

At this point you know the derivatives of a handful of “basic” functions. Your calculus life will be much easier if you have this list memorized for instant recall:

$$\begin{array}{ll} \frac{d}{dx}x^a = ax^{a-1} & \frac{d}{dx}a^x = a^x \ln a \\ \frac{d}{dx}\sin x = \cos x & \frac{d}{dx}\csc x = -\csc x \cot x \\ \frac{d}{dx}\cos x = -\sin x & \frac{d}{dx}\sec x = \sec x \tan x \\ \frac{d}{dx}\tan x = \sec^2 x & \frac{d}{dx}\cot x = -\csc^2 x \end{array}$$

You also know how to differentiate functions that are “pasted together” through addition, subtraction, multiplication, division, and composition:

Linearity: $\frac{d}{dx}(Af(x) + Bg(x)) = Af'(x) + Bg'(x)$ for constants A, B

Product Rule: $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

All the x 's in these rules may make them appear more complicated than they are. For instance, the product rule looks easier when written as $(fg)' = f'g + fg'$. In fact, this rule can be applied iteratively to a product of more than two functions, and the overall rule to remember is simply that each factor “gets its turn” at differentiation. For example, we have

$$(fghk)' = f'ghk + fg'hk + fgh'k + fghk'.$$

Likewise, the chain rule is best remembered informally as

$$\frac{d}{dx}f(\text{junk}) = f'(\text{junk}) \times [\text{derivative of junk}].$$

If the junk is itself a composition of functions, then we have to use the chain rule more than once. For example, to calculate the derivative of $f(x) = \cos^4(2e^x + 1)$, we first observe that

$$f(x) = (\text{junk})^4, \quad \text{where} \quad \text{junk} = \cos(2e^x + 1).$$

The chain rule therefore gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\text{junk})^4 \\ &= 4(\text{junk})^3 \cdot \frac{d}{dx}(\text{junk}) \\ &= 4\cos^3(2e^x + 1) \frac{d}{dx}\cos(2e^x + 1). \end{aligned}$$

To compute $\frac{d}{dx} \cos(2e^x + 1)$, we replace $2e^x + 1$ with “crud” and apply the chain rule again:

$$\begin{aligned} \frac{d}{dx} \cos(2e^x + 1) &= \frac{d}{dx} \cos(\text{crud}) \\ &= -\sin(\text{crud}) \cdot \frac{d}{dx}(\text{crud}) \\ &= -\sin(2e^x + 1) \frac{d}{dx}(2e^x + 1) \\ &= -\sin(2e^x + 1) \cdot 2e^x. \end{aligned}$$

So altogether we have

$$\begin{aligned} f'(x) &= 4 \cos^3(2e^x + 1)(-\sin(2e^x + 1) \cdot 2e^x) \\ &= -8e^x \cos^3(2e^x + 1) \sin(2e^x + 1). \end{aligned}$$

To differentiate a complicated function, we typically require multiple applications of the chain rule, along with the other differentiation rules. For instance:

