Math 1210: Exponential and Logarithmic Functions

Winter 2012

The following is a summary of our investigations into exponential and logarithmic functions. Read Sections 1.5 and 1.6 of your text for important review material.

- We tried to find the derivative of the general exponential function $f(x) = a^x$ by looking at the usual limit definition of the derivative.
- We discovered that $\frac{d}{dx}a^x$ was simply a^x multiplied by the value of the *fundamental limit*

$$\lim_{h \to 0} \frac{a^h - 1}{h}.$$

- We fantasized about a world in which this limit evaluates to 1, and then made our fantasies come true by *defining* the constant e to be the unique real number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

- This definition resulted in the wonderfully simple (and monumentally important) rule

$$\frac{d}{dx}e^x = e^x.$$

- We defined the *natural logarithm* to be the logarithm with respect to base e. We denote the natural logarithm of a positive number x by the symbol "$\ln x$" (pronounced "lawn x"). Thus

$$y = \ln x \quad \text{is equivalent to} \quad e^y = x.$$

  In words: The natural logarithm of x is the exponent to which e must be raised in order to obtain x. (So $e^{\ln 3} = 3$ and $\ln e^{-2} = -2$, by definition.)

- By rewriting a^x as $e^{x \ln a}$ and using the chain rule, we found the general differentiation rule

$$\frac{d}{dx}a^x = a^x \ln a.$$

- To find the derivative of $\ln x$, we began by writing the defining equation $e^{\ln x} = x$. We then took the derivative of both sides of this equation, making careful use of the chain rule on the left-hand side. This gave

$$e^{\ln x} \frac{d}{dx} \ln x = 1.$$

  But since $e^{\ln x} = x$, this equation readily yielded the important rule

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

- Repeating the process above with $\log_a x$ in place of $\ln x$, we instead obtained the general rule

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}.$$

Problems:

1. Differentiate the following. Simplify first if it seems helpful (and afterward if it seems fruitful).

   (a) $y = 2\ln x^4$

   (b) $y = \dfrac{\ln 2x}{e^{2x}}$

   (c) $y = 3^{4x}5^{3\sqrt{x}}$

   (d) $y = \ln \dfrac{x-1}{x+1}$

   (e) $y = \ln \sqrt{x^2+1}$

   (f) $y = \dfrac{3^x+1}{3^{-x}+1}$

   (g) $y = \log_{10}\left(x^3(2x^5+1)^4\right)$

   (h) $y = \ln e^{\sin x}\sqrt[3]{x+1}$

   (i) $y = \dfrac{\ln(e^x+1)}{x}$

   (j) $y = \log_2(\log_3(\log_4 x))$

2. Use logarithmic differentiation to compute y', where:

   (a) $y = 3x^3\sqrt[4]{1+\sin x}(2x+1)^{10}$

   (b) $y = \dfrac{2x^3(1+\sqrt{x})\tan^2 x}{(3x+1)^5}$

   (c) $y = \dfrac{2(\sin x+\cos x)^2}{x^5\sqrt{1+x^5}(1+x)^3}$

   (d) $y = (2x+1)^x$

   (e) $y = (\sin x)^{\sqrt{x}}$.

3. Find all points at which the tangent to the following curves is horizontal:

   (a) $y = x^3 - 2\ln(3x)$

   (b) $y = e^x \sin x$

   (c) $y = xe^{x^2}$

   (d) $y = x^2 \ln x$

   (e) $y = 4e^{x^2+3x+2}$

Answers:

1. (a) Simplify to get $y = 8\ln x$. Then $y' = \dfrac{8}{x}$.

   (b) Quotient rule gives

$$y' = \frac{\frac{1}{2x}\cdot 2\cdot e^{2x} - e^{2x}\cdot 2\cdot \ln 2x}{e^{4x}} = \frac{1-2x\ln x}{e^{2x}}.$$

   Alternatively, since $y = e^{-2x}\ln(2x)$, product rule gives

$$y' = e^{-2x}(-2)\ln 2x + e^{-2x}\cdot\frac{1}{2x}\cdot 2 = \frac{1}{xe^{2x}}\left(1-2x\ln 2x\right).$$

   Note: When using the chain rule to calculate the derivative of $\ln 2x$, we get $\frac{1}{2x}\cdot 2 = \frac{1}{x}$. The cancellation of these 2's can also be seen via log-rules. In particular, we have $\ln 2x = \ln 2 + \ln x$, and since $\ln 2$ is a constant it follows that $\frac{d}{dx}\ln 2x = 0 + \frac{1}{x} = \frac{1}{x}$.

   (c) $y' = (3^{4x}\ln 3\cdot 4)5^{3\sqrt{x}} + 3^{4x}\left(5^{3\sqrt{x}}\ln 5\cdot\dfrac{3}{2\sqrt{x}}\right)$

   (d) Simplify to get $y = \ln(x-1) - \ln(x+1)$, so that $y' = \dfrac{1}{x-1} - \dfrac{1}{x+1} = \dfrac{2}{x^2-1}$.

