

Name: SOLUTIONS

ID: _____

Math 2321: Linear Algebra II

Midterm Test

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Instructions:

- Calculators are permitted provided they don't dim the lights when you turn them on.
- There are 5 pages plus this cover page. Check that your test paper is complete.
- There are a total of 80 marks. The value of each question is indicated in the margin.
- Show all your work. Insufficient justification will result in a loss of marks.

Page	Maximum	Your Score
1	12	
2	21	
3	17	
4	15	
5	15	
Total	80	

[12] 1. Give precise definitions for the following terms and notation. Throughout, A is an $n \times n$ matrix.

(a) The *coordinates* of $\vec{x} \in \mathbb{R}^n$ with respect to the basis $\{v_1, \dots, v_n\}$.

The vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ such that $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$.

(b) An *eigenvector* of A .

A non-zero vector \vec{x} such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ .

(c) The *characteristic polynomial* of A .

$$p(\lambda) = \det(A - \lambda I)$$

(d) The *eigenspace* corresponding to eigenvalue λ of A

$$E_\lambda := \text{null}(A - \lambda I)$$

(e) The *geometric multiplicity* of the eigenvalue λ of A .

$$\dim E_\lambda, \text{ or } \text{nullity}(A - \lambda I)$$

(f) An *orthogonal basis* of \mathbb{R}^n .

A basis $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ of \mathbb{R}^n such that $\vec{u}_i \cdot \vec{u}_j = 0$ for all $i \neq j$.

- [12] 2. Answer **true** or **false** or **admit ignorance** to the following by circling your choice. You will receive 2 points for a correct response, 1 point for no response, and 0 for an incorrect response. *Throughout, A is an $n \times n$ matrix.*

TRUE FALSE UNSURE If A row reduces to R , then $\det A = \det R$.

TRUE FALSE UNSURE A is invertible if and only if 0 is an eigenvalue of A .

TRUE FALSE UNSURE Similar matrices have the same eigenvalues.

TRUE FALSE UNSURE If A has n distinct eigenvalues then A is diagonalizable.

TRUE FALSE UNSURE If λ is an eigenvalue of A then $-\lambda$ is an eigenvalue of A^{-1} .

TRUE FALSE UNSURE An orthogonal set of vectors is always linearly independent.

- [9] 3. Short answer. **No justification is required**, but partial credit may be awarded if you show some work.

(a) Suppose A and B are 4×4 matrices with $\det A = 2$ and $\det B = 3$. Determine $\det(A^2 B^{-1})$.

$$\det(A^2 B^{-1}) = (\det A)^2 \cdot \frac{1}{\det B} = \boxed{\frac{4}{3}}$$

(b) Suppose the eigenvalues of a matrix A are $-1, 1,$ and 2 . Find the eigenvalues of $3A^2 + 2I$.

$$\begin{aligned} \text{They will be: } & 3(-1)^2 + 2 = 5 \\ & 3(1)^2 + 2 = 5 \\ & 3(2)^2 + 2 = 14 \end{aligned}$$

$$\boxed{5, 14}$$

(c) Give an example of a 2×2 matrix that is invertible but not diagonalizable.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(Note: The char poly is $(\lambda - 1)^2$, so $\lambda = 1$ is an e-value with dg. mult. 2 and geom. mult. nullity $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1$. Thus the matrix is not diagonalizable. But it is invertible, since its determinant is 1)

[6]

$$4. \text{ Evaluate } \det \begin{bmatrix} 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 & 1 \\ -1 & 1 & 3 & 1 & -1 \\ 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & -1 & -2 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & -3 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 2 & 5 & 2 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -4 & -3 & -4 \end{bmatrix} \begin{array}{l} R_1 - 2R_4 \\ R_2 - R_4 \\ R_3 + R_4 \\ R_5 - 2R_4 \end{array}$$

$$= \det \begin{bmatrix} 0 & 0 & -2 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & -5 & -2 & -4 \end{bmatrix} \begin{array}{l} R_1 - R_2 \\ R_3 - 2R_2 \\ R_4 - R_2 \\ R_5 + R_2 \end{array}$$

$$= -\det \begin{bmatrix} 0 & -2 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & -5 & -2 & -4 \end{bmatrix} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \text{expand along 1st col.}$$

$$= \det \begin{bmatrix} -2 & -1 & 1 \\ 7 & 0 & 0 \\ -5 & -2 & -4 \end{bmatrix} \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{expand along 1st col}$$

$$= -7 \cdot \det \begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix} \left. \begin{array}{l} \\ \end{array} \right\} \text{expand along 2nd row}$$

$$= -7 \cdot 6$$

$$= \boxed{-42}$$

[11]

5. Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k .

$$\det(A - \lambda I) = \det \begin{pmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{pmatrix} = (7-\lambda)(1-\lambda) + 8 = \lambda^2 - 8\lambda + 15 = (\lambda-3)(\lambda-5)$$

So the e-values are $\lambda=3$ and $\lambda=5$.

$$E_3 = \text{null}(A - 3I) = \text{null} \begin{pmatrix} 4 & 2 \\ -4 & -2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

$$E_5 = \text{null}(A - 5I) = \text{null} \begin{pmatrix} 2 & 2 \\ -4 & -4 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$\text{Let } P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \text{ so that } P^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \text{ and } A = P \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} P^{-1}$$

$$\text{Then } A^k = P \begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3^k & -3^k \\ 2 \cdot 5^k & 5^k \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2(3^k - 5^k) & 2 \cdot 3^k + 5^k \end{bmatrix}$$

- [15] 6. For each of the following matrices A , determine whether A is diagonalizable. If so, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

$$(a) A = \begin{bmatrix} 0 & -2 & 2 \\ 2 & -4 & 3 \\ 2 & -3 & 2 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & -2 & 2 \\ 2 & -4-\lambda & 3 \\ 2 & -3 & 2-\lambda \end{vmatrix} = -\lambda \left[-(4+\lambda)(2-\lambda) + 9 \right] \\ &\quad - 2 \left[-2(2-\lambda) + 6 \right] \\ &\quad + 2 \left[-6 + 2(4+\lambda) \right] \\ &= -\lambda(\lambda^2 + 2\lambda + 1) - 2(2\lambda + 2) + 2(2\lambda + 2) \\ &= -\lambda(\lambda + 1)^2 \end{aligned}$$

So the e -values are $\lambda = 0$ and $\lambda = -1$, with alg. mult. 1 and 2, resp.

$$E_{-1} = \text{null}(A + I) = \text{null} \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 3 \\ 2 & -3 & 3 \end{pmatrix} = \text{null} \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $\dim E_{-1} = 1$, which is less than the alg. mult. of $\lambda = -1$.

Hence A is not diagonalizable.

$$(b) A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 2-\lambda & 0 & -2 \\ 1 & 3-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda) \det \begin{pmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix} \\ &= (3-\lambda)^2 (2-\lambda) \end{aligned}$$

So the e -values are $\lambda = 2$ and $\lambda = 3$, with alg. mult. 1 and 2, resp.

$$E_3 = \text{null}(A - 3I) = \text{null} \begin{pmatrix} -1 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \text{null} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$E_2 = \text{null}(A - 2I) = \text{null} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \text{null} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

So the alg. mult. equal the geom. mult., and A is diagonalizable.

$$\text{Let } P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then $D = P^{-1}AP$.