this problem unless we assume the the various sections of the track to meet smoothly. In this case, the straight sections $\overline{FR}, \overline{TS}$ are parallel. But the length FT is $8 + (8 + 7)\cos(30°) + 10 = 30.99\ldots$ metres, while $RS = 32$ metres.) Thus the diagram is inconsistent.

Page 268. "The circles in a variation of curling . . . " Does this bizarre introduction (which ties in neither with the text nor the real world) have any purpose except as a pretext to insert a sports picture?

5.5 Chapter 5. Probability

The chapter title should be "Counting and Probability." Coverage of the first topic is fine, and has some nice touches. Coverage of the second topic, probability, demonstrates recurring confusion between probability and proportion. Many of the questions in this chapter can be saved by explicit reference to the device of randomly selecting a subject, or in some cases by replacing "calculate" by "estimate."

The probability is not the same thing as the proportion. One can use proportion (calculated from a sample, perhaps) to *estimate* an underlying (possibly unknown) probability, or one can (as in Focus E) ask about the probabilities pertaining to a randomly selected subject. See Page 312 ff, Focus C, Questions 23, 27, 30, 47, 48, 52–55, 64–66, etc.

Investigation 15 is so badly designed that it should not be used in the classroom.

Page 318. Question 42(d). This question makes an unwritten (and unrealistic) assumption that the expenditures follow a uniform distribution.

Page 318. Question 43. The wording is misleading. What is apparently intended is something along the lines of: "For the next meteorite that hits the Earth, calculate the probability of each event: (a) The meteorite lands in North America" etc.

Page 333. Focus G. While both notations are valid, the $\binom{n}{r}$ notation is more widely used by mathematicians and statisticians.

Page 342. Question 7, Teacher's Resource. The question asks for a proof.

The Teacher's Resource suggests accepting "a numerical approach to the proof," which presumably means testing a few cases and saying "yes, it's true." This suggests a deep misunderstanding of the concept of "proof" on the part of the authors. The proof of part (b) is easy, and the proof of part (a) provides an excellent review of the rules for adding fractions.

Page 346. Investigation 15, Focus J, Teacher's Resource.

The inference exercise that is the focal point of this item is so badly written that it should never be used in the classroom. The Teacher's Resource entry for Question 3 makes very nearly every conceivable mistake in one paragraph, encouraging misconceptions the students may have developed.

- "The probability was about 13% ..." The probability that should be used for such an inference is the two-tailed probability that the test statistic is *equal to or more extreme than* the observed value. This would be on the order of 32%, not 13%.
- "...which is high..." High compared to what?

- "...but not high enough to prove conclusively that bias was present." In hypothesis testing, a hypothesis is rejected when the probability associated with the test statistic under the null hypothesis is *low*, not *high*.
- "It is likely but not certain that the company has used biased hiring practices." The process that has just been attempted is a hypothesis test, which has supposedly yielded a p-value of 0.13, which would occur one time in eight by random chance. No competent statistician, or user of statistics, would interpret such a p-value as making the alternative hypothesis "likely but not certain." Rather, "Slight evidence in favour of the alternative hypothesis" is about as strongly as anybody might put it.
- (Q9, Page 347; Teacher's Resource) "This does not conclusively prove bias but it does give another piece of evidence that bias might be present." The two tests are almost identical, one using algebra and the other, simulation. No new evidence is provided by what is in effect a repetition of the same test.

Page 355, Example 4. This question is badly phrased. It might be better written as "...that rain is *falling but* not affecting the east side." The answer to the question as stated is 0.622, as the conditions are met when it does not rain.

Page 358. Sidebar. Top. This note is useful but might be better placed on or about page 333.

Page 365. Question 35. As the assumption (used here) that the event of rain on one day is independent of the event of rain on the day before is not wholly realistic, it ought to be stated.

6 Mathematical Modeling 4

6.1 Chapter 1. Sequences and Series

Page 6. Question 11. The fact that the terms *continue* to decrease beyond those shown is nontrivial, and should not be glossed over. There are values (all those below -6) for which the recursive step *would* increase the value, and it is only because such values for t_n are never reached that the sequence never increases; this should be mentioned explicitly. The problem should be revisited after students have learned how to sum infinite geometric progressions; they will then see why -6 is so important.