   (e) Simplify to $y = \frac{1}{2}\ln(x^2+1)$, so that $y' = \dfrac{1}{2}\cdot\dfrac{1}{x^2+1}\cdot 2x = \dfrac{x}{x^2+1}$.

   (f) There's a tricky simplification here that works wonders. Factor 3^x out of the numerator to rewrite y as

$$y = 3^x\frac{1+3^{-x}}{3^{-x}+1} = 3^x.$$

   Life is now very easy! We immediately get $y' = 3^x\ln 3$.

   If you didn't see this simplification and used quotient rule instead, you should find that your answer simplifies to $3^x\ln 3$. (It's good practice to work through the algebra and confirm this.)

   (g) Simplify to $y = 3\log_{10}x + 4\log_{10}(2x^5+1)$. Then

$$y' = 3\cdot\frac{1}{x\ln 10} + 4\cdot\frac{1}{(2x^5+1)\ln 10}\cdot 10x^4 = \frac{46x^5+3}{x(2x^5+1)\ln 10}.$$

   (h) Simplify to $y = \ln e^{\sin x} + \ln\sqrt[3]{x+1}$. But $\ln(e^{\text{junk}}) = \text{junk}$, so this further simplifies to

$$y = \sin x + \frac{1}{3}\ln(x+1),$$

   from which we get $y' = \cos x + \dfrac{1}{3(x+1)}$.

   (i) Quotient rule gives

$$y' = \frac{\frac{1}{e^x+1}\cdot e^x\cdot x - 1\cdot\ln(e^x+1)}{x^2} = \frac{xe^x - (e^x+1)\ln(e^x+1)}{x^2(e^x+1)}.$$

   All we could do here in terms of simplification was to eliminate the fraction-within-fraction situation. In particular, notice that $\ln(e^x+1)$ cannot be simplified!

(j) Chain rule gives

$$y' = \frac{1}{\log_3(\log_4 x)\ln 2} \cdot \frac{1}{(\log_4 x)\ln 3} \cdot \frac{1}{x \ln 4}.$$

We can't really do anything to simplify a product of logarithms, so we'll just leave the answer in this nasty form.

2. (a) First simplify matters by calculating the logarithm of y. Doing so gives:

$$\begin{aligned}\ln y &= \ln 3 + \ln x^3 + \ln \sqrt[4]{1+\sin x} + \ln(2x+1)^{10} \\ &= \ln 3 + 3\ln x + \frac{1}{4}\ln(1+\sin x) + 10\ln(2x+1).\end{aligned}$$

Now differentiate to get

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= 0 + \frac{3}{x} + \frac{1}{4(1+\sin x)} \cdot \cos x + \frac{10}{2x+1} \cdot 2 \\ &= \frac{3}{x} + \frac{\cos x}{4(1+\sin x)} + \frac{20}{2x+1}.\end{aligned}$$

Now "solve for $\frac{dy}{dx}$" by moving y to the right-hand-side:

$$\begin{aligned}\frac{dy}{dx} &= y\Big(\frac{3}{x} + \frac{\cos x}{4(1+\sin x)} + \frac{20}{2x+1}\Big) \\ &= 3x^3 \sqrt[4]{1+\sin x}(2x+1)^{10}\Big(\frac{3}{x} + \frac{\cos x}{4(1+\sin x)} + \frac{20}{2x+1}\Big).\end{aligned}$$

(b) First calculate $\ln y$ to get:

$$\ln y = \ln 2 + 3\ln x + \ln(1+\sqrt{x}) + 2\ln(\tan x) - 5\ln(3x+1).$$

Differentiate to get

$$\begin{aligned}\frac{1}{y} \cdot y' &= \frac{3}{x} + \frac{1}{1+\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} + 2\frac{1}{\tan x}\sec^2 x - \frac{5}{3x+1} \cdot 3 \\ &= \frac{3}{x} + \frac{1}{2(\sqrt{x}+x)} + 2\cot x \sec^2 x - \frac{15}{3x+1}.\end{aligned}$$

Finally, solve for y' to arrive at

$$y = \frac{2x^3(1+\sqrt{x})\tan^2 x}{(3x+1)^5}\Big(\frac{3}{x} + \frac{1}{2(\sqrt{x}+x)} + 2\cot x \sec^2 x - \frac{15}{3x+1}\Big).$$

Note: There is generally a handful of reasonable ways to express a product of trigonometric functions. For instance, you should check that the expression $\cot x \sec^2 x$ appearing above could also have been written in any of the following forms:

$$\csc x \sec x \qquad \text{or} \qquad \frac{1}{\sin x \cos x} \qquad \text{or} \qquad \frac{2}{\sin 2x}.$$

All of these are simpler than the $\cot x \sec^2 x$, but none of them will cause the final answer here to look pleasant so I just left things in a raw form. If I were trying to do something with this derivative, such as set it to 0, then I would think much harder about simplification. (But in this case I would have no hope at a simple solution!)

