Math 2321: Recitation #6 Solutions

1. Read about the QR factorization in Section 5.3 of your text.

   Solution: Done!

2. Find the QR factorization of $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

   Solution: We can assume from the statement of the problem that the QR-factorization of A exists. However, we should generally verify this before we start by checking that the columns of A are linearly independent. This is readily done by performing a couple row or column operations.

   We now run he Gram-Schmidt orthogonalization process on the columns of A. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Set $\mathbf{u}_1 = \mathbf{v}_1$. Then compute

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 1 \end{bmatrix} - \frac{12}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

   and

$$\begin{aligned}
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2/3 \\ 1/3 \\ 0 \\ 2/3 \end{bmatrix}.
\end{aligned}$$

   Normalize $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ to get vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, and create a matrix $Q = [\mathbf{q}_1\, \mathbf{q}_2\, \mathbf{q}_3]$ with these columns:

$$Q = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & -2/3 \\ 2/\sqrt{6} & 0 & 1/3 \\ -1/\sqrt{6} & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 2/3 \end{bmatrix}$$

   Now find R by computing

$$R = Q^{-1}A = Q^TA = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ -2/3 & 1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{6} & 2\sqrt{6} & 2/\sqrt{6} \\ 0 & \sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Let A be a square matrix with QR factorization $A = QR$. Prove that A is similar to RQ.

Solution: We must show that there exists an invertible matrix P such that $RQ = P^{-1}AP$. Let $P = Q$, noting that P is invertible because Q is orthogonal (and hence invertible). Then we have

$$P^{-1}AP = Q^{-1}(QR)Q = (Q^{-1}Q)RQ = RQ,$$

as desired.