Page 6. Question 11(c) and Teacher's Resource. The question asks for a situation that could be modeled; the Teacher's Resource suggests a (rather artificial) answer which breaks down in the sixth week.

Page 6. Question 12(d) and Teacher's Resource. Unless one is using a spreadsheet or some similar device to multiply $2 \times 3 \times \cdots \times n$, one does have to compute intermediate values, though one doesn't have to actually look at them or key them in.

Page 8. Sidebar and text. A series is not the result of adding the terms of a sequence. The text is definitely wrong, and the sidebar ambiguous.

This is a subtle but important distinction. If we want to avoid terms such as "formal sum", it is probably best to say that a series is an *expression* such as

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{10} \quad \text{or}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i} + \cdots$$

Page 16. Question 11. Wouldn't it be better to let students discover the slightly remarkable fact that this shape of rectangle (and only this one) can be bisected to yield two rectangles similar to the original?

Page 20. Question 24 and Teacher's Resource Page 35. The step example is good but the reason why it works is probably more obscure than many things that are explained. A hint (every way to get to step N corresponds to exactly one of the following: a way to get to step $N-1$ with one further step, or a way to get to step $N-2$ with 2 further steps) would be appropriate in the Teacher's Resource.

Page 20. Question 29. For F_6, the use of this formula is definitely doing it the hard way. Why not get students to compute F_{30} both ways, showing why the formula is better? (A much better choice for this than the factorial in Question 12, page 6.)

Page 21. Chapter Project. A good idea but (with reference to the Teacher's Resource suggestions) it would be better to encourage the art project to be structured so as to reveal, not conceal, the series.

Page 22 ff. Section 1.3. In general, Section 1.3 could do with more worked examples; the exercises on pages 26–27 are quite challenging compared with those worked in the text.

Page 24. Example 2. Should be "the square *of the* number that tells you ..."

Page 30. Question 8 The lawyer's claim is not the same as the mathematical expression. A lawyer who lost one case in every 20 could claim to win "almost 100%" of her cases too.

Page 31. Question 18(b) and especially Question 20(c).

The graphing calculator *suggests* that the limit exists (and will suggest, equally persuasively, that the limit of the divergent series

$$1 + \frac{1}{11} + \frac{1}{21} + \frac{1}{31} + \cdots + \frac{1}{10j+1} + \cdots$$

exists). It is misleading to suggest that one "knows" a limit because a few calculations suggested a trend.

Page 32. Investigation 5. Fractals need not be self-similar, and the Koch snowflake curve is in fact an excellent counterexample. (The part of the Koch curve that comes from one edge of the original triangle *is* self-similar.)

Page 33. Question 22 and Teacher's Resource. The Teacher's Resource answer is completely erroneous, based upon two fallacies. Sloppy definitions should be corrected to make these two points perfectly clear:

- The Koch curve is the curve, not its interior. The curve itself is *not* made up of triangles.
- It does not follow that because a shape is made up of parts that are similar to each other that it is self-similar.

Page 33. Last line. Suggestion: include the "lim-sigma" form as well (see next comment):

$$S = \sum_{i=1}^{\infty} t_i = t_1 + t_2 + \cdots = \lim_{n\to\infty} \sum_{i=1}^{n} t_i = \lim_{n\to\infty} S_n$$

Page 36. Question 34. The Teacher's Resource explanation is unnecessarily vague, and suggests that something different is happening with $0.999\ldots$ than happens with $0.333\ldots$. Back on page 33 of the text an apparently deliberate effort was made to avoid the combination of a limit and sigma notation. Its introduction here, in combination with the words "for all practical purposes ..." suggests that there is something not quite valid about the alternative notation $0.999\ldots$ for 1. (Also: instead of $0.9(0.1)^{j-1}$, why not use $9(0.1)^j$, more in the spirit of place value notation?)

Page 36. Question 35(b) and Teacher's Resource. There is a fallacy in the suggested answer. There are two opposing factors, one (the first move) working in Lee's favour, one (higher odds) in Ann's. If Lee had to throw anything but 2 or 12 to win, while Ann could win on anything if her turn came around, the odds would be overwhelmingly in Lee's favour, though Ann's chance of winning on her turn would be higher. The question is simply bad; there is *not* any intuitive reason to suppose one person is favoured. You just have to calculate.

