

## Math 2321: Linear Algebra II

### Midterm Test #1

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Instructor: J. Irving

#### Instructions:

- Calculators are permitted provided they don't dim the lights when you turn them on.
- There are 5 pages plus this cover page. Check that your test paper is complete.
- There are a total of 80 marks. The value of each question is indicated in the margin.
- Show all your work. Insufficient justification will result in a loss of marks.

Page	Maximum	Your Score
1	14	
2	26	
3	14	
4	14	
5	12	
Total	80	

[14]

1. Give precise definitions for the following terms and notation. Throughout,  $A$  is an  $n \times n$  matrix.

(a) The *column space* of  $A$

The span of the columns of  $A$ .

(b) The *nullity* of  $A$ .

The dimension of the null space of  $A$ .

(c) The *coordinates* of  $x \in \mathbb{R}^n$  with respect to the basis  $\mathcal{B} = \{v_1, \dots, v_n\}$ .

The vector  $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$  such that  $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

(d) An *eigenvector* of  $A$ .

A nonzero vector  $x$  such that  $Ax = \lambda x$   
for some scalar  $\lambda$ .

(e) The *characteristic polynomial* of  $A$ .

$$\det(A - \lambda I)$$

(f) The *eigenspace* corresponding to eigenvalue  $\lambda$  of  $A$

$$E_\lambda = \text{null}(A - \lambda I)$$

(g) The *geometric multiplicity* of the eigenvalue  $\lambda$  of  $A$ .

$$\dim E_\lambda$$

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2. Answer true or false or admit ignorance to the following by circling your choice. You will receive 2 points for a correct response, 1 point for no response, and 0 for an incorrect response. Throughout,  $A$  is an  $n \times n$  matrix.

TRUE     FALSE     UNSURE    The rank and nullity of  $A$  sum to  $n$ .

TRUE     FALSE     UNSURE    If  $A$  has column rank  $n$  then  $\det A \neq 0$ .

TRUE     FALSE     UNSURE     $\det(A^2 - I) = \det(A)^2 - 1$ .

TRUE     FALSE     UNSURE     $A$  is invertible if and only if 0 is not an eigenvalue of  $A$ .

TRUE     FALSE     UNSURE    Similar matrices have the same eigenvectors.

TRUE     FALSE     UNSURE    The eigenvectors of  $A$  are linearly independent.

TRUE     FALSE     UNSURE    If  $\lambda$  is an eigenvalue of  $A$  then  $\lambda^2$  is an eigenvalue of  $A^2$ .

TRUE     FALSE     UNSURE    If  $A$  is triangular and invertible then  $A$  is diagonalizable.

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3. Short answer. No justification is required.

- (a) If  $A$  is a  $4 \times 7$  matrix, what are the possible values for the nullity of  $A$ ?

$$3, 4, 5, 6, 7$$

- (b) Suppose  $A$  and  $B$  are  $3 \times 3$  matrices with  $\det A = -1$  and  $\det B = 2$ . Determine  $\det(3A^5B^{-1})$ .

$$3^3 \cdot (-1)^5 \cdot \frac{1}{2} = -\frac{27}{2}$$

- (c) Suppose the eigenvalues of a matrix  $A$  are  $-1, 1$ , and  $2$ . Find the eigenvalues of  $2A^3 - I$ .

$$-3, 1, 15 \quad [\text{Evaluate } 2\lambda^3 - 1 \text{ at } \lambda = \pm 1, 2]$$

- (d) Let  $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ . Find the eigenvalues and eigenspaces of  $A$  over  $\mathbb{Z}_5$ , and use this information to find the eigenvalues and eigenspaces of  $2A + I$  over  $\mathbb{Z}_5$ .

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2 \\ &= \lambda^2 - 5\lambda + 4 \\ &= \lambda^2 - 1 \quad \text{in } \mathbb{Z}_5. \end{aligned}$$

So e-vals of  $A$  are  $\pm 1$ , and  $E_1 = \text{null} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$   
 $E_{-1} = \text{null} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

$\therefore$  e-values of  $2A + I$  are  $2(1) + 1 = 3$  and  $2(-1) + 1 = -1$ ,  
with  $E_3 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$  and  $E_{-1} = \text{span} \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ .

