Math 2321: Recitation #9 Solutions

1. (a) Extend the set $\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}$ to a basis for $\mathcal{M}_{2\times 2}$.

Solution: The given matrices have coordinates $[1, 0, -1, 0]^T$, $[1, 1, 1, 1]^T$, and $[1, 1, 0, 1]^T$ with respect to the standard basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $\mathcal{M}_{2\times 2}$. Put these coordinates in a matrix and row reduce to get

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Clearly $(0, 0, 0, 1)$ is not in span of these rows. It follows that

$$\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

is linearly independent. Since $\mathcal{M}_{2\times 2}$ has dimension 4, this set must be a basis.

(b) Reduce the set $\{1 + x + x^3,\ 2 - x + 2x^3,\ x + x^3,\ x^2 - x,\ 1 - x + x^2,\ x^3\}$ to a basis for $\mathcal{P}_3$.

Solution: Row reduce the matrix whose columns are the coordinate vectors of the given polynomials with respect to the standard basis $\{1, x, x^2, x^3\}$ of $\mathcal{P}_3$. This gives

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -3 & 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The first 4 columns contain pivots, so the first four polynomials from the given set form a linearly independent set. That is,

$$\{1 + x + x^3,\ 2 - x + 2x^3,\ x + x^3,\ x^2 - x\}$$

is linearly independent in $\mathcal{P}_3$. Since $\mathcal{P}_4$ has dimension 4, this must be a basis.

(c) Find a basis for span $\{1, \sin^2 x, \cos^2 x, \sin 2x, \cos 2x\}$ in $\mathcal{F}$.

Solution: Since $\cos^2 x = 1 - \sin^2 x$ and $\cos 2x = 1 - 2\sin^2 x$, we have

$$\operatorname{span}\{1, \sin^2 x, \cos^2 x, \sin 2x, \cos 2x\} = \operatorname{span}\{1, \sin^2 x, \sin 2x\}.$$

We claim that $\{1, \sin^2 x, \sin 2x\}$ is linearly independent in $\mathcal{F}$. To see this, suppose that

$$a + b\sin^2 x + c\sin 2x = 0.$$

This identity must hold for all x. In particular, setting $x = 0$ gives $a = 0$. Then setting $x = \frac{\pi}{2}$ in $b\sin^2 x + c\sin 2x = 0$ gives $b = 0$. Finally, setting $x = \frac{\pi}{4}$ in $c\sin 2x = 0$ gives $c = 0$. Therefore $\{1, \sin^2 x, \sin 2x\}$ is indeed linearly independent. It is therefore a basis for $\operatorname{span}\{1, \sin^2 x, \cos^2 x, \sin 2x, \cos 2x\}$.

2. Find a basis for the subspace $\{p(x) : xp'(x) = p(x)\}$ of $\mathcal{P}_3$.

Solution: Let $\mathcal{S} = \{p(x) \in \mathcal{P}_3 : xp'(x) = p(x)\}$. Then

$$\begin{aligned} p(x) = a + bx + cx^2 + dx^3 \in \mathcal{S} &\iff x(b + 2cx + 3dx^2) = a + bx + cx^2 + dx^3 \\ &\iff bx + 2cx^2 + 3dx^3 = a + bx + cx^2 + dx^3 \\ &\iff \{a = 0, b = b, c = 2c, d = 3d\} \\ &\iff a = c = d = 0. \end{aligned}$$

That is, $p(x) \in \mathcal{S}$ if and only if $p(x) = bx$ for some $b \in \mathbb{R}$. Therefore $\mathcal{S} = \text{span}\{x\}$, so that $\{x\}$ is a basis for $\mathcal{S}$.

3. Find a formula for the dimension of the vector space of symmetric $n \times n$ matrices over $\mathbb{R}$.

Solution: The idea is that not all the entries in a symmetric matrix are independent of each other, since the entries in the lower half of the matrix must match those in the upper half. For example, every 3×3 matrix is of the form

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = aE_{11} + dE_{22} + fE_{33} + b(E_{12} + E_{21}) + c(E_{13} + E_{31}) + e(E_{23} + E_{32}).$$

So the space of 3×3 symmetric matrices has dimension 6, with basis

$$\{E_{11}, E_{22}, E_{33}, E_{12} + E_{21}, E_{13} + E_{31}, E_{23} + E_{32}\}.$$

Notice that $6 = 3 + 2 + 1$ is the number of matrix entries on or above the diagonal.

In general, the space of $n \times n$ symmetric matrices will have dimension $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$, with basis

$$\{E_{ii} : 1 \leq i \leq n\} \cup \{E_{ij} + E_{ji} : 1 \leq i < j \leq n\}.$$

4. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of the vector space V. Prove that

$$\{\mathbf{v}_1,\ \mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \ldots, \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n\}$$

is a also a basis of V.

Solution: Let $\mathbf{u}_k = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k$, for $k = 1, 2, \ldots, n$. Then the coordinates of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are

$$[\mathbf{u}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad [\mathbf{u}_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad [\mathbf{u}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad [\mathbf{u}_n]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

These coordinates are clearly linearly independent in $\mathbb{R}^n$, and therefore form a basis for $\mathbb{R}^n$. It follows that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for V.