

Math 2321: Linear Algebra II

Midterm Test #2

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Instructions:

- Calculators are permitted provided they don't dim the lights when you turn them on.
- There are 6 pages plus this cover page. Check that your test paper is complete.
- There are a total of 100 marks. The value of each question is indicated in the margin.
- Show all your work. Insufficient justification will result in a loss of marks.

Page	Maximum	Your Score
1	16	
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Total	100	

- [15] 1. Give precise definitions for the following terms and notation. Throughout, \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n and W is a subspace of \mathbb{R}^n .

- (a) An *orthogonal set* of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$.

$\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is orthogonal if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$

- (b) An *orthogonal matrix* Q .

A square matrix Q is orthogonal if its columns form an orthonormal set.

- (c) The *projection* of \mathbf{u} onto \mathbf{v} .

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$$

- (d) The *orthogonal complement* W^\perp of the subspace $W \subseteq \mathbb{R}^n$.

$$W^\perp = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W \}$$

- (e) A *QR factorization* of the $m \times n$ matrix A .

$A = QR$, where Q is an $m \times n$ matrix with orthonormal columns and R is $n \times n$ upper triangular.

- [7] 2. Give precise statements of the following theorems.

- (a) The *Orthogonal Decomposition Theorem*.

Let W be a subspace of \mathbb{R}^n . For any $\mathbf{v} \in \mathbb{R}^n$ there exist unique vectors $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$.

- (b) The *Spectral Theorem* for real symmetric matrices.

A matrix is orthogonally diagonalizable if and only if it is real symmetric.

[10]

3. Short answer. No justification is required.

- (a) For a certain
- 2×2
- matrix
- A
- and vector
- $\mathbf{v} \in \mathbb{R}^2$
- we have

$$A\mathbf{v} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \quad A^2\mathbf{v} = \begin{bmatrix} 17 \\ 32 \end{bmatrix}, \quad A^3\mathbf{v} = \begin{bmatrix} 83 \\ 168 \end{bmatrix}, \quad A^4\mathbf{v} = \begin{bmatrix} 417 \\ 832 \end{bmatrix}, \quad A^5\mathbf{v} = \begin{bmatrix} 2083 \\ 4168 \end{bmatrix}.$$

Find the dominant eigenvalue of A and a corresponding eigenvector.

Eigenvalue is 5 ($\approx \frac{2083}{4168}$), with eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- (b) Let
- Q
- be an orthogonal matrix. What are the possible values of
- $\det Q$
- ?

$$\det Q = \pm 1$$

- (c) Let
- W
- be a subspace of
- \mathbb{R}^5
- spanned by three vectors. What are the possible dimensions of
- W^\perp
- ?

$$\dim W + \dim W^\perp = 5, \text{ so } \dim W^\perp = 2, 3, \text{ or } 4$$

(because $\dim W = 1, 2, \text{ or } 3$)

- (d) Suppose
- A
- and
- B
- are orthogonally diagonalizable
- $n \times n$
- matrices. Which of the following matrices are also necessarily orthogonally diagonalizable? (Circle your choices.)

 $A+B$ A^3 $5A$ AB $A^2 - I$

[16]

4. Answer
- true**
- or
- false**
- to the following by circling your choice. You will receive 2 points for a correct response, 1 point for no response, and 0 for an incorrect response.
- Throughout, A is an $n \times n$ real matrix and W is a subspace of \mathbb{R}^n .*

TRUE FALSE Every orthonormal set of vectors is linearly independent.

TRUE FALSE Every nonzero subspace of \mathbb{R}^n has an orthogonal basis.

TRUE FALSE If λ is an eigenvalue of an orthogonal matrix then $\lambda = \pm 1$.

TRUE FALSE Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

TRUE FALSE $(\text{row}(A))^\perp = \text{null}(A)$.

TRUE FALSE $(W^\perp)^\perp = W$

TRUE FALSE For any $\mathbf{v} \in \mathbb{R}^n$, we have $\text{proj}_W(\text{proj}_W(\mathbf{v})) = \text{proj}_W(\mathbf{v})$.

TRUE FALSE Every orthogonally diagonalizable matrix is invertible.

