

(1)

MATH 2321: RECITATION #2  
SOLUTIONS

$$\begin{aligned} \textcircled{1} \text{ (a) We have } \det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & 4 \\ -1 & 5-\lambda \end{bmatrix} \\ &= -\lambda(5-\lambda) + 4 \\ &= \lambda^2 - 5\lambda + 4 \\ &= (\lambda-4)(\lambda-1) \end{aligned}$$

So  $\det(A - \lambda I) = 0 \Leftrightarrow \lambda = 4 \text{ or } \lambda = 1$

These are the eigenvalues of A.

$$\begin{aligned} \text{(b) First, } E_4 &= \text{nullspace}(A - 4I) \\ &= \text{nullspace} \begin{bmatrix} -4 & 4 \\ -1 & 1 \end{bmatrix} \\ &= \text{nullspace} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Thus  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is an eigenvector with eigenvalue 4, and  $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$  is a basis of  $E_4$ .

$$\begin{aligned} \text{Now } E_1 &= \text{nullspace}(A - I) \\ &= \text{nullspace} \begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix} \\ &= \text{nullspace} \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \\ &= \text{span} \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

So  $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$  is an eigenvector with eigenvalue 1, and  $\left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$  is a basis of  $E_1$ .

(2)

$$\textcircled{D} \quad \text{we have } \det(B - \lambda I) = \det \begin{bmatrix} 3-\lambda & 5 \\ -1 & -1-\lambda \end{bmatrix}$$

$$= (3-\lambda)(-1-\lambda) + 5$$

$$= \lambda^2 - 2\lambda + 2$$

(a) Over  $\mathbb{R}$ ,  $\lambda^2 - 2\lambda + 2 = 0$  has no solutions, since its discriminant is negative.

So  $B$  has no eigenvalues over  $\mathbb{R}$ .

(b) Over  $\mathbb{C}$ , we have  $\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{2^2 - 8}}{2} = 1 \pm i$   
by the quadratic formula.

So the eigenvalues of  $B$  over  $\mathbb{C}$  are  $\lambda = 1+i$  and  $\lambda = 1-i$

$$\text{Let } E_{1+i} = \text{null space}(B - (1+i)I)$$

$$= \text{null space} \begin{bmatrix} 2-i & 5 \\ -1 & -2-i \end{bmatrix}$$

$$= \text{null space} \begin{bmatrix} 2-i & 5 \\ 0 & 0 \end{bmatrix}$$

$$= \text{span} \left\{ \begin{bmatrix} -5 \\ 2-i \end{bmatrix} \right\}$$

$$\text{Similarly, } E_{1-i} = \text{null space} \begin{bmatrix} 2+i & 5 \\ -1 & -2+i \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -5 \\ 2+i \end{bmatrix} \right\}$$

(c) Over  $\mathbb{Z}_5$ , we get  $\lambda^2 - 2\lambda + 2 = 0 \Leftrightarrow \lambda = 3$  or  $\lambda = 4$   
(by inspection). So the eigenvalues here are 3 and 4

$$\text{Then } E_3 = \text{null space} \begin{bmatrix} 0 & 5 \\ -1 & -4 \end{bmatrix} = \text{null space} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

(3)

$$\text{and } E_4 = \text{null space} \begin{bmatrix} 1 & 5 \\ -1 & -5 \end{bmatrix} = \text{null space} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

③ a) Note that  $J \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Thus  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is an eigenvector with eigenvalue 5.

Also observe that  $J$  has rank 1, so  $Jx=0$  has nontrivial solutions. Thus 0 is an e-value of  $J$ .

So 0 and 5 are e-values of  $J$ .

$$\begin{aligned} b) \text{ Let } E_0 &= \text{null space}(J - 0 \cdot I) \\ &= \text{null space}(\mathbb{I}) \\ &= \text{null space} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{by row reduction}) \end{aligned}$$

So a basis for  $E_0$  is  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ .

$$\text{Also } E_5 = \text{null space}(J - 5I)$$

$$= \text{null space} \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$$

$$= \text{null space} \begin{bmatrix} -5 & 0 & 0 & 0 & 5 \\ 0 & -5 & 0 & 0 & 5 \\ 0 & 0 & -5 & 0 & 5 \\ 0 & 0 & 0 & -5 & 5 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix} \quad (\text{Replace } R_i \text{ with } R_i + R_5 \text{ for } i=1,2,3,4)$$

(4)

$$= \text{nullspace} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix} \quad (\text{Replace } R_i \text{ with } \frac{1}{5}R_i \text{ for } i=1,2,3,4)$$

$$= \text{nullspace} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{Replace } R_5 \text{ with } R_5 - R_1 - R_2 - R_3 - R_4)$$

$$= \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

So a basis for  $E_5$  is  $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$