$$\begin{aligned} \frac{d}{dx} \sqrt{3^{2x+1} + 5 \sec x^2} &= \frac{d}{dx} \sqrt{\text{junk}} \quad [\text{junk} = 3^{2x+1} + 5 \sec x^2] \\ &= \frac{1}{2\sqrt{\text{junk}}} \cdot \frac{d}{dx}(\text{junk}) \\ &= \frac{1}{2\sqrt{3^{2x+1} + 5 \sec x^2}} \frac{d}{dx}(3^{2x+1} + 5 \sec x^2) \\ &= \frac{1}{2\sqrt{3^{2x+1} + 5 \sec x^2}} \left(\frac{d}{dx} 3^{2x+1} + 5 \frac{d}{dx} \sec x^2 \right) \quad [\text{by linearity}] \\ &= \frac{1}{2\sqrt{3^{2x+1} + 5 \sec x^2}} \left(\frac{d}{dx} 3^{\text{crud}} + 5 \frac{d}{dx} \sec(\text{blah}) \right) \quad [\text{crud} = 2x + 1, \quad \text{blah} = x^2] \\ &= \frac{1}{2\sqrt{3^{2x+1} + 5 \sec x^2}} \left(3^{\text{crud}} \ln 3 \cdot \frac{d}{dx}(\text{blah}) + 5 \sec(\text{blah}) \tan(\text{blah}) \cdot \frac{d}{dx}(\text{blah}) \right) \\ &= \frac{1}{2\sqrt{3^{2x+1} + 5 \sec x^2}} \left(3^{2x+1} \ln 3 \cdot \frac{d}{dx}(2x + 1) + 5 \sec x^2 \tan x^2 \cdot \frac{d}{dx} x^2 \right) \\ &= \frac{1}{2\sqrt{3^{2x+1} + 5 \sec x^2}} \left(3^{2x+1} \ln 3 \cdot 2 + 5 \sec x^2 \tan x^2 \cdot 2x \right) \\ &= \frac{2 \ln 3 \cdot 3^{2x+1} + 10x \sec x^2 \tan x^2}{2\sqrt{3^{2x+1} + 5 \sec x^2}} \end{aligned}$$

As another example, take a moment to convince yourself that

$$y = \sqrt[3]{1 + x^3 \sin^2(\pi x)}$$

has the following messy derivative:

$$\frac{dy}{dx} = \frac{1}{3} \left(1 + x^3 \sin^2(\pi x) \right)^{-2/3} \left(3x^2 \sin^2(\pi x) + x^3 \cdot 2 \sin(\pi x) \cdot \cos(\pi x) \cdot \pi \right).$$

Problems:

1. Differentiate the following. Simplify first if it seems helpful (and afterward if it seems fruitful).

(a) $y = 7(x + 3)\sqrt{x + 1}$

(b) $y = \frac{2 \sin x}{1 + 3e^{5x}}$

(c) $y = 5x^2 e^{x^2} \sqrt{x + 1}$

(d) $y = x^2 \sqrt{x}(3 + x^4)$

(e) $y = \sec x \tan x$

(f) $y = \frac{2x^3 + 1}{7x^2 - 3x + 2}$

(g) $y = \frac{x^2 \sin x + x}{2x\sqrt{x} + 1}$

(h) $y = \frac{3}{2x^3 + 3 \cos x}$

(i) $y = 3 \cos x \sin x$

(j) $y = \frac{\csc x}{\cot x - 1}$

(k) $y = \frac{1 + x^2}{\sqrt[4]{x - 1}}$

(l) $y = (3x^3 + 2x + 1)^{10}$

(m) $y = 3^{2x} \sqrt{2^{3x} + 7}$

(n) $y = 5 \sin^2 3x$

(o) $y = \frac{3}{\sqrt[3]{x + \sin x}}$

(p) $y = \frac{1}{(2 - e^{\cos x})^3}$

(q) $y = (2x + \cos^2 x)^{2/3}$

(r) $y = \tan^3(2x)$

(s) $y = \sin(x^2)(1 + x^3)^4$

(t) $y = \sqrt{\frac{1 + x}{1 - x}}$

(u) $y = \sec^3\left(\sqrt[3]{1 + x^2}\right)$

2. Find all points at which the tangent to the following curves is horizontal:

(a) $y = x\sqrt{x + 1}$

(b) $y = \frac{x}{1 + 2x^3}$

(c) $y = e^x(e^x - 1)$.

(d) $y = \sin^3(2x)$

Answers:

Warning! I have looked over these answers quickly, but I assume there are still typos lurking about. So don't be too alarmed if you see something that baffles you — it may just be my mistake! Please let me know if you find any errors.

