Math 2321: Recitation #7 Solutions

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These problems come from Section 6.0 of your text.

1. We create a Fibonacci-type sequence $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ by choosing any values for x_0 and x_1, and then using the rule $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$. For instance:

$$\begin{aligned}
\mathbf{x} &= (1, 1, 2, 3, 5, 8, 13, \ldots) \\
\mathbf{y} &= (2, 0, 2, 2, 4, 6, 10, \ldots) \\
\mathbf{z} &= (1, -1, 0, -1, -1, -2, -3, \ldots).
\end{aligned}$$

2. (a) From our examples in Problem 1, note that

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (3, 1, 4, 5, 9, 14, 23, \ldots) \\
\mathbf{y} + \mathbf{z} &= (3, -1, 2, 1, 3, 4, 7, \ldots).
\end{aligned}$$

   Both of these are Fibonacci-type sequences.

   (b) Again from Problem 1 we have

$$\begin{aligned}
2\mathbf{x} &= (2, 2, 4, 6, 10, 16, 26, \ldots) \\
-3\mathbf{y} &= (-6, 0, -6, -6, -12, -18, -30, \ldots).
\end{aligned}$$

   Both of these are Fibonacci-type sequences.

3. (a) Let $\mathbf{x} = (x_0, x_1, \ldots)$ and $\mathbf{y} = (y_0, y_1, \ldots)$ be two Fibonnaci-type sequences, and let $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Then $\mathbf{z} = (z_0, z_1 \ldots)$, where $z_n = x_n + y_n$. We wish to show $z_n = z_{n-1} + z_{n-2}$ for $n \geq 2$. To do so, we simply calculate:

$$\begin{aligned}
z_n &= x_n + y_n \\
&= (x_{n-1} + x_{n-2}) + (y_{n-1} + y_{n-2}) && \text{[since } \mathbf{x} \text{ and } \mathbf{y} \text{ are Fibonacci-like]} \\
&= (x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2}) && \text{[rearrange]} \\
&= z_{n-1} + z_{n-2}.
\end{aligned}$$

   (b) Let $\mathbf{x} = (x_0, x_1, \ldots)$ be a Fibonnaci-type sequences, and let $\mathbf{z} = c\mathbf{x}$ for some scalar c. Then $\mathbf{z} = (z_0, z_1 \ldots)$, where $z_n = cx_n$. We again wish to show $z_n = z_{n-1} + z_{n-2}$ for $n \geq 2$. To do so, we calculate:

$$\begin{aligned}
z_n &= cx_n \\
&= c(x_{n-1} + x_{n-2}) && \text{[since } \mathbf{x} \text{ is Fibonacci-like]} \\
&= (cx_{n-1}) + (cx_{n-2}) && \text{[rearrange]} \\
&= z_{n-1} + z_{n-2}.
\end{aligned}$$

4. Yes, *Fib* satisfies all of the "vector" properties of $\mathbb{R}^n$. The Fibonacci-type analogue of the zero vector is $\mathbf{0} = (0, 0, 0, \ldots)$. If $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ is in *Fib*, then $-\mathbf{x} = (-x_0, -x_1, -x_2, \ldots)$ is also in *Fib*.

5. The natural analogue of $\mathbf{e}_1$ in *Fib* should have 1 as its first coordinate and 0 as its second coordinate. But these two values completely determine the remaining coordinates. So we take $\mathbf{e} = (1, 0, 1, 1, 2, 3, \ldots)$.

6. We claim that $\mathbf{x} = x_0\mathbf{e} + x_1\mathbf{f}$. This is equivalent to the identity

$$(x_0, x_1, x_2, \ldots) = x_0(1, 0, 1, \ldots) + x_1(0, 1, 1, \ldots) = (x_0, x_1, x_0 + x_1, \ldots).$$

To prove that this is true, we need only observe that the first two coordinates match up on the left- and right-hand sides. Here we are using the fact that two Fibonacci-type sequences are equal if and only if they agree first two coordinates, which is true simply because the first two elements of a Fibonacci-type sequence *completely determine the sequence.*

7. Suppose $\mathbf{0} = c\mathbf{e} + d\mathbf{f}$, that is

$$(0, 0, \ldots) = c(1, 0, \ldots) + d(0, 1, \ldots) = (c, d, \ldots).$$

Clearly this implies $c = d = 0$.

8. It is sensible to say that the dimension of *Fib* is 2, because we have seen that the two "vectors" $\mathbf{e}$ and $\mathbf{f}$ span the space and are linearly independent. (Problems 6 and 7, respectively.)

9. Notice that $(1, r, r^2, \ldots) \in \mathit{Fib}$ if and only if $r^n = r^{n-1} + r^{n-2}$ for all $n \geq 2$. Divide this equation by r^{n-2} (allowed since clearly $r \neq 0$) to get the quadratic

$$r^2 = r + 1.$$

Solve this equation with the quadratic formula to find that

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

So there are only two geometric sequences contained in *Fib*, namely $(1, r, r^2, \ldots)$ for $r = \frac{1}{2}(1 + \sqrt{5})$ or $r = \frac{1}{2}(1 - \sqrt{5})$.

10. Let $r = \frac{1}{2}(1 + \sqrt{5})$ and $s = \frac{1}{2}(1 - \sqrt{5})$. Then $\mathbf{r} = (1, r, r^2, \ldots)$ and $\mathbf{s} = (1, s, s, \ldots)$ are elements of *Fib*, by Problem 10. Moreover, it is easy to check that they are linearly independent, since $a\mathbf{r} + b\mathbf{s} = \mathbf{0}$ is the same as

$$(0, 0, \ldots) = a(1, r, \ldots) + b(1, s, \ldots) = (a + b, ar + bs).$$

This equality requires $a + b = 0$ and $ar + bs = 0$, which forces $a = b = 0$.

11. Let $\mathbf{f} = (f_0, f_1, f_2, \ldots) = (0, 1, 1, 2, 3, \ldots)$ be the Fibonacci sequence. Let us express $\mathbf{f}$ in terms of the basis $\{\mathbf{r}, \mathbf{s}\}$ from Problem 10. To do so, we suppose $\mathbf{f} = a\mathbf{r} + b\mathbf{s}$. This gives

$$(0, 1, \ldots) = (a + b, ar + bs, \ldots),$$

so that $a + b = 0$ and $ar + bs = 1$. Solving this system for a and b yields $a = 1/(r - s)$ and $b = -1/(r - s)$. But $r = \frac{1}{2}(1 + \sqrt{5})$ and $s = \frac{1}{2}(1 - \sqrt{5})$, so we have $r - s = \sqrt{5}$. Therefore

$$\begin{aligned}
\mathbf{f} &= a\mathbf{r} + b\mathbf{s} \\
&= \frac{1}{\sqrt{5}}\mathbf{r} - \frac{1}{\sqrt{5}}\mathbf{s} \\
&= \frac{1}{\sqrt{5}}\big((1, r, r^2, r^3, \ldots) - (1, s, s^2, s^3, \ldots)\big) \\
&= \frac{1}{\sqrt{5}}(0, r - s, r^2 - s^2, r^3 - s^3, \ldots).
\end{aligned}$$

Comparing coordinates on both sides of this equation gives $f_n = \frac{1}{\sqrt{5}}(r^n - s^n)$, as desired.