Saint Mary's University
DEPARTMENT OF MATHEMATICS
AND COMPUTING SCIENCE

Name: SOLUTIONS

Signature: ____________

ID: ____________

Math 1210: Winter 2012
Midterm Test

February 15, 10:00–11:15am

Instructor: J. Irving

Instructions:

- *No electronic devices, or aids of any kind, are permitted.*
- *There are 5 pages of questions plus this cover page. Check that your test paper is complete.*
- *There are a total of 70 marks. The value of each question is indicated in the margin.*
- *Answer in the spaces provided, using backs of pages for additional space if necessary.*
- *Show all your work. Insufficient justification will result in a loss of marks.*

Page	Maximum	Your Score
1	18	
2	16	
3	12	
4	8	
5	16	
Total	**70**	

[18] 1. Find the derivative of each of the following functions. **Do not** simplify your answers.

(a) $f(x) = 2\sqrt[4]{x} + \dfrac{7}{(x+1)^2} + 6\sec 3x + 2\sin^{-1} x - 2e^{-1}$

$$f'(x) = \frac{1}{2}x^{-3/4} - \frac{14}{(x+1)^3} + 6\sec 3x \tan 3x \cdot 3 + \frac{2}{\sqrt{1-x^2}}$$

(b) $f(x) = \tan^{-1}(\sqrt{x})$

$$f'(x) = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}}$$

(c) $f(x) = \dfrac{\sin x}{1+\cos(3^x)}$

$$f'(x) = \frac{\cos x\left(1+\cos 3^x\right) - \left(-\sin(3^x)\cdot 3^x \ln 3\right)\sin x}{\left(1+\cos 3^x\right)^2}$$

(d) $f(x) = \log_5(x^2+1)\tan^3 x$

$$f'(x) = \frac{1}{(x^2+1)\ln 5} \cdot 2x \cdot \tan^3 x + \log_5(x^2+1)\cdot 3\tan^2 x \sec^2 x$$

[8] 2. Use **logarithmic differentiation** to determine $\frac{dy}{dx}$, where $y = \frac{x^6(1+x^4)^{2x}}{e^{3x}\sqrt{1+x^2}}$.

$$\ln y = 6\ln x + 2x\ln(1+x^4) - 3x - \frac{1}{2}\ln(1+x^2)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{6}{x} + 2\ln(1+x^4) + 2x\cdot\frac{1}{1+x^4}\cdot 4x^3 - 3 - \frac{1}{2}\cdot\frac{1}{1+x^2}\cdot 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^6(1+x^4)^{2x}}{e^{3x}\sqrt{1+x^2}}\left[\frac{6}{x} + 2\ln(1+x^4) + \frac{8x^4}{1+x^4} - 3 - \frac{x}{1+x^2}\right]$$

[8] 3. Consider the curve defined by $x^2y^2 = 3x - 2y^3$.

(a) Use **implicit differentiation** to find a formula for $\frac{dy}{dx}$ in terms of x and y.

Diff. wrt x to get $2xy^2 + x^2\cdot 2y\frac{dy}{dx} = 3 - 6y^2\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}(2x^2y + 6y^2) = 3 - 2xy^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3-2xy^2}{2x^2y + 6y^2}$$

(b) Find the equation of the tangent line to the curve at the point $(x, y) = (1, 1)$.

At $(1,1)$, get $\frac{dy}{dx} = \frac{3-2}{2+6} = \frac{1}{8}$.

So the tangent line is $y - 1 = \frac{1}{8}(x-1)$

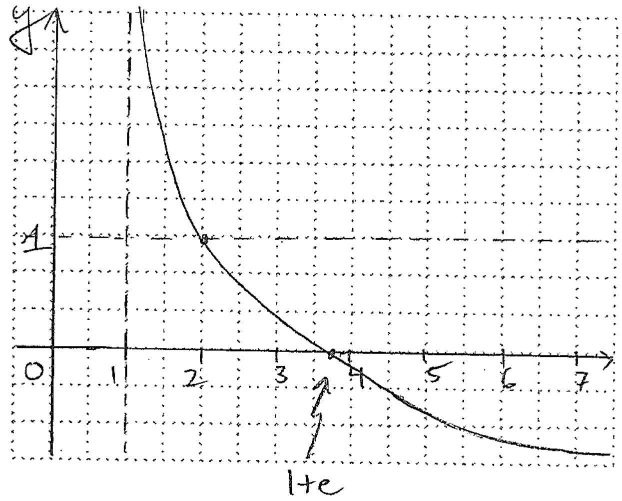
[6] 4. **Sketch the following curves.** Choose your scale wisely so that your sketch fills most of the given grid. Label all x- and y-intercepts.

