Math 1210: Recitation Test #2 Winter 2012

Name: SOLUTIONS | A#:

1. Consider the function $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ \sqrt{x+1} & \text{if } 0 \le x < 3 \\ 5 - x & \text{if } x \ge 3. \end{cases}$

(a) Determine all points x at which $f(x)$ is discontinuous. **Justify** your answer fully.

Clearly $f(x)$ is continuous at all x except possibly $x=0,3$.

Now: $\left.\begin{array}{l} \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0 \\ \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x+1} = 1 \end{array}\right\} \therefore \lim_{x\to 0} f(x)$ does not exist

And: $\left.\begin{array}{l} \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{x+1} = 2 \\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (5-x) = 2 \end{array}\right\} \therefore \lim_{x\to 3} f(x) = 2 = f(3)$

Therefore f is discontinuous only at $\boxed{x=0}$.

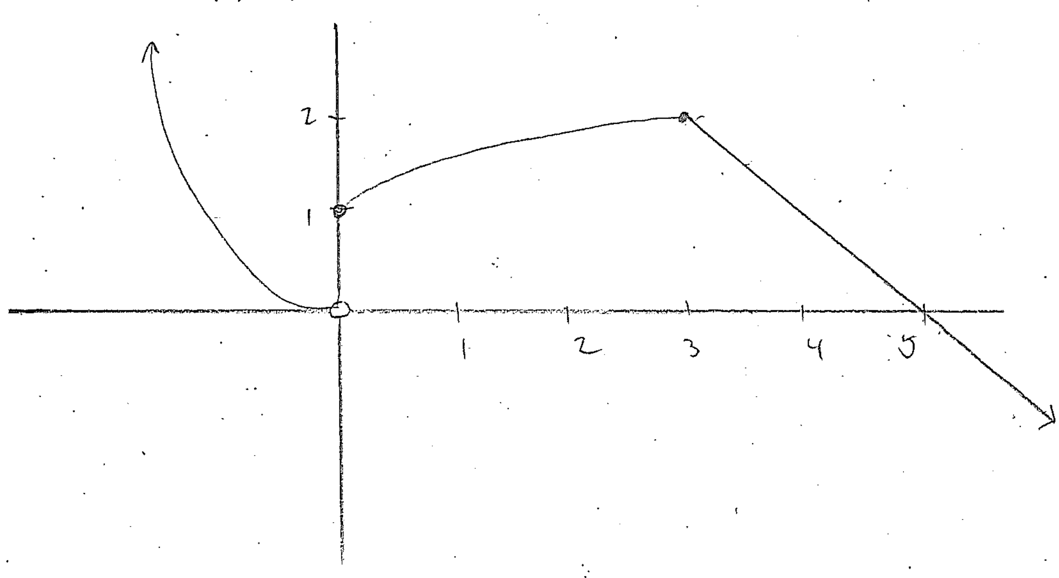
(b) Decide whether f is continuous from the left or from the right at each of the points of discontinuity found in (a).

Notice that $f(0) = \sqrt{0+1} = 1$.

Since $\lim_{x\to 0^-} f(x) = 0 \neq 1$ and $\lim_{x\to 0^+} f(x) = 1 = f(0)$,

we see that f is continuous only from the right at $x=0$

(c) Sketch the graph of $y = f(x)$.



2. Evaluate the following limits, if they exist. If a limit does not exist, decide whether it tends to $\pm\infty$. **Justify** your answers fullly.

(a) $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$

$$= \lim_{x \to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9}$$

$$= \lim_{x \to 9} (\sqrt{x}+3) = \boxed{6}$$

(b) $\lim_{x \to 2} \frac{|x-2|}{x^2-2x}$ DOES NOT EXIST

$$\lim_{x \to 2^-} \frac{|x-2|}{x^2-2x} = \lim_{x \to 2^-} \frac{-(x-2)}{x(x-2)} = \lim_{x \to 2^-} -\frac{1}{x} = -\frac{1}{2}$$

$$\lim_{x \to 2^+} \frac{|x-2|}{x^2-2x} = \lim_{x \to 2^+} \frac{x-2}{x(x-2)} = \lim_{x \to 2^+} \frac{1}{x} = \frac{1}{2}.$$

UNEQUAL

(c) $\lim_{t \to \pi/2} \frac{1-\cos t}{1-\sin t}$

Note that $1-\sin t \to 0^+$ as $t \to \frac{\pi}{2}$ (from either side!)

while $1-\cos t \to 1$ as $t \to \frac{\pi}{2}$.