6.2 Chapter 2. Functions: A New Perspective

Page 56. The assumption that somebody works variable hours for two employers, but that the times worked are always equal, seems very artificial.

Page 57. Question D and Teacher's Resource page 98. The domain is an attribute of the *function*, not of the graph.

Page 57. Question 3. What is meant by a "reasonable" relationship? Given that the original functions were not required to show a "reasonable" relationship, why should their combination?

Page 58. Question 5. The use of polynomial regression to fit a function to a set of exact values is valid (probably more so than most applications of polynomial regression by nonexperts). So is the use of finite differences (Teacher's Resource page 100, sidebar) to check order. But why not

- use residuals to check the adequacy of the model; or
- simply use a cubic model and observe which coefficients are 0; or
- follow though on the finite-differences calculation instead of using regression?

We repeat points raised in Book 1. If you know that as set of points lie on the graph of a polynomial function, then the calculator will readily find an equation describing that function. If you have a scatter plot of points (data on the circumferences and weights of pumpkins, perhaps), then the calculator will, just as readily, fit a straight line or a curve to those points. The TI calculator doesn't "know" how the input numbers were obtained. It just does the calculations. In the first example, you will have used the calculator to avoid manipulations of formulae for polynomial functions.

In the second case, you will have fitted a model to data that *varies randomly about some unknown model*, and you are obliged to give information about how well you think the fitted model describes the underlying model. That is, you must *quantify the doubt* associated with the proposed model.

Page 59. Question 10 and Teacher's Resource, page 104. The tabular presentation suggests that the domain might not extend beyond the finite set $\{0, 1, 2, 5, 15, 30\}$. (The table is also artificial, in that Donny has clearly derived the table from a function.)

Why does the graph of $p(t)$ extend beyond the stated domain and range of the function?

Page 61. Teacher's Resource page 106-7 Both the textbook and Teacher's Resource are inconsistent about domains. This would not be a problem if the student was not expected to be able to derive "the" domain.

Page 62. Question 20; Teacher's Resource page 109. The linearity of the lakeshore is implied, but should probably be made explicit. It does not help that the diagram on page 109 of the Teacher's Resource shows the lake as elliptical.

Page 63. The use of a grey box for the constant coefficient (presumably to get around the problem of using a, b, c instead of a_n, a_{n-1}, a_{n-2}) is nonstandard, ugly, and not easily emulated with a pencil. The standard notation with subscripts should be used, though we admit that such notation takes a little while to learn.

Page 63. Sidebar note 2. The real numbers are a subset of the complex numbers. It would be more correct to say that "most complex numbers are not real numbers." Better yet: "Recall that a complex number has the form $a + bi$ where $i = \sqrt{-1}$, and that a complex number is also a real number only when $b = 0$."

Page 63, 64. "Roots" were defined in Book 3 as values; multiplicity of roots was not properly explained there and the concept is poorly explained here. After Book 3, students might think that all roots are either single or double. So Book 4 must make clear that higher degree polynomials may have roots of high order. We suggest, for example, "a triple real root" rather than "three equal real roots."

Page 81 Question 51 The answer given, while mathematically correct, is absurd (the garden roller is a slightly elongated sphere). Why not give a minimum volume for which the shape is plausible?

Page 87. Sidebar. The function $x \mapsto x^2$ does not have an inverse function. The Teacher's Resource introduces the restriction of the domain to $x \geq 0$; the problem as given in the textbook is unanswerable and will cause unnecessary confusion.

Page 106, Question 7 (b). "How do imaginary zeros of a polynomial function relate to the number of critical points?" is answered in the Teacher's Resource (page 210) by "The number of imaginary roots will affect the shape of the graph and the number of critical points." Firstly, this does not answer the question. Secondly, do the authors mean "complex?" Thirdly, the statement in the Teacher's Resource is simply incorrect: the polynomials $x^3 - x$ and $x^3 - x - 17$ have 0 and 2 nonreal roots (the only possibilities for a real cubic), the same shape, and the same number of critical points.

Page 109. For "semicircular," read "hemispherical."

6.3 Chapter 3. Functions, Part II

Page 128. Sidebar. "Did you know?" A newton is *not* "about 0.1 kg." The newton is a unit of force, the kilogram, of mass.

