Math 1210: Fun With Derivatives

Winter 2012

At this point you know the derivatives of a handful of "basic" functions. Your calculus life will be much easier if you have this list memorized for instant recall:

$$\frac{d}{dx}x^a = ax^{a-1} \qquad \frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

You also know how to differentiate functions that are "pasted together" through addition, subtraction, multiplication, division, and composition:

Linearity: $\frac{d}{dx}(Af(x) + Bg(x)) = Af'(x) + Bg'(x)$ for constants A, B

Product Rule: $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

All the x's in these rules may make them appear more complicated than they are. For instance, the product rule looks easier when written as $(fg)' = f'g + fg'$. In fact, this rule can be applied iteratively to a product of more than two functions, and the overall rule to remember is simply that each factor "gets its turn" at differentiation. For example, we have

$$(fghk)' = f'ghk + fg'hk + fgh'k + fghk'.$$

Likewise, the chain rule is best remembered informally as

$$\frac{d}{dx}f(\text{junk}) = f'(\text{junk}) \times [\text{derivative of junk}].$$

If the junk is itself a composition of functions, then we have to use the chain rule more than once. For example, to calculate the derivative of $f(x) = \cos^4(2e^x + 1)$, we first observe that

$$f(x) = (\text{junk})^4, \qquad \text{where} \quad \text{junk} = \cos(2e^x + 1).$$

The chain rule therefore gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\text{junk})^4 \\ &= 4(\text{junk})^3 \cdot \frac{d}{dx}(\text{junk}) \\ &= 4\cos^3(2e^x + 1)\,\frac{d}{dx}\cos(2e^x + 1). \end{aligned}$$

To compute $\frac{d}{dx}\cos(2e^x+1)$, we replace $2e^x+1$ with "crud" and apply the chain rule again:

$$\begin{aligned}\frac{d}{dx}\cos(2e^x+1) &= \frac{d}{dx}\cos(\text{crud})\\ &= -\sin(\text{crud})\cdot\frac{d}{dx}(\text{crud})\\ &= -\sin(2e^x+1)\,\frac{d}{dx}(2e^x+1)\\ &= -\sin(2e^x+1)\cdot 2e^x.\end{aligned}$$

So altogether we have

$$\begin{aligned}f'(x) &= 4\cos^3(2e^x+1)(-\sin(2e^x+1)\cdot 2e^x)\\ &= -8e^x\cos^3(2e^x+1)\sin(2e^x+1).\end{aligned}$$

To differentiate a complicated function, we typically require multiple applications of the chain rule, along with the other differentiation rules. For instance:

$$\begin{aligned}\frac{d}{dx}\sqrt{3^{2x+1}+5\sec x^2} &= \frac{d}{dx}\sqrt{\text{junk}} \qquad [\text{junk} = 3^{2x+1}+5\sec x^2]\\ &= \frac{1}{2\sqrt{\text{junk}}}\cdot\frac{d}{dx}(\text{junk})\\ &= \frac{1}{2\sqrt{3^{2x+1}+5\sec x^2}}\,\frac{d}{dx}(3^{2x+1}+5\sec x^2)\\ &= \frac{1}{2\sqrt{3^{2x+1}+5\sec x^2}}\left(\frac{d}{dx}3^{2x+1}+5\frac{d}{dx}\sec x^2\right) \qquad [\text{by linearity}]\\ &= \frac{1}{2\sqrt{3^{2x+1}+5\sec x^2}}\left(\frac{d}{dx}3^{\text{crud}}+5\frac{d}{dx}\sec(\text{blah})\right) \qquad [\text{crud} = 2x+1, \quad \text{blah} = x^2]\\ &= \frac{1}{2\sqrt{3^{2x+1}+5\sec x^2}}\left(3^{\text{crud}}\ln 3\cdot\frac{d}{dx}(\text{blah})+5\sec(\text{blah})\tan(\text{blah})\cdot\frac{d}{dx}(\text{blah})\right)\\ &= \frac{1}{2\sqrt{3^{2x+1}+5\sec x^2}}\left(3^{2x+1}\ln 3\cdot\frac{d}{dx}(2x+1)+5\sec x^2\tan x^2\cdot\frac{d}{dx}x^2\right)\\ &= \frac{1}{2\sqrt{3^{2x+1}+5\sec x^2}}\left(3^{2x+1}\ln 3\cdot 2+5\sec x^2\tan x^2\cdot 2x\right)\\ &= \frac{2\ln 3\cdot 3^{2x+1}+10x\sec x^2\tan x^2}{2\sqrt{3^{2x+1}+5\sec x^2}}\end{aligned}$$