Thus $\lim_{t \to \pi/2} \frac{1-\cos t}{1-\sin t} = \boxed{+\infty}$

(d) $\lim_{z \to 1} \frac{e^z}{\sqrt[3]{1-z}}$

As $z \to 1^-$, we have $e^z \to e$ and $\sqrt[3]{1-z} \to 0^+$.

Thus $\lim_{z \to 1^-} \frac{e^z}{\sqrt[3]{1-z}} = +\infty$.

But as $z \to 1^+$, get $e^z \to e$ and $\sqrt[3]{1-z} \to 0^-$,

so that $\lim_{z \to 1^+} \frac{e^z}{\sqrt[3]{1-z}} = +\infty$.

Therefore $\lim_{z \to 1} \frac{e^z}{\sqrt[3]{1-z}}$ does not exist.

3. State the Mean Value Theorem as precisely as you can, and draw a picture that illustrates its meaning.

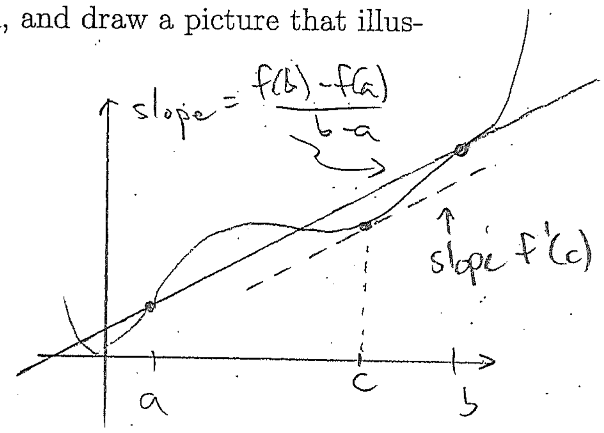
Let f be a function that is

① continuous on $[a,b]$

② differentiable on (a,b).

Then there is some $c \in (a,b)$

such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$



4. (a) Find all critical numbers of the function $f(x) = \sqrt[3]{x^2 - 2x - 3}$

$$f'(x) = \frac{1}{3}(x^2-2x-3)^{-2/3}(2x-2)$$

$$= \frac{2}{3} \cdot \frac{x-1}{((x-3)(x+1))^{2/3}}$$

So $f'(x)$ is undefined $\Leftrightarrow$ $x=3$ or $x=-1$

and $f'(x)=0 \Leftrightarrow x=1$.

$\therefore$ Critical numbers are $\boxed{x=-1,\ 1,\ 3}$

(b) Find the absolute maximum and minimum values of $f(x)$ on the interval $[-2, 2]$.

Of the critical numbers in (a), only $x=\pm 1$ are in $[-2,2]$.

Evaluate: $f(-2)=\sqrt[3]{5}$, $f(-1)=0$, $f(1)=-\sqrt[3]{4}$, $f(2)=-\sqrt[3]{3}$.

Thus the absolute max is $\sqrt[3]{5}$ (at $x=-2$)

and the absolute min is $-\sqrt[3]{4}$ (at $x=1$)

5. For the the function $g(x) = 2x^3 - 3x^2 - 12x$, do the following:

- Determine where the function is **increasing** and **decreasing**, and find all local **maxima** and **minima**.
- Determine where the function is **concave up** and **concave down**, and find all **inflection points**.
- Find all x- and y- **intercepts**.
- Use this information to sketch the curve $y = g(x)$.

Calculate: $g'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$

$g''(x) = 12x - 6 = 6(2x-1)$

Then $g'(x) = 0 \iff x = 2$ OR $x = -1$.

		-1	2
$x-2$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$
$g'(x)$	$+$	$-$	$+$

From table, see that g is increasing on $(-\infty, -1) \cup (2, \infty)$ and decreasing on $(-1, 2)$. Thus there is a local max at $x = -1$ (where $f(-1) = 7$) and a local min at $x = 2$ (where $f(2) = -20$)

Also, $g''(x) = 0 \iff x = \frac{1}{2}$, while $g''(x) < 0$ for $x < \frac{1}{2}$ and $g''(x) > 0$ for $x > \frac{1}{2}$.

So g is concave down on $(-\infty, \frac{1}{2})$ and concave up on $(\frac{1}{2}, \infty)$, with an inflection point at $x = \frac{1}{2}$. (where $f(\frac{1}{2}) = -\frac{13}{2}$)

The y-intercept is $y(0) = 0$.

For x-intercepts: $2x^3 - 3x^2 - 12x = 0 \iff x(2x^2 - 3x - 12) = 0$

$\iff x = 0$ OR $x = \frac{3 \pm \sqrt{105}}{4} \approx \frac{-7}{4}, \frac{13}{4}$

Sketch: