NOTES

Crackpot Angle Bisectors!

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This note is dedicated to the memory of William Thurlow, physician, astronomer, and lifelong student.

A taxi, travelling from the point (x_1, y_1) to the point (x_2, y_2) through a rectangular grid of streets, must cover a distance $|x_1 - x_2| + |y_1 - y_2|$. Using this metric, rather than the Euclidean "as the crow flies" distance, gives an interesting geometry on the plane, often called the *taxicab geometry*. The lines and points of this geometry correspond to those of the normal Euclidean plane. However, the "circle"—the set of all points at a fixed distance from some center—is a square oriented with its edges at 45° to the horizontal (FIGURE 1). As can be seen, there are more patterns of intersection for these than there are in the Euclidean plane.

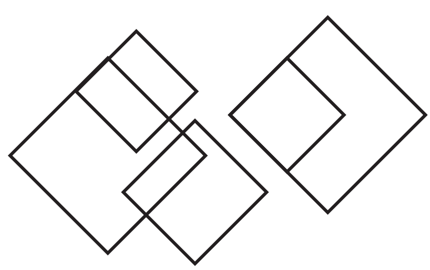

Figure 1 Circles of a taxicab geometry.

There is nothing particularly special about squares and (Euclidean) circles in this context. Any other centrally-symmetric convex body also has a geometry in which it plays the rôle of the circle, with the unit distance in any direction given by the parallel radius of the body. The reader curious about these "Minkowski geometries" is referred to A.C. Thompson's book [**11**].

While taxicab geometry has many applications in advanced mathematics, it is also studied at an elementary level as a foil for Euclidean geometry: a geometry that differs enough from that of Euclid that it enables students to see the function of some fundamental axioms. (As Kipling might have put it, "What do they know of Euclid who only Euclid know?") This use, in undergraduate courses, goes back at least to Martin [**10**] and Krause [**9**], both in 1975. (Byrkit's 1971 article [**5**], while also influential, deals with the taxicab metric on the integer lattice, axiomatically a very different system.)

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In this paper we survey some basic facts about axiomatic taxicab geometry. We give particular consideration to the question of angle measure (an active research area in its own right [**3, 4, 6, 7**]), and show that the taxicab geometry sheds some light on Wantzel's famous result that the angle cannot be trisected by classical methods, a high point of many undergraduate geometry courses.

Following Martin and Krause, we consider the taxicab metric in the context of a set of axioms for Euclidean geometry, based loosely on those of Birkhoff [**1**], which can be summarized as follows. The terms "point" and "line" and the relation "on" are undefined, and the field of real numbers is axiomatized separately. Comments and definitions are interspersed.

INCIDENCE AXIOM. *Two points are on a unique line, and there are three points not all on the same line.*

The unique line through two points A and B is represented as $\overleftrightarrow{AB}$. This axiom allows a line to be identified with the set of points that lie on it.

RULER AXIOM. *For every line l there is a bijection f_l between the points of l and the real numbers.*

A *line segment* is a set of the form $\{x : a \leq f_l(x) \leq b\}$, and its *endpoints* are the points that are mapped to a and b by f_l. A set S is *convex* if it contains any line segment that has both endpoints in S. The *distance* $d(A, B)$ between two distinct points A, B is defined to be $|f(A) - f(B)|$ for the bijection f whose domain is the line $\overleftrightarrow{AB}$. Two line segments $\overline{AB}$ and $\overline{CD}$ are *congruent* if $d(A, B) = d(C, D)$.

SEPARATION AXIOM. *The complement of any line may be partitioned into two convex sets, such that every line segment with one endpoint in each intersects the line.*

A *ray* is any set of points on a line l of the form $\{x : f_l(x) \leq a\}$ or $\{x : f_l(x) \geq a\}$; the point with $f_l(x) = a$ is called the *endpoint*. An *angle* is the configuration consisting of two rays with a common endpoint, not both subsets of a common line. Angles are *supplementary* if they share one ray, and the union of the other two rays is a line.