As another example, take a moment to convince yourself that

$$y = \sqrt[3]{1+x^3\sin^2(\pi x)}$$

has the following messy derivative:

$$\frac{dy}{dx} = \frac{1}{3}\Big(1+x^3\sin^2(\pi x)\Big)^{-2/3}\big(3x^2\sin^2(\pi x)+x^3\cdot 2\sin(\pi x)\cdot\cos(\pi x)\cdot\pi\big).$$

Problems:

1. Differentiate the following. Simplify first if it seems helpful (and afterward if it seems fruitful).

   (a) $y = 7(x+3)\sqrt{x+1}$

   (b) $y = \dfrac{2\sin x}{1+3e^{5x}}$

   (c) $y = 5x^2 e^{x^2}\sqrt{x+1}$

   (d) $y = x^2\sqrt{x}(3+x^4)$

   (e) $y = \sec x \tan x$

   (f) $y = \dfrac{2x^3+1}{7x^2-3x+2}$

   (g) $y = \dfrac{x^2\sin x + x}{2x\sqrt{x}+1}$

   (h) $y = \dfrac{3}{2x^3+3\cos x}$

   (i) $y = 3\cos x\ \sin x$

   (j) $y = \dfrac{\csc x}{\cot x - 1}$

   (k) $y = \dfrac{1+x^2}{\sqrt[4]{x-1}}$

   (l) $y = (3x^3+2x+1)^{10}$

   (m) $y = 3^{2x}\sqrt{2^{3x}+7}$

   (n) $y = 5\sin^2 3x$

   (o) $y = \dfrac{3}{\sqrt[3]{x+\sin x}}$

   (p) $y = \dfrac{1}{(2-e^{\cos x})^3}$

   (q) $y = (2x+\cos^2 x)^{2/3}$

   (r) $y = \tan^3(2x)$

   (s) $y = \sin(x^2)(1+x^3)^4$

   (t) $y = \sqrt{\dfrac{1+x}{1-x}}$

   (u) $y = \sec^3\left(\sqrt[3]{1+x^2}\right)$

2. Find all points at which the tangent to the following curves is horizontal:

   (a) $y = x\sqrt{x+1}$

   (b) $y = \dfrac{x}{1+2x^3}$

   (c) $y = e^x(e^x-1)$.

   (d) $y = \sin^3(2x)$

Answers:

Warning! I have looked over these answers quickly, but I assume there are still typos lurking about. So don't be too alarmed if you see something that baffles you — it may just be my mistake! Please let me know if you find any errors.

1. (a) Product rule yields $y' = 7\sqrt{x+1} + \dfrac{7(x+3)}{2\sqrt{x+1}}$.

   (b) Quotient rule gives $y' = \dfrac{2\cos x(1+3e^{5x}) - (3e^{5x}\cdot 5)(2\sin x)}{(1+3e^{5x})^2} = \dfrac{2\cos x + 6e^{5x}(\cos x - 5\sin x)}{(1+3e^{5x})^2}$.
   The simplification step I have taken here is somewhat arbitrary. The important thing at this point is to get the *correct* derivative, written in any form.

   (c) Product rule gives $y' = 10xe^{x^2}\sqrt{x+1} + 5x^2e^{x^2}(2x)\cdot\sqrt{x+1} + 5x^2e^{x^2}\dfrac{1}{2\sqrt{x+1}}$.

   (d) The simplest approach is to write $y = 3x^{5/2} + x^{13/2}$ so that $y' = \frac{15}{2}x^{3/2} + \frac{13}{2}x^{11/2}$.
   The same result is obtained by applying the product rule to $y = x^{5/2}(3+x^4)$, or by applying the "3-fold" product rule to $y = x^2\cdot\sqrt{x}\cdot(3+x^4)$.

   (e) The product rule gives $y' = (\sec x\tan x)(\tan x) + \sec x(\sec^2 x) = \sec x(\tan^2 x + \sec^2 x)$.
   It smells like this should simplify, but about the best we can do is apply the identity $1+\tan^2 x = \sec^2 x$ to write $y' = \sec x\,(2\sec^2 x - 1)$.