1. (a) Product rule yields $y' = 7\sqrt{x+1} + \frac{7(x+3)}{2\sqrt{x+1}}$.
- (b) Quotient rule gives $y' = \frac{2 \cos x(1 + 3e^{5x}) - (3e^{5x} \cdot 5)(2 \sin x)}{(1 + 3e^{5x})^2} = \frac{2 \cos x + 6e^{5x}(\cos x - 5 \sin x)}{(1 + 3e^{5x})^2}$.
The simplification step I have taken here is somewhat arbitrary. The important thing at this point is to get the *correct* derivative, written in any form.
- (c) Product rule gives $y' = 10xe^{x^2}\sqrt{x+1} + 5x^2e^{x^2}(2x) \cdot \sqrt{x+1} + 5x^2e^{x^2}\frac{1}{2\sqrt{x+1}}$.
- (d) The simplest approach is to write $y = 3x^{5/2} + x^{13/2}$ so that $y' = \frac{15}{2}x^{3/2} + \frac{13}{2}x^{11/2}$.
The same result is obtained by applying the product rule to $y = x^{5/2}(3 + x^4)$, or by applying the “3-fold” product rule to $y = x^2 \cdot \sqrt{x} \cdot (3 + x^4)$.
- (e) The product rule gives $y' = (\sec x \tan x)(\tan x) + \sec x(\sec^2 x) = \sec x(\tan^2 x + \sec^2 x)$.
It smells like this should simplify, but about the best we can do is apply the identity $1 + \tan^2 x = \sec^2 x$ to write $y' = \sec x(2 \sec^2 x - 1)$.
- (f) This is a messy but straightforward application of quotient rule:

$$y' = \frac{6x^2(7x^2 - 3x + 2) - (14x - 3)(2x^3 + 1)}{(7x^2 - 3x + 2)^2} = \frac{14x^4 - 12x^3 + 12x^2 - 14x + 3}{(7x^2 - 3x + 2)^2}.$$

- (g) Apply quotient rule, noting along the way that $\frac{d}{dx}x^2 \sin x = 2x \sin x + x^2 \cos x$ and $\frac{d}{dx}x\sqrt{x} = \frac{d}{dx}x^{3/2} = \frac{3}{2}\sqrt{x}$. Arrive at

$$y' = \frac{(2x \sin x + x^2 \cos x + 1)(2x\sqrt{x} + 1) - 3\sqrt{x}(x^2 \sin x + x)}{(2x\sqrt{x} + 1)^2}.$$

- (h) Applying the quotient rule gives

$$y' = \frac{0 \cdot (2x^3 + 3 \cos x) - 3(6x^2 - 3 \sin x)}{(2x^3 + 3 \cos x)^2} = \frac{-3(6x^2 - 3 \sin x)}{(2x^3 + 3 \cos x)^2}.$$

This is correct, but it is a little silly to use the quotient rule when the numerator is a constant. Instead, recognize that the given function is simply $y = 3(2x^3 + 3 \cos x)^{-1}$. Then $y' = -3(2x^3 + 3 \cos x)^{-2}(6x^2 - 3 \sin x)$ by chain rule.

- (i) Applying the product rule gives $y' = 3((-\sin x) \sin x + \cos x (\cos x))$, which can be simplified to $y' = 3 \cos 2x$ through the identity $\cos^2 x - \sin^2 x = \cos 2x$.
A cleaner approach is simplify *before* differentiating. Use $\sin 2x = 2 \sin x \cos x$ to write $y = \frac{3}{2} \sin 2x$. Then the chain rule gives $y' = \frac{3}{2}(\cos 2x)(2)$.
- (j) The product and quotient rules give

$$y' = \frac{\csc x \cot x (\cot x - 1) - (-\csc^2 x) \csc x}{(\cot x - 1)^2}.$$

This expression is correct, and we could now try to simplify. However, it is always good to be on the lookout for basic simplifications *before* differentiating.

In this case, you will quickly find that $y = (\cos x - \sin x)^{-1}$ upon expressing $\csc x$ and $\cot x$ in terms of $\sin x$ and $\cos x$. So the chain rule (or quotient rule) immediately gives

$$y' = \frac{-(-\sin x - \cos x)}{(\cos x - \sin x)^2},$$

If you take the time to convert everything to $\sin x$ and $\cos x$, you will see that our first expression for y' does match this one.

