SAINT MARY'S UNIVERSITY
DEPARTMENT OF MATHEMATICS
AND COMPUTING SCIENCE

Name: ______________________________

ID: ______________________________

Math 2321: Linear Algebra II

Midterm Test #2

March 14, 2014

Instructor: J. Irving

Instructions:

- *Calculators are permitted provided they don't dim the lights when you turn them on.*
- *There are 6 pages plus this cover page. Check that your test paper is complete.*
- *There are a total of 100 marks. The value of each question is indicated in the margin.*
- *Show all your work. Insufficient justification will result in a loss of marks.*

Page	Maximum	Your Score
1	16	
2	26	
3	14	
4	14	
5	16	
6	14	
Total	**100**	

[10] 1. Give precise definitions for the following terms and notation. Throughout, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$.

(a) An *orthogonal set* of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$.

(b) An *orthogonal matrix* Q.

(c) The *projection* of $\mathbf{u}$ onto $\mathbf{v}$.

(d) The *orthogonal complement* $W^\perp$ of the subspace $W \subseteq \mathbb{R}^n$.

(e) A *QR factorization* of the $m \times n$ matrix A.

[6] 2. Give precise statements of the following theorems.

(a) The *Orthogonal Decomposition Theorem.*

(b) The *Spectral Theorem* for real symmetric matrices.

[10] 3. Short answer. **No justification is required**.

(a) For a certain 2×2 matrix A and vector $\mathbf{v}\in\mathbb{R}^2$ we have

$$A\mathbf{v}=\begin{bmatrix}3\\8\end{bmatrix},\qquad A^2\mathbf{v}=\begin{bmatrix}17\\32\end{bmatrix},\qquad A^3\mathbf{v}=\begin{bmatrix}83\\168\end{bmatrix},\qquad A^4\mathbf{v}=\begin{bmatrix}417\\832\end{bmatrix},\qquad A^5\mathbf{v}=\begin{bmatrix}2083\\4168\end{bmatrix}.$$

Find the dominant eigenvalue of A and a corresponding eigenvector.

(b) Let Q be an orthogonal matrix. What are the possible values of $\det Q$?

(c) Let W be a subspace of $\mathbb{R}^5$ spanned by three vectors. What are the possible dimensions of $W^\perp$?

(d) Suppose A and B are orthogonally diagonalizable $n\times n$ matrices. Which of the following matrices are also necessarily orthogonally diagonalizable? (Circle your choices.)

$A+B$ $\qquad A^3$ $\qquad 5A$ $\qquad AB$ $\qquad A^2-I$

[16] 4. Answer **true** or **false** to the following by circling your choice. You will receive 2 points for a correct response, 1 point for no response, and 0 for an incorrect response. *Throughout, A is an $n\times n$ real matrix and W is a subspace of $\mathbb{R}^n$.*

TRUE FALSE Every orthonormal set of vectors is linearly independent.

TRUE FALSE Every nonzero subspace of $\mathbb{R}^n$ has an orthogonal basis.

TRUE FALSE If λ is an eigenvalue of an orthogonal matrix then $\lambda=\pm1$.

TRUE FALSE Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

TRUE FALSE $(\mathrm{row}(A))^\perp=\mathrm{null}(A)$.

TRUE FALSE $(W^\perp)^\perp=W$

TRUE FALSE For any $\mathbf{v}\in\mathbb{R}^n$, we have $\mathrm{proj}_W(\mathrm{proj}_W(\mathbf{v}))=\mathrm{proj}_W(\mathbf{v})$.

TRUE FALSE Every orthogonally diagonalizable matrix is invertible.

5. Consider the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

[6] (a) Apply the Gram-Schmidt procedure to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to obtain an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

[4] (b) Find the coordinates of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to your orthogonal basis from (a).

[4] (c) Find a QR factorization of the matrix $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

6. Let $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, and consider the subspace $W = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ of $\mathbb{R}^4$.

[5] (a) Find a basis for $W^\perp$.

[6] (b) Find the orthogonal decomposition of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ with respect to W.

[3] (c) Find an orthogonal basis of $\mathbb{R}^4$ that contains $\mathbf{w}_1$ and $\mathbf{w}_2$. [*Hint:* You can use your work from (a).]

7. Let $\mathbf{x} = (x_1, x_2, x_3)^T$ and consider the quadratic form

$$f(\mathbf{x}) = 3x_1^2 + 3x_2^2 + 3x_3^2 + 4x_1x_2 - 4x_1x_3 + 4x_2x_3.$$

[2] (a) State the matrix A such that $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

[12] (b) Find an orthogonal matrix Q such that the change of variables $\mathbf{x} = Q\mathbf{y}$ transforms $f(\mathbf{x})$ into a form $g(\mathbf{y})$ with no cross-product terms. State both Q and $g(\mathbf{y})$.

[2] (c) Find the minimum value of $f(\mathbf{x})$ subject to the constraint $||\mathbf{x}|| = 1$, and determine the value of $\mathbf{x}$ at which this minimum occurs.

[4] 8. Let Q be an orthogonal 2×2 matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. If θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, prove that θ is also the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$.

[6] 9. Let W be a subspace of $\mathbb{R}^n$ and suppose $W = \text{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$. Prove that $\mathbf{v} \in W^{\perp}$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \ldots, k$.

[4] 10. Let A be a real symmetric matrix. Prove that if every eigenvalue of A is nonnegative then $A = B^2$ for some real symmetric matrix B.