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MATH 2321: RECITATION # 1
SOLUTIONS

① First row reduce A to get $R = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

- a) $\text{rank}(A) = 3$, since R has 3 nonzero rows
- b) $\text{nullity}(A) = 2$, since $\text{rank}(A) + \text{nullity}(A) = \# \text{columns}(A) = 5$.
- c) $\{(1, 0, -3, 5, 0), (0, 1, 2, -1, 0), (0, 0, 0, 0, 1)\}$ is a basis for the row space of A, since these are the nonzero rows of R.
- d) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ -8 \end{bmatrix} \right\}$ is a basis for the column space of A, since these are the columns of A corresponding to the pivots of R.
- e) The null space of A is the same as the null space of R, which is seen to be the set of vectors

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5 : \begin{array}{l} x_1 = 3x_3 - 5x_4 \\ x_2 = -2x_3 + x_4 \\ x_5 = 0 \end{array} \right\} = \text{span} \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Thus $\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for the null space of A

(2)

$$\textcircled{2} \text{ a) } \text{rank}(A^T) = \text{row rank}(A^T)$$

$$= \text{column rank}(A)$$

$$= \text{rank}(A)$$

$$= 3$$

$$\text{b) } \text{nullity}(A^T) = 1, \text{ since } \text{rank}(A^T) + \text{nullity}(A^T) = \#\text{columns}(A^T) = 4.$$

$$\text{c) Same answer as (1d), since } \text{rowspace}(A^T) = \text{column space}(A)$$

$$\text{d) Same answer as (1c) since } \text{columnspace}(A^T) = \text{row space}(A)$$

e) Row reduce A^T to get

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 4/7 \\ 0 & 1 & 0 & -19/14 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Then a basis for the null space is

$$\left\{ \begin{bmatrix} -8 \\ 19 \\ 2 \\ 14 \end{bmatrix} \right\}$$

Note: Since we know that $\text{nullity}(A^T) = 1$, any nonzero vector in the null space of A^T serves as a basis.

(3) Consider the matrix $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ whose columns are the given vectors.

Row reduce B to get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

is a basis for the span.

(3)

④ a) The columns of A give 5 vectors in 3-dimensional space \mathbb{R}^3 . Therefore they are dependent.

b) The rank of A is at most 3. Since $\text{rank}(A) + \text{nullity}(A) = 5$, it follows that $\text{nullity}(A)$ is 2, 3, 4, or 5.

⑤ a) Let $\{b_1, b_2, \dots, b_n\}$ be the columns of B . Then $\{Ab_1, Ab_2, \dots, Ab_n\}$ are the columns of AB .

$$\begin{aligned} \text{Thus } \text{rank}(B) &= \text{dimension of column space of } B \\ &= \dim \text{span}\{b_1, \dots, b_n\} \end{aligned}$$

$$\text{and } \text{rank}(AB) = \dim \text{span}\{Ab_1, \dots, Ab_n\}$$

But any dependence relationship amongst the $\{b_i\}$ gives a dependence relationship amongst the $\{Ab_i\}$, since

$$\begin{aligned} \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n = 0 &\Rightarrow A(\lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n) = 0 \\ &\Rightarrow \lambda_1 (Ab_1) + \lambda_2 (Ab_2) + \dots + \lambda_n (Ab_n) = 0 \end{aligned}$$

$$\text{It follows that } \dim \text{span}\{b_1, b_2, \dots, b_n\} \geq \dim \text{span}\{Ab_1, \dots, Ab_n\}.$$

$$\text{That is, } \text{rank}(B) \geq \text{rank}(AB).$$

A similar argument shows $\text{rank}(A) \geq \text{rank}(AB)$.

[Simply let $\{a_1, a_2, \dots, a_n\}$ be the rows of A , so that $\{a_1 B, a_2 B, \dots, a_n B\}$ are the rows of AB]

(4)

b) Part (a) gives the general rule $\text{rank}(ST) \leq \text{rank}(T)$ for any non matrices S and T .

Setting $S=A$ and $T=B$ gives $\text{rank}(AB) \leq \text{rank}(B)$

If A is invertible, then setting $S=A^{-1}$ and $T=AB$ gives

$$\text{rank}(A^{-1}(AB)) \leq \text{rank}(AB) \Rightarrow \text{rank}(B) \leq \text{rank}(AB)$$

Thus $\text{rank}(AB) = \text{rank}(B)$.

A similar argument shows $\text{rank}(BA) = \text{rank}(B)$.

(6) Clearly A has rank 1 if and only if it has at least one nonzero row, say $a = (\alpha_1, \alpha_2, \dots, \alpha_n)$, and each of its rows is a scalar multiple of a — that is, the rows of A are $\beta_1 a, \beta_2 a, \dots, \beta_m a$ for some scalars $\beta_1, \beta_2, \dots, \beta_m$.

But this is the same as saying $A = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} [\alpha_1, \alpha_2, \dots, \alpha_n]$

Letting $u = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$ and $v = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ to get $A = uv^T$ with $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

NOTE: There is a slight error in the question as posed - we must be sure that we have nonzero vectors u and v with $A = uv^T$, as otherwise $A = 0$ and $\text{rank}(A) \neq 0$ is other than 1.