   (f) This is a messy but straightforward application of quotient rule:
   $$y' = \frac{6x^2(7x^2-3x+2) - (14x-3)(2x^3+1)}{(7x^2-3x+2)^2} = \frac{14x^4 - 12x^3 + 12x^2 - 14x + 3}{(7x^2-3x+2)^2}.$$

   (g) Apply quotient rule, noting along the way that $\frac{d}{dx}x^2\sin x = 2x\sin x + x^2\cos x$ and $\frac{d}{dx}x\sqrt{x} = \frac{d}{dx}x^{3/2} = \frac{3}{2}\sqrt{x}$. Arrive at
   $$y' = \frac{(2x\sin x + x^2\cos x + 1)(2x\sqrt{x}+1) - 3\sqrt{x}(x^2\sin x + x)}{(2x\sqrt{x}+1)^2}.$$

   (h) Applying the quotient rule gives
   $$y' = \frac{0\cdot(2x^3+3\cos x) - 3(6x^2 - 3\sin x)}{(2x^3+3\cos x)^2} = \frac{-3(6x^2-3\sin x)}{(2x^3+3\cos x)^2}.$$
   This is correct, but it is a little silly to use the quotient rule when the numerator is a constant. Instead, recognize that the given function is simply $y = 3(2x^3+3\cos x)^{-1}$. Then $y' = -3(2x^3+3\cos x)^{-2}(6x^2 - 3\sin x)$ by chain rule.

   (i) Applying the product rule gives $y' = 3((-\sin x)\sin x + \cos x\,(\cos x))$, which can be simplified to $y' = 3\cos 2x$ through the identity $\cos^2 x - \sin^2 x = \cos 2x$.
   A cleaner approach is simplify *before* differentiating. Use $\sin 2x = 2\sin x\cos x$ to write $y = \frac{3}{2}\sin 2x$. Then the chain rule gives $y' = \frac{3}{2}(\cos 2x)(2)$.

   (j) The product and quotient rules give
   $$y' = \frac{-\csc x\,\cot x(\cot x - 1) - (-\csc^2 x)\csc x}{(\cot x - 1)^2}.$$

This expression is correct and can be simplified by common factoring $-cscx$ in the numerator and applying the identity $1 + \cot^2 x = \csc^2 x$.

However, it is always good to be on the lookout for basic simplifications *before* differentiating. In this case, upon expressing $\csc x$ and $\cot x$ in terms of $\sin x$ and $\cos x$, we find that the given expression for y simplifies to $y = (\cos x - \sin x)^{-1}$. So the chain rule (or quotient rule) immediately gives

$$y' = \frac{-(-\sin x - \cos x)}{(\cos x - \sin x)^2},$$

If you take the time to convert everything to $\sin x$ and $\cos x$, you will see that our first expression for y' does match this one.

With a little more trig trickery we notice that

$$\begin{aligned}(\cos x - \sin x)^2 &= (\cos^2 x + \sin^2 x) - 2\cos x \sin x \\ &= 1 - \sin 2x.\end{aligned}$$

Therefore

$$y' = \frac{\sin x + \cos x}{1 - 2\sin x}.$$

Again, the important thing right now is to get a *correct* derivative. However, we will later need to be able to simply expressions as above.

(k) Applying the quotient rule gives

$$y' = \frac{2x\sqrt[4]{x-1} - \frac{1}{4}(x-1)^{-3/4}(1+x^2)}{(\sqrt[4]{x-1})^2}.$$

Alternatively, write $y = (1+x^2)(x-1)^{-1/4}$ and use the product rule to get

$$y' = 2x(x-1)^{-1/4} + (1+x^2)(-\tfrac{1}{4}(x-1)^{-5/4}).$$

Take a moment to convince yourself that these expressions for y' are equivalent.

(l) Chain rule gives $y' = 10(3x^3 + 2x + 1)^9 \cdot (9x^2 + 2)$.