(c) Same as always. Start by calculating $\ln y$:

$$\ln y = \ln 2 + 2\ln(\sin x + \cos x) - 5\ln x - \frac{1}{2}\ln(1+x^5) - 3\ln(1+x).$$

Then

$$\frac{y'}{y} = \frac{2}{\sin x + \cos x}(\cos x - \sin x) - \frac{5}{x} - \frac{5x^4}{2(1+x^5)} - \frac{3}{1+x}.$$

So we have

$$y' = \frac{2(\sin x + \cos x)^2}{x^5\sqrt{1+x^5}(1+x)^3}\Big(\frac{2(\cos x - \sin x)}{\sin x + \cos x} - \frac{5}{x} - \frac{5x^4}{2(1+x^5)} - \frac{3}{1+x}\Big).$$

Note: This is a fine answer, but it's good practice to simplify the trig fraction. For instance, notice that

$$\begin{aligned}\frac{\cos x - \sin x}{\sin x + \cos x} &= \frac{(\cos x - \sin x)(\sin x + \cos x)}{(\sin x + \cos x)^2}\\ &= \frac{\cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x + 2\sin x\,\cos x}\\ &= \frac{\cos 2x}{1+\sin 2x}.\end{aligned}$$

That's lovely, and I feel very proud of myself... until I notice that I should have simplified *before* differentiating. In this case, the factor $(\sin x + \cos x)^2$ found in the numerator of y could have been replaced by $1 + \sin 2x$ right from the start. (Why?)

(d) This is one of the few instances where logarithmic differentiation isn't just a convenience, it's essentially a necessity.

Take the logarithm as usual to get $\ln y = x\ln(2x+1)$. Then differentiate, using product rule, to get

$$\frac{1}{y}\cdot y' = \ln(2x+1) + x\cdot\frac{1}{2x+1}\cdot 2.$$

It follows that

$$y' = (2x+1)^x\left(\ln(2x+1) + \frac{2x}{2x+1}\right).$$

Note: It is a very (very!) common mistake to claim the derivative of $(2x+1)^x$ is either $x(2x+1)^{x-1}$ or $(2x+1)^x\ln(2x+1)$. But both of these are attempts to use a known differentiation rule in a case where it does not apply. Make sure you can explain why these answers are not valid.

(e) Take the logarithm to get $\ln y = \sqrt{x}\ln(\sin x)$. Then

$$\frac{1}{y}\cdot y' = \frac{1}{2\sqrt{x}}\ln(\sin x) + \sqrt{x}\frac{1}{\sin x}\cos x = \frac{1}{2\sqrt{x}}\Big(\ln(\sin x) + 2x\,\cot x\Big),$$

so that

$$y' = \frac{(\sin x)^{\sqrt{x}}}{2\sqrt{x}}\Big(\ln(\sin x) + 2\cot x\Big),$$

3. (a) Differentiate to get $y' = 3x^2 - \frac{2}{x}$. Then set $y' = 0$ as follows:

$$0 = 3x^2 - \frac{2}{x} = \frac{3x^3 - 2}{x}.$$

Hence

$$y' = 0 \iff 3x^3 - 2 = 0 \iff x = \sqrt[3]{\frac{2}{3}}.$$

So the tangent is horizontal only at $x = \sqrt[3]{2/3}$.

(b) Differentiate to get $y' = e^x \sin x + e^x \cos x = e^x(\cos x + \sin x)$. Hence $y' = 0$ if and only if $e^x = 0$ or $\cos x + \sin x = 0$. But $e^x = 0$ is impossible (exponentials never equal 0), so we have $\cos x + \sin x = 0$. Rearrange to get $\cos x = -\sin x$, and divide by $\cos x$ to arrive at $\tan x = -1$. This occurs when $x = \frac{3\pi}{4} + k\pi$ for any integer k. It is at these values of x that the tangent is horizontal.

(c) Differentiate to get $y' = e^{x^2} + xe^{x^2} \cdot 2x = e^{x^2}(1 + 2x^2)$. Since neither e^{x^2} nor $1 + 2x^2$ can equal 0 (why?), we conclude that $y' = 0$ is impossible. Therefore the tangent line to this curve is never horizontal.

(d) First compute $y' = 2x \ln x + x^2 \cdot \frac{1}{x} = x(2 \ln x + 1)$. Thus $y' = 0$ when $x = 0$ or $2 \ln x + 1 = 0$. But $x = 0$ is not in the domain of the original function (since $\ln 0$ is not defined), so we exclude it from consideration and focus on $2 \ln x + 1 = 0$. From here we get $\ln x = -\frac{1}{2}$, so that $x = e^{-1/2} = 1/\sqrt{e}$. Thus the tangent to this curve is horizontal only at $x = 1/\sqrt{e}$.

(e) Calculate $y' = 4e^{x^2+3x+2}(2x + 3)$, so that $y' = 0$ if and only if $2x + 3 = 0$. (Again, e^{junk} is never zero.) So the tangent is horizontal precisely when $x = -\frac{3}{2}$.