[8]

4. Evaluate  $\det \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \end{bmatrix}$ .

$$= \left| \begin{array}{ccccc|c} 6 & 0 & 0 & 0 & 5 & R_1 - R_2 + R_3 \\ 0 & 2 & 4 & 4 & -1 & R_2 + R_5 \\ 0 & 2 & 2 & 0 & 0 & R_3 + R_4 + R_5 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \end{array} \right|$$

$$= 5 \cdot 2 \cdot 2 \left| \begin{array}{cccc|c} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{array} \right| \quad (\text{Expand along first row})$$

$$= 20 \left| \begin{array}{ccc|c} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{array} \right|$$

$$= 20 \left( 1 \cdot \left| \begin{array}{cc|c} 1 & 2 \\ 1 & 0 \end{array} \right| - 1 \cdot \left| \begin{array}{cc|c} 1 & 1 \\ 1 & 1 \end{array} \right| \right) \boxed{= -20}$$

[6]

5. Find all values of  $k$  such that the matrix  $\begin{bmatrix} k & k+1 & 1 \\ 0 & 2k & k-1 \\ k & 3k & k-2 \end{bmatrix}$  is invertible.

$$\begin{aligned} \det \begin{bmatrix} k & k+1 & 1 \\ 0 & 2k & k-1 \\ k & 3k & k-2 \end{bmatrix} &= \det \begin{bmatrix} k & k+1 & 1 \\ 0 & 2k & k-1 \\ 0 & 2k-1 & k-3 \end{bmatrix} \\ &= k \det \begin{bmatrix} 2k & k-1 \\ -1 & -2 \end{bmatrix} \\ &= k(-4k + k - 1) \\ &= -k(3k + 1) \end{aligned}$$

So the determinant is 0 only for  $k=0$  and  $k = -\frac{1}{3}$ .

Thus the matrix is invertible for all  $k \in \mathbb{R}$  except  $k=0$  and  $k = -\frac{1}{3}$ .

[14]

6. For each of the following matrices  $A$ , determine whether  $A$  is diagonalizable. If so, find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $D = P^{-1}AP$ .

$$(a) A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \\ &= \det \begin{bmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 0 & 1-\lambda & 1-\lambda \end{bmatrix} \\ &= (1-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} \\ &= (1-\lambda)(-\lambda) \begin{bmatrix} 2-\lambda & 1 \\ 1 & 1 \end{bmatrix} \\ &= -\lambda(1-\lambda)^2 \end{aligned}$$

So the eigenvalues are  $\lambda = 0$  and  $\lambda = 1$ .

$$\text{Now } E_1 = \text{null} \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

$$\text{And } E_0 = \text{null} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$(b) A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

Thus  $A$  is diagonalizable,  
and we have  $D = P^{-1}AP$ , where

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 1 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)^2(3-\lambda) \end{aligned}$$

So the eigenvalues are  $\lambda = 2$  and  $\lambda = 3$ .

Now  $E_2 = \text{null} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$ , which is of dimension 1  
(since this matrix has rank 2).

Since the geometric multiplicity of  $\lambda = 2$  is 1 and its algebraic multiplicity is 2, we conclude that  $A$  is not diagonalizable.

[8]

7. Let  $A$  and  $B$  be  $n \times n$  matrices with eigenvalues  $\lambda$  and  $\mu$ , respectively.

- (a) Give an example to show that  $\lambda\mu$  need not be an eigenvalue of  $AB$ .

$$\text{Let } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then  $A$  and  $B$  have char poly  $(1-x)^2$ , hence e-values  $\lambda = \mu = 1$ .

But  $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  has char poly  $x^2 - 3x + 1$ , and  $\lambda\mu = 1$  is not a root of this poly.

- (b) Suppose  $\lambda$  and  $\mu$  correspond to the *same* eigenvector  $x$ . Show that, in this case,  $\lambda\mu$  is an eigenvalue of  $AB$ .

Suppose  $Ax = \lambda x$  and  $Bx = \mu x$ , with  $x \neq 0$ .

$$\begin{aligned} \text{Then } ABx &= A(Bx) \\ &= A(\mu x) \\ &= \mu(Ax) \\ &= \mu(\lambda x) \\ &= \mu\lambda x \end{aligned}$$

Thus  $x$  is an e-vector of  $AB$ , with e-value  $\mu\lambda$ .

(In particular,  $\mu\lambda$  is an e-value of  $AB$ , as required.)

[4]

8. Let  $A$  and  $B$  be invertible matrices of the same dimensions. Show that  $AB$  and  $BA$  are similar.

To show  $AB$  and  $BA$  are similar, we must exhibit an invertible matrix  $P$  such that

$$AB = P^{-1}BAP.$$

Since  $B$  is invertible, we can take  $P = B^{-1}$ .

Then  $P^{-1}BAP = B^{-1}BAB = AB$ , as required.