(m) Product and chain rules give $y' = 3^{2x} \ln 3 \cdot 2 \cdot \sqrt{2^{3x} + 7} + 3^{2x} \cdot \frac{1}{2}(2^{3x} + 7)^{-1/2} \cdot 2^{3x} \ln 2 \cdot 3$

(n) Chain rule gives $y' = 5 \cdot 2\sin(3x) \cdot \cos(3x) \cdot 3 = 30 \sin 3x \cos 3x = 15 \sin 6x$.

(o) Write $y = 3(x + \sin x)^{-1/3}$ and use chain rule to get

$$y' = 3 \cdot (-\tfrac{1}{3})(x + \sin x)^{-4/3} \cdot (1 + \cos x) = -\frac{1 + \cos x}{(x + \sin x)^{4/3}}.$$

(p) Write $y = (2 - e^{\cos x})^{-3}$. Chain rule gives

$$y' = -3(2 - e^{\cos x})^{-4} \cdot (-e^{\cos x})(-\sin x) = \frac{-3e^{\cos x} \sin x}{(2 - e^{\cos x})^4}.$$

(q) Chain rule gives $y' = \frac{2}{3}(2x + \cos^2 x)^{-1/3} \cdot (2 + 2\cos x(-\sin x)) = \dfrac{2(2 - \sin 2x)}{3\sqrt[3]{2x + \cos^2 x}}$.

(r) Chain rule gives $y' = 3\tan^2(2x) \cdot \sec^2(2x) \cdot 2 = 6(\sec 2x \tan 2x)^2$.

(s) Chain and product rules give $y' = \cos(x^2) \cdot 2x \cdot (1+x^3)^4 + \sin(x^2) \cdot 4(1+x^3)^3 \cdot 3x^2$. Not a lot can be done to simplify here, though we can common factor to write

$$y' = 2x(1+x^3)^3\Big((1+x^3)\cos x^2 + 6x \sin x^2\Big).$$

(t) Chain and quotient rules give

$$\begin{aligned} y' &= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2} \cdot \frac{1(1-x)-(-1)(1+x)}{(1-x)^2} \\ &= \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \cdot \frac{2}{(1-x)^2} \\ &= \frac{1}{(1-x)\sqrt{1-x^2}}, \end{aligned}$$

where the last line follows from $(1-x)(1+x) = 1-x^2$.

(u) A few applications of the chain rule are needed here:

$$\begin{aligned} y' &= 3\sec^2\left(\sqrt[3]{1+x^2}\right) \cdot \left(\sec\sqrt[3]{1+x^2}\tan\sqrt[3]{1+x^2}\right) \cdot \tfrac{1}{3}(1+x^2)^{-2/3} \cdot 2x \\ &= \frac{2x\sec^3\left(\sqrt[3]{1+x^2}\right)\tan\left(\sqrt[3]{1+x^2}\right)}{(1+x^2)^{2/3}}. \end{aligned}$$

2. (a) The derivative is

$$y' = \sqrt{x+1} + \frac{1}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}},$$

so we have $y' = 0$ if and only if $3x+2=0$. Hence the tangent line is horizontal only at $x = -\frac{2}{3}$.

(b) The derivative is

$$y' = \frac{1 \cdot (1+2x^3) - 6x^2 \cdot x}{(1+2x^3)^2} = \frac{1-4x^3}{(1+2x^3)^2}.$$

Therefore $y' = 0$ only when $1 - 4x^3 = 0$, which occurs when $x = \sqrt[3]{\frac{1}{4}}$. So the tangent is horizontal only at $x = 1/\sqrt[3]{4}$.

(c) The derivative works out to $y' = 2e^{2x} - e^x = e^x(2e^x - 1)$. This is zero when either $e^x = 0$ or $2e^x - 1 = 0$. Note that $e^x = 0$ is impossible. On the other hand, $2e^x - 1 = 0 \iff e^x = \frac{1}{2} \iff x = \ln\frac{1}{2}$. So the tangent is horizontal only at $x = \ln\frac{1}{2} = -\ln 2$.

(d) The derivative is

$$y' = 3\sin^2(2x) \cdot \cos(2x) \cdot 2 = 6\sin^2(2x)\cos(2x).$$

Thus $y' = 0$ if and only if $\sin 2x = 0$ or $\cos 2x = 0$. The first condition occurs when $2x$ is a multiple of π, and the second occurs when $2x$ is an odd multiple of $\frac{\pi}{2}$. Thus $y' = 0$ precisely when $x = k\pi/2$ for some integer k.