

Saint Mary's University
DEPARTMENT OF MATHEMATICS
AND COMPUTING SCIENCE

Name: SOLUTIONS

Signature: _____

ID: _____

Math 1211: Winter 2014
Midterm Test #1 (Version A)

February 5, 2014

Instructions:

- All aids and electronic devices (such as cell phones, calculators, notes) are prohibited and should not be in reach during the test. Possession of such items will be construed as an act of academic dishonesty. Put away all cell phones, iPods, calculators, course notes, etc.
- There are 4 pages plus this cover page. Check that your test paper is complete.
- There are a total of 75 marks. The value of each question is indicated in the margin.
- Answer in the spaces provided, using backs of pages for additional space if necessary.
- Show all your work. Insufficient justification will result in a loss of marks.

Page	Maximum	Your Score
1	20	
2	20	
3	20	
4	15	
Total	75	

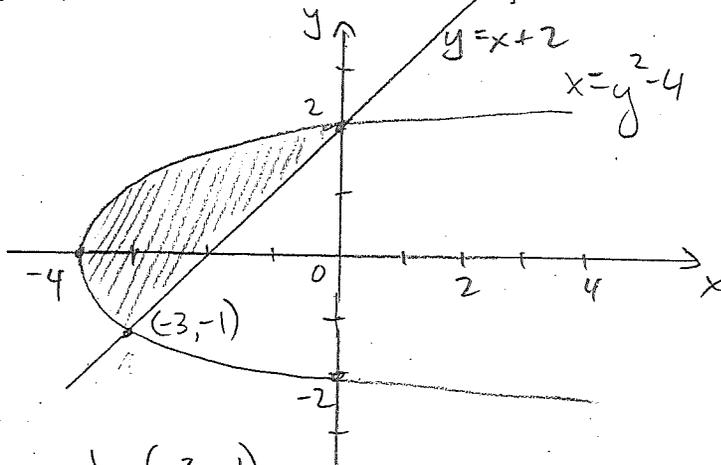
1. Let \mathcal{R} be the region bounded between the curves $x = y^2 - 4$ and $y = x + 2$

[6] (a) Find the points where these curves intersect, and sketch the region \mathcal{R} .

[Note: Your sketch must be appropriately scaled and labelled to receive full credit.]

$$\begin{aligned}
 y &= y^2 - 4 + 2 \\
 \Rightarrow y^2 - y - 2 &= 0 \\
 \Rightarrow (y - 2)(y + 1) &= 0 \\
 \Rightarrow y = 2 \text{ or } y = -1
 \end{aligned}$$

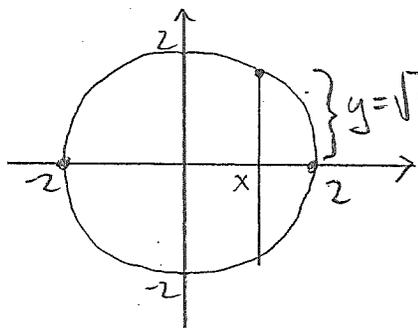
\therefore intersection pts are $(0, 2)$ and $(-3, -1)$



[6] (b) Find the area of \mathcal{R} .

$$\begin{aligned}
 \text{Area} &= \int_{-1}^2 ((y - 2) - (y^2 - 4)) dy \\
 &= \int_{-1}^2 (y - y^2 + 2) dy \\
 &= \left[\frac{1}{2}y^2 - \frac{1}{3}y^3 + 2y \right]_{-1}^2 \\
 &= \left(2 - \frac{8}{3} + 4 \right) - \left(\frac{1}{2} + \frac{1}{3} - 2 \right) \\
 &= 8 - \frac{9}{3} - \frac{1}{2} \\
 &= \frac{9}{2}
 \end{aligned}$$

[8] 2. The base of a solid is the circle $x^2 + y^2 = 4$, and each cross section perpendicular to the x -axis is a rectangle whose height is half its width. Determine the volume of this solid.



CROSS SECTION AT x

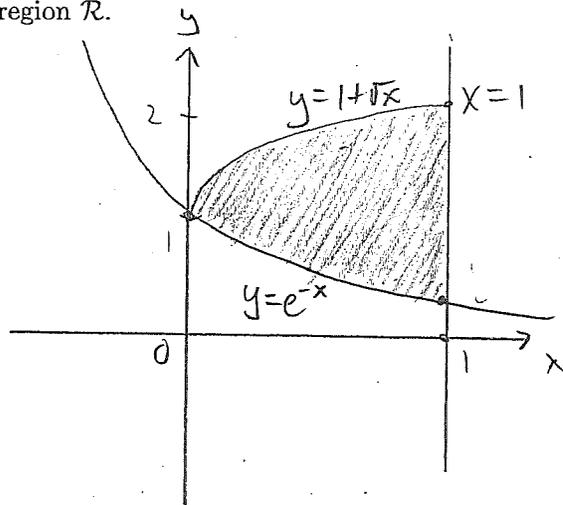


$$\begin{aligned}
 \text{Area} &= 2\sqrt{4 - x^2} \cdot \sqrt{4 - x^2} \\
 &= 2(4 - x^2)
 \end{aligned}$$

$$\begin{aligned}
 \text{So volume} &= \int_{-2}^2 2(4 - x^2) dx \\
 &= 2 \left(4x - \frac{x^3}{3} \right) \Big|_{-2}^2 \\
 &= 2 \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] = \frac{64}{3}
 \end{aligned}$$

3. Let \mathcal{R} be the region bounded between the curves $y = e^{-x}$, $y = 1 + \sqrt{x}$, and $x = 1$.

[4]

(a) Sketch the region \mathcal{R} .