5. Consider the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of \mathbb{R}^3 , where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

[6]

(a) Apply the Gram-Schmidt procedure to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to obtain an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\ \mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \\ \mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 3/10 \\ 3/2 \end{bmatrix} \\ &\qquad\qquad\qquad \xrightarrow{\text{scale}} \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \\ \text{So } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} &= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\} \end{aligned}$$

[4] (b) Find the coordinates of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to your orthogonal basis from (a).

$$\frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{5}{5} = 1$$

$$\frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{3}{6} = \frac{1}{2}$$

\therefore coordinates are

$$\begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\frac{\mathbf{v} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} = \frac{15}{30} = \frac{1}{2}$$

[4] (c) Find a QR factorization of the matrix $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ whose columns are $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 .

Normalize $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to get $\mathbf{Q} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{6} & -2/\sqrt{30} \\ 2/\sqrt{5} & -1/\sqrt{6} & 1/\sqrt{30} \\ 0 & 1/\sqrt{6} & 5/\sqrt{30} \end{bmatrix}$

$$\begin{aligned} \text{Then } \mathbf{R} &= \mathbf{Q}^T \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ -2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{5} & \sqrt{5} & 3/\sqrt{5} \\ 0 & \sqrt{6} & 3/\sqrt{6} \\ 0 & 0 & 9/\sqrt{30} \end{bmatrix} \end{aligned}$$

6. Let $w_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, and consider the subspace $W = \text{span}\{w_1, w_2\}$ of \mathbb{R}^4 .

[5]

- (a) Find a basis for W^\perp .

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & -1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -7 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/7 & -4/7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2/7 & 1/7 \\ 0 & 1 & -1/7 & -4/7 \end{bmatrix}$$

$$\text{So } W^\perp = \text{null} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & -1 & 1 & 1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -2/7 \\ 1/7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/7 \\ 4/7 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A basis for W^\perp is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 0 \\ 7 \end{bmatrix} \right\}$

[6]

- (b) Find the orthogonal decomposition of $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ with respect to W .

$$\begin{aligned} \text{Then: } \text{proj}_W(v) &= \frac{v \cdot w_1}{w_1 \cdot w_1} w_1 + \frac{v \cdot w_2}{w_2 \cdot w_2} w_2 \\ &= \frac{2}{6} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + \frac{4}{12} \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix} \\ \text{and } \text{perp}_W(v) &= v - \text{proj}_W(v) = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \\ 1 \end{bmatrix} \end{aligned}$$

The orthogonal decomposition is $v = \text{proj}_W(v) + \text{perp}_W(v)$.

[3]

- (c) Find an orthogonal basis of \mathbb{R}^4 that contains w_1 and w_2 . [Hint: You can use your work from (a).]

Orthogonalize the vectors for (a):

$$\begin{aligned} \begin{bmatrix} -1 \\ 4 \\ 0 \\ 7 \end{bmatrix} &- \frac{(-1, 4, 0, 7) \cdot (-2, 1, 7, 0)}{(-2, 1, 7, 0) \cdot (-2, 1, 7, 0)} \begin{bmatrix} -2 \\ 1 \\ 7 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 4 \\ 0 \\ 7 \end{bmatrix} - \frac{6}{54} \begin{bmatrix} -2 \\ 1 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} -7/9 \\ 35/9 \\ -7/9 \\ 7/9 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} -1 \\ 5 \\ -1 \\ 9 \end{bmatrix} \end{aligned}$$

So an orthogonal basis for \mathbb{R}^4 is $\left\{ \underbrace{\begin{bmatrix} -2 \\ 1 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 0 \\ 7 \end{bmatrix}}_{W^\perp}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}}_{W} \right\}$

7. Let $\mathbf{x} = (x_1, x_2, x_3)^T$ and consider the quadratic form

$$f(\mathbf{x}) = 3x_1^2 + 3x_2^2 + 3x_3^2 + 4x_1x_2 - 4x_1x_3 + 4x_2x_3.$$

- [2] (a) State the matrix A such that $f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$.

