Math 1211: Series Comparison Tests — Winter 2014

In the past couple lectures, we have considered two very special types of series, namely *geometric series* and *p-series*. Here's a recap of what we know.

- In the case of **geometric series**, we have perfect information: We know precisely when they converge, and if they do converge then their sum is given by a very simple formula:

$$\sum_{n=1}^{\infty} cr^{n-1} = c + cr + cr^2 + cr^3 + \cdots \qquad \begin{cases} \text{converges to } \frac{c}{1-r} \text{ if } |r| < 1 \\ \text{diverges if } |r| \geq 1. \end{cases}$$

  For example, the series

$$\sum_{n\geq 1} \frac{5 \cdot 2^n}{3^{n+1}} = \frac{5 \cdot 2}{3^2} + \frac{5 \cdot 2^2}{3^3} + \frac{5 \cdot 2^3}{3^4} + \cdots$$

  is geometric with initial term $c = \frac{10}{9}$ and common ratio $r = \frac{2}{3}$. Since $\frac{2}{3} < 1$, this series converges, and its value is

$$\sum_{n\geq 1} \frac{5 \cdot 2^n}{3^{n+1}} = \frac{c}{1-r} = \frac{\frac{10}{9}}{1 - \frac{2}{3}} = \frac{10}{3}.$$

  On the other hand, the series

$$\sum_{n\geq 1} \frac{2^{2n}}{3^n} = \frac{4}{3} + \frac{16}{9} + \frac{64}{27} + \cdots$$

  does not converge, since it is geometric with common ratio $r = \frac{4}{3} > 1$.

- In the case of ***p*-series**, we don't have perfect information: The integral test tells us precisely when such series converge, but it does not tell us to what number! Recall that we have the following rule:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \qquad \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

  For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

  converges, because it is a p-series with $p = 2 > 1$. But it takes some very clever analysis to prove that the sum of this series is actually $\pi^2/6$. Sometimes no amount of cleverness will do. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$$

  also must converge (p-series, $p = 3 > 1$), but in this case we know amazingly little about the value of the sum. This mysterious number, known as *Apéry's constant*, pops up occasionally in various branches of math and physics. We know its approximate value (1.20205...), but we have no idea if it can be expressed as a combination of "known" numbers such as $\sqrt{2}$ or π or e.

Over the next couple of lectures we're going to develop some tools to determine when series converge. But none of these tools will tell us *anything* about the actual numerical value of a convergent series. This may seem strange, but it turns out to be very worthwhile. It is exactly this type of analysis that will permit us to express functions as *Taylor series*, which are essentially "infinite degree" Taylor polynomials.

We have emphasized geometric series and p-series at the outset partly because they are easy to analyse, but mostly because they serve as important "measuring sticks" against which we will compare various other series.

Comparison Theorem: Suppose $\sum a_n$ and $\sum b_n$ are two series with positive terms. Further, suppose that $a_n \leq b_n$ for all n. Then

- if $\sum b_n$ converges then $\sum a_n$ converges, and
- if $\sum a_n$ diverges then $\sum b_n$ diverges.

This theorem should remind you of comparison of improper integrals. If the terms of a given series are smaller than those of a convergent series, then the given series must also converge. If the terms of a given series are bigger than those of a divergent series, then the given series must diverge.

Examples:

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n-\sqrt{n}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n+4^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^6+1}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n2^n}$
- $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

The comparison test is very important, but sometimes it isn't quite the right tool. For example, the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$$

should converge, because when n is very large we have $\frac{1}{n^2-n} \approx \frac{1}{n^2}$ and we know $\sum \frac{1}{n^2}$ converges. But it is not straightforward to use direct comparison, because

$$\frac{1}{n^2-n} > \frac{1}{n^2}.$$

The above inequality shows that our series is actually *bigger* than the convergent series $\sum \frac{1}{n^2}$, and this doesn't tell us anything. So we turn to a more powerful tool.

Limit Comparison Theorem: Suppose $\sum a_n$ and $\sum b_n$ are two series with positive terms, and let

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Then

- if $L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges,
- if $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges, and
- if $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.

Don't try to memorize this theorem. Instead, ponder what it says until it makes sense.

For instance, imagine that $\frac{a_n}{b_n} \to 5$ as $n \to \infty$. This just means that the terms of $\sum a_n$ eventually become approximately 5 times those of $\sum b_n$. Multiplying the terms of a series by 5 can't possibly affect convergence. Therefore if one series converges, so must the other; and if one diverges then so must the other.

If instead the ratio $\frac{a_n}{b_n}$ approaches 0, then the terms of $\sum a_n$ are eventually tiny in comparison to those of $\sum b_n$. So if $\sum b_n$ converges, then the smaller series $\sum a_n$ must also converge. Similar logic applies when $\frac{a_n}{b_n}$ approaches ∞.

Examples:

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{3n^5+2n+1}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{4n^3+1}}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{2n+2^n}{3n+3^n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2}$