[16]

(b) Give expressions, in terms of definite integrals, for the solids obtained by revolving \mathcal{R} around each of the following axes. Do not evaluate your integrals!

i. The x -axis.

$$\pi \int_0^1 ((1 + \sqrt{x})^2 - e^{-2x}) dx$$

ii. The y -axis.

$$2\pi \int_0^1 x(1 + \sqrt{x} - e^{-x}) dx$$

iii. The line $x = 2$.

$$2\pi \int_0^1 (2 - x)(1 + \sqrt{x} - e^{-x}) dx$$

iv. The line $y = -2$.

$$\pi \int_0^1 ((3 + \sqrt{x})^2 - (2 + e^{-x})^2) dx$$

[20]

4. Evaluate the following integrals:

(a) $\int \frac{e^x}{\sqrt[3]{1-e^x}} dx$

Let $u = 1 - e^x$, so $du = -e^x dx$

$$= - \int \frac{du}{\sqrt[3]{u}}$$

$$= -\frac{3}{2} u^{2/3} + C$$

$$= -\frac{3}{2} (1 - e^x)^{2/3} + C$$

(b) $\int \sin^5 x \cos^2 x dx$

$$= \int (\sin^2 x)^2 \cos^2 x \sin x dx$$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x dx$$

$$= - \int (1 - u^2)^2 u^2 du \quad [u = \cos x, du = -\sin x dx]$$

$$= - \int (u^2 - 2u^4 + u^6) du$$

$$= -\frac{1}{3} u^3 + \frac{2}{5} u^5 - \frac{1}{7} u^7 + C = -\frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C$$

(c) $\int \cos^{-1} 4x dx$

Use parts, with $u = \cos^{-1} 4x$, $dv = dx$

$$du = \frac{-4}{\sqrt{1-(4x)^2}} dx, \quad v = x$$

$$\text{Get } \int \cos^{-1} 4x dx = x \cos^{-1} 4x + 4 \int \frac{x}{\sqrt{1-16x^2}} dx$$

$$= x \cos^{-1} 4x - \frac{1}{4} \sqrt{1-16x^2} + C$$

(d) $\int_{-1}^0 x \sqrt{1+x} dx$

Substitute $u = 1+x$, so $du = dx$, $x = u-1$.

$$\text{Get } \int_{-1}^0 x \sqrt{1+x} dx = \int_0^1 (u-1) \sqrt{u} du$$

$$= \int_0^1 (u^{3/2} - u^{1/2}) du$$

$$= \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_0^1$$

$$= \frac{2}{5} - \frac{2}{3}$$

$$= -\frac{4}{15}$$

[15]

5. Evaluate the following integrals:

(a) $\int_0^2 \frac{2x+1}{x^2+4} dx$

$$= \int_0^2 \frac{2x}{x^2+4} dx + \int_0^2 \frac{dx}{x^2+4}$$

$$= \ln(x^2+4) \Big|_0^2 + \frac{1}{4} \int_0^2 \frac{dx}{\left(\frac{x}{2}\right)^2+1} \quad (\text{let } u = \frac{x}{2}, \text{ so } du = \frac{1}{2} dx)$$

$$= \ln(8) - \ln(4) + \frac{1}{2} \int_0^1 \frac{du}{u^2+1}$$

$$= \ln\left(\frac{8}{4}\right) + \frac{1}{2} \left[\tan^{-1} u \right]_0^1$$

$$= \ln 2 + \frac{1}{2} \tan^{-1}(1) - \frac{1}{2} \tan^{-1}(0)$$

$$= \ln 2 + \frac{\pi}{8}$$

(b) $\int \frac{\ln x}{\sqrt{x}} dx$ Hint: Use integration by parts.

Let $u = \ln x$ $dv = \frac{1}{\sqrt{x}} dx$
 $du = \frac{1}{x} dx$ $v = 2\sqrt{x}$

Get $\int \frac{\ln x}{\sqrt{x}} dx = 2\sqrt{x} \ln x - \int 2\sqrt{x} \cdot \frac{1}{x} dx$

$$= 2\sqrt{x} \ln x - 2 \int \frac{1}{\sqrt{x}} dx$$

$$= 2\sqrt{x} \ln x - 4\sqrt{x} + C$$

(c) $\int \tan^5 2x \cos^3 2x dx$ Note: Don't panic. This is intended as a challenge.

$$= \int \frac{\sin^5 2x}{\cos^5 2x} \cdot \cos^3 2x dx$$

$$= \int \frac{\sin^5 2x}{\cos^2 2x} dx$$

$$= \int \frac{(\sin^2 2x)^2}{\cos^2 2x} \sin 2x dx$$

$$= \int \frac{(1 - \cos^2 2x)^2}{\cos^2 2x} \sin 2x dx$$

$$= -\frac{1}{2} \int \frac{(1-u^2)^2}{u^2} du \quad \left[\begin{array}{l} u = \cos 2x \\ du = -2 \sin 2x dx \end{array} \right]$$

$$= -\frac{1}{2} \int \left(\frac{1}{u^2} - 2 + u^2 \right) du$$

$$= -\frac{1}{2} \left(-\frac{1}{u} - 2u + \frac{1}{3} u^3 \right) + C$$

$$= \frac{1}{2 \cos 2x} + \cos 2x - \frac{1}{6} \cos^3 2x + C$$