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & 2 \\ -2 & 2 & 3 \end{bmatrix}$$

- [12] (b) Find an orthogonal matrix Q such that the change of variables $\mathbf{x} = Q\mathbf{y}$ transforms $f(\mathbf{x})$ into a form $g(\mathbf{y})$ with no cross-product terms. State both Q and $g(\mathbf{y})$.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & -2 \\ 2 & 3-\lambda & 2 \\ -2 & 2 & 3-\lambda \end{bmatrix} \\ &= (3-\lambda)[(3-\lambda)^2 - 4] - 2[6 - 2\lambda + 4] - 2[4 + 6 - 2\lambda] \\ &= (3-\lambda)(1-\lambda)(5-\lambda) - 4(5-\lambda) - 4(5-\lambda) \\ &= (5-\lambda)[\lambda^2 - 4\lambda + 3 - 4 - 4] \\ &= (5-\lambda)^2(1+\lambda) \\ E_5 &= \text{null} \begin{bmatrix} -2 & 2 & -2 \\ 2 & -2 & 2 \\ -2 & 2 & -2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \\ E_{-1} &= \text{null} \begin{bmatrix} 4 & 2 & -2 \\ 2 & 4 & 2 \\ -2 & 2 & 4 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

$$\text{Orthogonalize } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ to get } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{(-1)}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \\ = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}$$

$$\text{Normalize to let } Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$\text{Then } g(\mathbf{y}) = (Q\mathbf{y})^T A (Q\mathbf{y}) = 5y_1^2 + 5y_2^2 - y_3^2$$

- [2] (c) Find the minimum value of $f(\mathbf{x})$ subject to the constraint $\|\mathbf{x}\| = 1$, and determine the value of \mathbf{x} at which this minimum occurs.

The minimum is -1 (the smallest e-value of A)

It is attained at $\mathbf{x} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ (a normalized e-vector with e-value -1)

[4]

8. Let Q be an orthogonal 2×2 matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. If θ is the angle between \mathbf{x} and \mathbf{y} , prove that θ is also the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$.

$$\text{We know } \mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cos \theta$$

Let α be the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$.

$$\text{Then } Q\mathbf{x} \cdot Q\mathbf{y} = \|Q\mathbf{x}\| \|Q\mathbf{y}\| \cos \alpha$$

But $Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ and $\|Q\mathbf{x}\| = \|\mathbf{x}\|$, $\|Q\mathbf{y}\| = \|\mathbf{y}\|$ (as Q is orthogonal)

$$\text{Thus } \mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cos \alpha$$

Therefore $\cos \alpha = \cos \theta$, so $\alpha = \theta$.

[6]

9. Let W be a subspace of \mathbb{R}^n and suppose $W = \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$. Prove that $\mathbf{v} \in W^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \dots, k$.

If $\mathbf{v} \in W^\perp$ then $\mathbf{v} \cdot \mathbf{w}_i = 0 \ \forall i$ because $\mathbf{w}_i \in W \ \forall i$.

Now suppose $\mathbf{v} \cdot \mathbf{w}_i = 0 \ \forall i$. Then let $\mathbf{w} \in W$.

We can write \mathbf{w} as $\mathbf{w} = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \dots + \alpha_k \mathbf{w}_k$ for some scalars α_i .

$$\begin{aligned} \text{Then } \mathbf{v} \cdot \mathbf{w} &= (\alpha_1 \mathbf{w}_1 + \dots + \alpha_k \mathbf{w}_k) \cdot \mathbf{v} = \alpha_1 (\mathbf{v} \cdot \mathbf{w}_1) + \dots + \alpha_k (\mathbf{v} \cdot \mathbf{w}_k) \\ &= 0 + \dots + 0 \\ &= 0 \end{aligned}$$

So $\mathbf{v} \cdot \mathbf{w} = 0 \ \forall \mathbf{w} \in W$, meaning $\mathbf{v} \in W^\perp$, as required.

[4]

10. Let A be a real symmetric matrix. Prove that if every eigenvalue of A is nonnegative then $A = B^2$ for some real symmetric matrix B .

Let A be a real symmetric matrix whose eigenvalues are non-negative; say $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$.

Since A is real symmetric, there exists an orthogonal matrix Q such that $A = Q^T D Q$, where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

Let $\hat{D} = \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_n} \end{bmatrix}$ and set $B = Q^T \hat{D} Q$.

Then B is symmetric and $B^2 = Q^T \hat{D} Q Q^T \hat{D} Q = Q^T \hat{D}^2 Q = Q^T D Q = A$.