(a) $y = 1 - \ln(x-1)$

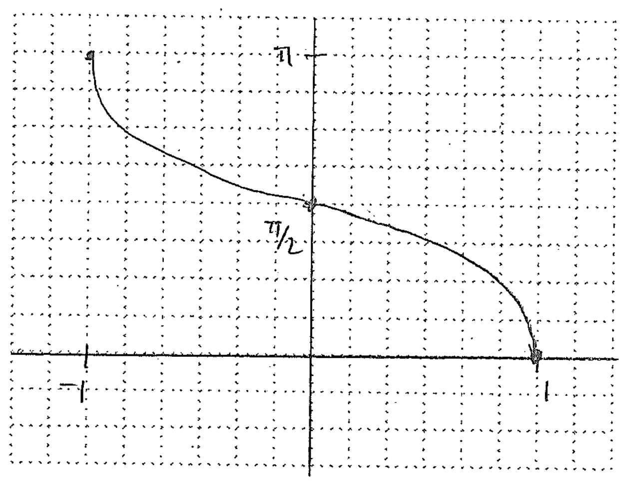
x-int. when

$\ln(x-1) = 1$

$\Leftrightarrow x = 1 + e$



(b) $y = \cos^{-1} x$



[6] 5. Consider the function $f(x) = \sqrt[3]{1 + e^{2x}}$ defined for all real numbers x.

(a) Find the inverse function $f^{-1}(x)$.

Let $y = \sqrt[3]{1+e^{2x}}$. Then $y^3 = 1 + e^{2x}$

$\Leftrightarrow e^{2x} = y^3 - 1$

$\Leftrightarrow 2x = \ln(y^3 - 1)$

$\Leftrightarrow x = \frac{1}{2}\ln(y^3 - 1)$.

So $f^{-1}(x) = \frac{1}{2}\ln(x^3 - 1)$

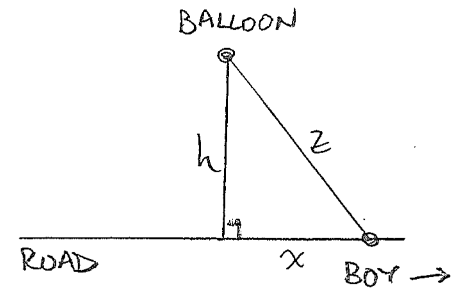
(b) What are the domain and range of $f^{-1}(x)$?

Domain of $f^{-1}(x)$ is $(1, \infty)$, since $\ln(x^3-1)$ is only sensible when $x^3 - 1 > 0$. (that is, $x > 1$).

Range of $f^{-1}(x)$ is the domain of $f(x)$, which is $\mathbb{R}$. (that is, all real numbers).

[8] 6. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 seconds later?

Note: Include a diagram that clearly illustrates the situation and introduces any variables used in your solution. Your solution must be clearly laid out to obtain full credit.



Let x, h and z be as indicated in the diagram. Then we know:

- $x^2 + h^2 = z^2$
- $\frac{dx}{dt} = 15$
- $\frac{dh}{dt} = 5$

We want $\frac{dz}{dt}$ when $x = 3 \cdot 15 = 45$ and $h = 45 + 3 \cdot 5 = 60$.

Differentiate $x^2 + h^2 = z^2$ wrt t. to get

$$\cancel{2}x \frac{dx}{dt} + \cancel{2}h \frac{dh}{dt} = \cancel{2}z \frac{dz}{dt}$$

$$\Rightarrow \quad 15x + 5h = z \frac{dz}{dt}.$$

When $x = 45$ and $h = 60$, get $z = \sqrt{45^2 + 60^2} = 75$

So $75 \frac{dz}{dt} = 15(45) + 5(60) \Rightarrow \frac{dz}{dt} = \frac{15 \cdot 45 + 5 \cdot 60}{75}$

$$= 13$$

Therefore the distance between the boy and the balloon is increasing at a rate of 13 ft/sec.

[6] 7. Use the **limit definition of the derivative** to determine $f'(x)$, where $f(x) = \dfrac{1}{2x^2} - 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{2(x+h)^2} \cancel{-1}\right) - \left(\frac{1}{2x^2} \cancel{-1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{2h\,x^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-2xh - h^2}{2hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-2x - h}{2x^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-2x}{2x^2x^2} = -\frac{1}{x^3}$$

[5] 8. **True or false.** Circle the correct choice.

TRUE FALSE $\dfrac{d}{dx}\cos^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$

TRUE FALSE $\dfrac{d}{dx}\csc x = -\csc x \cot x.$

TRUE FALSE $\dfrac{d}{dx} f(x)g(x) = f'(x)g'(x)$

TRUE FALSE The function $f(x) = x^2 - 1$, for $x \geq 0$, is one-to-one.

TRUE FALSE $\dfrac{d}{dx}|2x - 1| = 2$ for $x > 0$.

[5] 9. **Fill in the blanks.** Only your answer will be graded.

(a) If $f(x) = g(x^3)$ and $g'(x) = \dfrac{1}{x^2}$ then $f'(x) =$ $3/x^4$.

(b) $e^{3\ln 2}$ is most simply written as 8.

(c) $\sin^{-1}(-\frac{1}{2})$ is most simply written as $-\pi/6$

(d) $\cos(\tan^{-1}(2))$ is most simply written as $1/\sqrt{5}$.

(e) The function $f(x) = \sec x$ has an inverse when its domain is restricted to $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$