PROTRACTOR AXIOM. *There exists an additive measure on the angles at each point, such that the measures of two supplementary angles add to π.*

This axiom is actually (see [**10**, §14.2]) provable from the first three. However, it is often included for greater clarity. Angles are defined to be *congruent* if they have the same measure.

SAS CONGRUENCE AXIOM. *If two triangles have two sides and the included angle congruent, then the other side and angles are also congruent.*

PARALLEL AXIOM. *Given a line and a point not on the line, there exists a unique parallel to the line through the point.*

One of the main triumphs of axiomatic geometry is the fact that this axiom set is "categorical": every system obeying it is equivalent to the Euclidean plane. (See, for instance, [**10**, p. 322], for a discussion of this.)

Returning to the taxicab geometry, we see that the first three axioms (and thus the protractor axiom) are valid in the taxicab geometry, as is the parallel axiom. However, while this shows that an angle measure *exists*, it is not unique. In the usual development of an axiomatic system of this type, the function of the SAS axiom is to make both "rulers" and "protractors" invariant under translation (*homogeneous*) and rotation (*isotropic*), thus determining both the metric and the way in which angles are measured. However, no angle measure for the the taxicab geometry can be consistent with the SAS axiom.

This can be proved indirectly, by noting that with any angle measure, the taxicab geometry satisfies **Incidence, Ruler, Separation** and **Parallel**; if it also satisfied **SAS** it would be, as noted above, completely equivalent to the Euclidean geometry. But, whatever angle measure is chosen, two taxicab circles can intersect in a line segment, a thing impossible in the Euclidean plane.

However, there is a more satisfying direct proof. Firstly, we note that the SAS axiom implies (see, for instance, [**10**, §17.3]) the SSS congruence theorem in the presence of the first three axioms. In FIGURE 2, $\triangle OBC$ and $\triangle DOC$ are equilateral and hence (by SSS) have all three angles equal. But $m\angle OCB + m\angle DCO = \pi$, whereas $m\angle BOC + m\angle DOC < \pi$. (A similar, but less general, argument appears on p.195 of [**10**]).

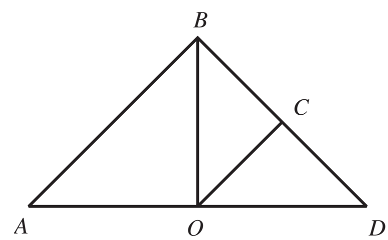

Figure 2 A counterexample to Euc.I.4 in the taxicab geometry

The taxicab geometry, then, demonstrates the function of the **SAS** axiom in somewhat the same way that hyperbolic geometry demonstrates the function of **Parallel**. However, the analogy is not perfect. Denying **Parallel** in the presence of Euclid's other axioms gives a unique alternative, hyperbolic geometry. Denying **SAS** does not let us derive the taxicab geometry. There are many geometries that satisfy the other axioms—for instance, all the two-dimensional Minkowski geometries do so. As observed above, no such geometry can also satisfy **SAS** unless it is Euclidean.

The taxicab metric is not isotropic, but it is homogeneous. Homogeneity can be axiomatized in a "non-**SAS**" geometry in various ways. For instance, **Parallel** could be replaced by the following, based on Euc.I.34:

PARALLELOGRAM AXIOM. *If both pairs of opposite sides of a quadrilateral are parallel, then they and the opposite angles are equal.*

In the presence of **Incidence, Ruler, Separation** and **SAS**, the axioms **Parallel** and **Parallelogram** are equivalent; but without **SAS**, **Parallelogram** is stronger. As well as implying the uniqueness of parallels, it also forces the geometry to be homogeneous; any figure can be translated to any point without distortion of length or angle measure. **Parallelogram** falls short of being a "taxicab **SAS**" axiom, however. It neither implies a taxicab metric (the axiom set {**Incidence, Ruler, Separation, Parallelogram**} is consistent with every Minkowski geometry) nor distinguishes between the various homogeneous taxicab geometries with different protractors.

