Determinants

In this section, we only consider square matrices. Suppose that A is an n by n matrix. We want to define a function "det(A)" whose value is a real number. To do this we could give an explicit formula, but it is a very long and complex formula that is seldom used in practice. Instead we will define det by its properties (much the same way in which we defined the inverse of a matrix).

The idea is simple: if A is 2 by 2, det(A) is the area of the parallelogram generated by the row vectors of A; if A is 3 by 3, det(A) is the volume of the parallelepiped spanned by the rows vectors of A, etc. The properties defining det all come from this idea:

Definition D.1 The function det from $\mathfrak{M}_{n,n}$ (the n by n matrices) to **F** (the set of scalars) is the function having the following Properties:

a) det(I) = 1

b) If $A \in \mathfrak{M}_{n,n}$ is obtained from $B \in \mathfrak{M}_{n,n}$ by switching two rows of B, then det(A) = –det(B).

c) If $A \in \mathfrak{M}_{n,n}$, det(A) is linear in each row of A.

Now this sort of definition is rather bold in that it asserts that there is such a function and that it is unique and the definer is obliged to show that both are indeed the case! Thus it may appear that we are "proving a definition". This is not the case; rather we are justifying these assertions to show that the definition makes sense.

This is accomplished by demonstrating how to compute det(A) in a unique way. No big surprise here – the method is strict Gaussian elimination!!

First let's examine the 3 parts of this definition in light of our goal to make det represent area/volume.

The area of the unit square in $\mathbf{R}^2$ is 1 and the volume of the unit cube in $\mathbf{R}^3$ is 1. This motivates det(I) = 1.

Secondly, for the purposes of calculus, you found that negative area/volume is essential for certain procedures to work. Part b) is there to guarantee that if orientation of an object is switched by switching left and right, then the sign of its area/volume also switches.

Thirdly, if one of the vectors generating a parallelogram is doubled, then the area of that parallelogram is doubled. Similar insight says that if one of the vectors generating a parallelogram is multiplied by α, then the area of that parallelogram is multiplied by α. The fact that det should also "respect" addition is easily demonstrated by a diagram [shown in class].

Now we proceed to deduce the properties of the determinant and to find a method to compute it.

Theorem D.1 If two rows of a square matrix A are the same, then det(A) = 0.

<u>Proof</u>. Interchange the two equal rows. Note that the matrix is unchanged, but by Definition D.1 b), det(A) = –det(A). Hence det(A) = 0. **QED**

Theorem D.2 If a multiple of a row i (say α times row i) of a square matrix A is added to a different row, say row j of A to obtain the matrix B, then det(A) = det(B).

<u>Proof</u>. Let C be the matrix obtained from A by replacing row j with row i. Then by Definition D.1 c), det(B) is linear in the j^{th} row, and hence we obtain det(B) = det(A) + α det(C). Since C has 2 equal rows, det(C) = 0 (by Theorem D.1) and thus det(A) = det(B). **QED**

Theorem D.3 If a row of a square matrix A is all 0's, then det(A) = 0.

<u>Proof.</u> Let row j be the row of zeros. Then by Definition D.1 c), det(B) is linear in the j^{th} row, and hence we obtain det(A) = det(A) + det(A). Hence det(A) = 0. **QED**

Lemma D.4 If the square matrix A is not invertible (**singular**), then det(A) = 0.

<u>Proof</u> By Theorem D.2, together with Definition D.1 b) if we perform strict Gaussian elimination on the matrix A and obtain the matrix B, we get det(B) = ±det(A). If A is noninvertible (**singular**) then reducing to echelon form via strict Gaussian elimination yields a matrix B with a zero row and hence det(A) = 0. **QED**

Theorem D.5 If the square matrix A is triangular, then det(A) is the product of the entries along the main diagonal of A.

<u>Proof</u> If A has a zero on the main diagonal then by the rank theorem, A is noninvertible (**singular**) and hence by Lemma D.4, det(A) = 0. But the product of the entries along the main diagonal of A is also 0 since it contains a factor of 0. On the other hand, if A has no zero on the main diagonal then we can row reduce A, via strict Gaussian elimination and using no row interchanges, to a diagonal matrix L whose entries are exactly the entries along the main diagonal of A. But then by Theorem D.2, det(A) = det(L), and using Definition D.1 c) on successive rows of L we get $\det(L) = a_{1,1}\, a_{2,2}\, a_{3,3} \ldots a_{n,n} \det(I) = a_{1,1}\, a_{2,2}\, a_{3,3} \ldots a_{n,n}$. (using Definition D.1 a) for the last step.) **QED**

NOTE At this point we note that we have found a method for computing the determinant of any square matrix A: reduce A to echelon form using strict Gaussian elimination. Then $\det(A) = (-1)^k\, p$ where p is the product of the pivots and k is the number of simple row exchanges that occurred in the elimination process.

Theorem D.6 The square matrix A is invertible (**nonsingular**), if and only if $\det(A) \neq 0$.

<u>Proof</u> In one direction, if A is noninvertible (**singular**), then, by Lemma D.4,

det(A) = 0.

In the other direction, if A is invertible (**nonsingular**) then reducing A to echelon form via strict Gaussian elimination yields a triangular matrix B with no zero pivot and hence by Theorem D.5 $\det(B) \neq 0$. As before, by Theorem D.2, together with Definition D.1 b) we get $\det(B) = \pm\det(A)$. Hence $\det(A) \neq 0$. **QED**

Theorem D.7 For square matrices R and S of the same size, we have: det(RS) = det(R)det(S).

Proof If S is singular, then both sides are 0 by Theorem D.6.

Assume that S is invertible (ie det(S) is not 0). Define the function g from $\mathfrak{M}_{n,n}$ to **F** (the set of scalars) by the formula: $g(X) = \frac{\det(XS)}{\det(S)}$. We will show that g(X) = det(X), by showing that g satisfies the three defining properties of det (Definition D.1). It will then follow that det(XS) = det(X)det(S) for every matrix X in $\mathfrak{M}_{n,n}$.

a) $g(I) = \frac{\det(IS)}{\det(S)} = \frac{\det(S)}{\det(S)} = 1$

b) If $A \in \mathfrak{M}_{n,n}$ is obtained from $B \in \mathfrak{M}_{n,n}$ by switching two rows of B, then observe that A = EB for some elementary matrix E that always switches two rows when it multiplies a matrix on the left. But then AS = (EB)S =E(BS), and hence AS is obtained from BS by switching two rows of BS. Hence:

$$g(A) = \frac{\det(AS)}{\det(S)} = -\frac{\det(BS)}{\det(S)} = -g(B)$$

c) If A, B $\in \mathfrak{M}_{n,n}$, are identical except perhaps along the j^{th} row, then AS and BS are identical except perhaps along the j^{th} row. Hence

$$\begin{aligned} g(\alpha A + \beta B) &= \frac{\det\big((\alpha A + \beta B)S\big)}{\det(S)} = -\frac{\det(\alpha AS + \beta BS)}{\det(S)} \\ &= \alpha\frac{\det(AS)}{\det(S)} + \beta\frac{\det(BS)}{\det(S)} \\ &= \alpha g(A) + \beta g(B). \end{aligned}$$

Hence, if A $\in \mathfrak{M}_{n,n}$, g(A) is linear in each row of A. **QED**

Theorem D.8 For the square matrix A we have: det(A) = det(A^T).