Pages 135 and 192. Chapter Project. This is neither good art nor good mathematics. The arcs shown are *not* graphs of linear, rational, etc. functions, but (for the most part) are the graphs of their *restrictions to bounded intervals.* To make matters still more confusing, circles and quartic ovals are shown, though they are not on the "menu" of functions suggested. (Nor can such curves be assembled piecewise, since none of the functions on the "menu" has a point of vertical tangency.) To make matters still more confusing, circles and quartic ovals are not functions at all (yet they are described as such on page 192). The word "function" is used incorrectly in Book 1, also.

Students would learn more from this game if they were forced to write formulae for the functions and to clearly specify their *restricted* domains. The text should set a good example, giving such information for one of the diagrams in the sidebar.

Page 149. 'Did You Know?' The Latin for "star" is *stella*, not *sterre* which is Middle English.

Page 168. Definition of $|x|$ and sidebar note. It should be made clear that these two definitions are equivalent; there is no conflict between them.

Page 179. Question 10 and Teacher's Resource page 315. See previous note. It should be noted also that the definitions may even overlap if they are not in conflict.

Page 212. Question 23. In light of the implicit precision on the amount of 0.01%, Perry's statement is correct.

6.4 Chapter 4. Trigonometry

This chapter is basically sound, except for the confusion about inverse trigonometric functions. In Book 2, Chapter 4 and Chapter 6, inverse trig functions were introduced with the notation "$\sin^{-1}$", etc. Now the student is told that these are relations and that the functions which give the principle values are "Sin^{-1}", etc. It does not help that relations (in this sense) have not been discussed since a short exploration in Book 1, or that looking up "relation" in the index of Book 4 directs the reader only to "recursive relations" - which are, in fact, functions. Finally, the "Sin^{-1}" usage is by no means universal; students will probably see other conventions in physics and calculus texts. We have no quarrel with the authors' choice of conventions, but they must be consistent and they should warn the students that there are other accepted conventions.

What is *not* correct or acceptable is the claim (on page 266) that $\sin^{-1} x$ returns a value in degrees while $\arcsin x$ returns one in radians. This is simply wrong. (We have checked with several standard reference works, as well as several mathematicians, an engineer, and a physicist, none of whom had ever heard of such a convention.) The notations have identical meanings, and the result of either is an *angle*, which may be measured in any unit one wishes. The use of the phrase "angle or radian measure" in the third paragraph suggests confusion at a very fundamental level.

It is interesting to note that the authors immediately proceed, on the same page, to violate this "convention," writing

$$\begin{aligned} y &= \mathrm{Sin}^{-1} x : \\ D &= \{x \mid -1 \le x \le 1, x \in R\}; \quad R = \{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \in R\} \end{aligned}$$

Finally, the theory of "biorhythms" (page 262) is an outdated piece of pseudoscience, repeatedly demonstrated to be worthless, and has no place in a serious educational textbook. Contrary to what is stated in the textbook, biorhythm believers look at the three cycles separately, with emphasis on the "low" periods and the "critical" days when the curves cross the t axis, not their sum.

6.5 Chapter 5. Numbers Most Complex!

There is some confusion in the section on complex numbers, in particular, mixing i and j in the same book is unnecessary, especially as i is not being used for impedance. (A sidebar saying "electrical engineers often use j instead of i" would be nice.)

The following statement, on page 282, is incorrect.

> "an angle of 90 degrees, or j, is used for inductive reactance X_L,"

and similarly for capacitative reactance. The complex number $\sqrt{-1}$ is not an angle and is certainly not the measure of an angle. On the same page, reactive components are labeled in *real* ohms, not henries and farads *or* imaginary ohms. This hides the crucial dependence on frequency and implies that the reactive impedance is an attribute of the component alone. There is no such component as a 12 ohm capacitor—check at Radio Shack!

Writing j *before* a numerical constant ("$2 + j3$" for what most would write as "$2 + 3i$") is not a practice we have ever come across through various courses in math, physics and electronics. Finally, after the hard work, all the students get is a complex number, with no discussion of what a complex impedance *means*. This is a disappointment, as it would be very easy at this point (had inductance and capacitance not been hidden) to explain about filters, resonance, etc. and have a real application.

Work on this report is still in progress. October 31, 2002.