The fact that obvious axiom sets don't fix the nature of the protractor is presumably one of the reasons why it has been observed (see [**3**, p. 279] or [**6**, p. 32]) that there is no one angle measure that is wholly natural to Minkowski geometry. Martin [**10**, p. 195] and Stahl [**13**, p. 24] consider a geometry in which the taxicab metric is endowed with Euclidean angle measure. Thompson and Dray [**12**] give an alternative model in which angles are measured in "t-radians", units based on the taxicab length subtended on the unit circle. Yet other definitions of angle are also given by A. C. Thompson [**11**] and B. Dekster [**6**]. Moreover, Busemann [**4**] and Glogovśkiĭ [**7**] each give different operations generalizing Euclidean angle bisection, which can be taken as bases for angle measurement.

Let's look at some easy theorems of Euclidean geometry, and see how they fare under the new axiom set. The construction of an equilateral triangle (Euc.I.1) is valid in the taxicab geometry (and with a vengeance for any base with a slope of $\pm 45°$, on which there are infinitely many equilateral triangles!) So do the "communication" theorems Euc.I.2 and Euc.I.3. in which a line segment is copied to a specified location and orientation. However, as seen above, the taxicab geometry does not have the SAS congruence property (Euc. I.4). Neither does the *Pons Asinorum* (Euc.I.5, "the base angles of an isosceles triangle are equal") nor its converse hold, the taxi presumably having rendered the donkey obsolete as a means of transportation! Moreover, there is no SSS congruence property; counterexamples are readily found to all of these.

Euclid's next proposition, I.9, is a construction bisecting an angle. Stahl [**13**, p. 59], who follows Martin in using Euclidean angle measure, gives as an exercise "Comment on [Euc. I.9] in the context of [the taxicab geometry]." Given the level of the textbook (undergraduate, with emphasis on prospective teachers), the location of the exercise (in the second chapter), and the lack of comments or hints, the comment is presumably intended to be on the validity of the Euclidean proof (which uses **SAS**) in the taxicab context. However, it is interesting to ask whether some other construction for bisecting "Martin angles" does work.

To pursue this question of angle bisection, we define a "taxi construction" in terms of the following operations:

1. Given two points, construct the straight line through them.
2. Given an ordered pair of points, construct the taxi circle ("diamond") with center at the first and passing through the second.
3. Given two straight lines, construct the point (if any) at which they meet.
4. Given a straight line and a taxi circle, construct their intersection (if any).
5. Given two taxi circles, construct their intersection (if any).

The intersections of a line and a taxi circle, or of two taxi circles, can be one or two points, or (as in FIGURE 1) may consist of a line segment, a point and a line segment, or two line segments. In cases with a line segment, we represent it by its two endpoints (though the fact that it is a segment may be used freely). Construction of any interior points of the intersection must be done separately. Two circles, or a circle and a line, which intersect in two points, will be said to be in *general position*.

We may wonder whether we should also allow the "corners" of a circle, or the horizontal and vertical lines through a point, to be constructed as primitive operations. It turns out that there is no need to do this, as a fairly simple construction using the listed operations gives these.

CONSTRUCTION. *Given a circle, to determine its four corners*

Choose five distinct points A_1, A_2, A_3, A_4, A_5 on the circle. Construct each of the ten lines determined by pairs A_i, A_j, and consider the intersection of each with the circle. By the pigeonhole principle, two of the points must lie on the same side of the circle, and the corresponding intersection will be that entire side. Its endpoints are two adjacent corners of the circle. The lines through each of these corners and the center of the circle intersect the circle again in the other two corners.

As a bonus, we have also constructed the horizontal and vertical lines through the center of the circle! We now come to the main result of the paper.

THEOREM. *There is no construction in the taxicab geometry that will bisect the Euclidean measure of an arbitrary angle.*

Proof. We define a point to be rational if both its coordinates are rational; a line to be rational if it has the form $ax + by = c$ for rational a, b, c; and a taxi circle to be rational if it has rational center and rational radius. It is easily verified that all five elementary constructions, given rational data, return (if anything) one or more rational elements.

Let $O = (0, 0)$, $A = (1, 0)$, and $B = (1, 1)$. The angle $\angle AOB$ has all elements rational; but the ray bisecting it has slope $\tan(\pi/8) = \sqrt{2} - 1$ and is not rational. Thus this angle cannot be bisected in the Martin taxicab geometry. ■

This proof is interesting in its own right; and pedagogically it is very useful as a warmup exercise to prepare students for Wantzel's much more important (but significantly more difficult) proof [**14**] that no Euclidean construction *trisects* an arbitrary angle. (See [**2**], or undergraduate geometry texts such as [**10**] or [**13**], for accessible modern presentations of this result.) In Wantzel's proof, the Hippasian numbers—those that can be obtained using addition, subtraction, multiplication, division, and the square root function—are shown to be closed under Euclidean construction, and to contain values defining a 60° angle but not a 20° angle.

Euclid uses the bisection of an angle in his next proposition, the bisection of an arbitrary straight line. He constructs, on the base $\overline{AB}$, an isosceles triangle $\triangle ABC$, bisects the angle $\angle ACB$, and shows that the bisector also bisects $\overline{AB}$. Clearly, this approach must be abandoned in Martin's taxicab geometry! However, a line segment can be bisected using the taxicab equivalent (FIGURE 3) of an alternative construction ascribed by Proclus to Apollonius [**8**, vol. 1, p. 268].

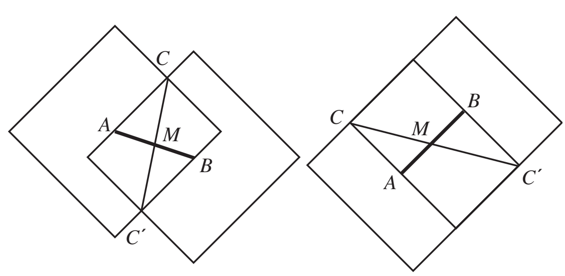

Figure 3 Bisecting the line segment $\overline{AB}$

CONSTRUCTION. *To bisect a line segment using taxicab constructions*

Given a line segment $\overline{AB}$ we construct taxi circles around A through B and around B through A. If the circles are in general position (that is, if the slope of the segment $\overline{AB}$ is not ± 1), we join the two intersection points C, C', and the line $\overline{CC'}$ bisects $\overline{AB}$. Otherwise a slight modification, joining opposite endpoints, effects the same construction.

As a consequence of this construction, we see that in the "t-radian" taxicab geometry of Thompson and Dray, an angle can always be bisected. (Indeed, as a referee of an earlier version of this paper pointed out, it can be trisected, or divided into any number of equal parts.) In many ways, this angle measure is more natural for a taxicab geometry; but the analogy between the failure of angle bisection in Martin's geometry and angle trisection in Euclid's suggests a valuable pedagogical reason for choosing Martin's definition of angle measure, if only one is to be used.

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Quadratic Residues and the Frobenius Coin Problem

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Recently I was struck by the fact that an odd prime p has $(p-1)/2$ quadratic residues mod p and that for relatively prime p and q, there are $(p-1)(q-1)/2$ non-representable Frobenius numbers. I found the presence of $(p-1)/2$ in both expressions curious. Is there some relationship between quadratic residues and the Frobenius numbers that accounts for the presence of $(p-1)/2$ in the two expressions?

As it so happens, there is. Square the non-representable Frobenius numbers for p and q. Mod p, these numbers consist of $q-1$ copies of each of the $(p-1)/2$ quadratic residues mod p, and, mod q, they consist of $p-1$ copies of each of the $(q-1)/2$ quadratic residues mod q. The situation for 5 and 7 is illustrated in the following table. The first row consists of the non-representable Frobenius numbers for 5 and 7, and the second the squares of these numbers. The third and fourth rows are the second row mod 5 and mod 7, respectively.

x	1	2	3	4	6	8	9	11	13	16	18	23
x^2	1	4	9	16	36	64	81	121	169	256	324	529
x^2 mod 5	1	4	4	1	1	4	1	1	4	1	4	4
x^2 mod 7	1	4	2	2	1	1	4	2	1	4	2	4