With a little more trig trickery we notice that

$$\begin{aligned}(\cos x - \sin x)^2 &= (\cos^2 x + \sin^2 x) - 2 \cos x \sin x \\ &= 1 - \sin 2x.\end{aligned}$$

Therefore

$$y' = \frac{\sin x + \cos x}{1 - 2 \sin x}.$$

Again, the important thing right now is to get a *correct* derivative. However, we will later need to be able to simplify expressions as above.

(k) Applying the quotient rule gives

$$y' = \frac{2x\sqrt[4]{x-1} - \frac{1}{4}(x-1)^{-3/4}(1+x^2)}{(\sqrt[4]{x-1})^2}.$$

Alternatively, write $y = (1+x^2)(x-1)^{-1/4}$ and use the product rule to get

$$y' = 2x(x-1)^{-1/4} + (1+x^2)(-\frac{1}{4}(x-1)^{-5/4}).$$

Take a moment to convince yourself that these expressions for y' are equivalent.

- (l) Chain rule gives $y' = 10(x^3 + 2x + 1)^9 \cdot (3x^2 + 2)$.
- (m) Product and chain rules give $y' = 3^{2x} \ln 3 \cdot 2 \cdot \sqrt{2^{3x} + 7} + 3^{2x} \cdot \frac{1}{2}(2^{3x} + 7)^{-1/2} \cdot 2^{3x} \ln 2 \cdot 3$
- (n) Chain rule gives $y' = 5 \cdot 2 \sin(3x) \cdot \cos(3x) \cdot 3 = 30 \sin 3x \cos 3x = 15 \sin 6x$.
- (o) Write $y = 3(x + \sin x)^{-1/3}$ and use chain rule to get

$$y' = 3 \cdot (-\frac{1}{3})(x + \sin x)^{-4/3} \cdot (1 + \cos x) = -\frac{1 + \cos x}{(x + \sin x)^{4/3}}.$$

(p) Write $y = (2 - e^{\cos x})^{-3}$. Chain rule gives

$$y' = -3(2 - e^{\cos x})^{-4} \cdot (-e^{\cos x})(-\sin x) = \frac{-3e^{\cos x} \sin x}{(2 - e^{\cos x})^4}.$$

(q) Chain rule gives $y' = \frac{2}{3}(2x + \cos^2 x)^{-1/3} \cdot (2 + 2 \cos x(-\sin x)) = \frac{2(2 - \sin 2x)}{3\sqrt[3]{2x + \cos^2 x}}$.

(r) Chain rule gives $y' = 3 \tan^2(2x) \cdot \sec^2(2x) \cdot 2 = 6(\sec 2x \tan 2x)^2$.

(s) Chain and product rules give $y' = \cos(x^2) \cdot 2x \cdot (1 + x^3)^4 + \sin(x^2) \cdot 4(1 + x^3)^3 \cdot 3x^2$. Not a lot can be done to simplify here, though we can common factor to write

$$y' = 2x(1 + x^3)^3 \left((1 + x^3) \cos x^2 + 6x \sin x^2 \right).$$

(t) Chain and quotient rules give

$$\begin{aligned}y' &= \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-1/2} \cdot \frac{1(1-x) - (-1)(1+x)}{(1-x)^2} \\&= \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \cdot \frac{2}{(1-x)^2} \\&= \frac{1}{(1-x)\sqrt{1-x^2}},\end{aligned}$$

where the last line follows from $(1-x)(1+x) = 1-x^2$.

(u) A few applications of the chain rule are needed here:

$$\begin{aligned}y' &= 3 \sec^2(\sqrt[3]{1+x^2}) \cdot (\sec \sqrt[3]{1+x^2} \tan \sqrt[3]{1+x^2}) \cdot \frac{1}{3}(1+x^2)^{-2/3} \cdot 2x \\&= \frac{2x \sec^3(\sqrt[3]{1+x^2}) \tan(\sqrt[3]{1+x^2})}{(1+x^2)^{2/3}}.\end{aligned}$$

2. (a) The derivative is

$$y' = \sqrt{x+1} + \frac{1}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}},$$

so we have $y' = 0$ if and only if $3x+2 = 0$. Hence the tangent line is horizontal only at $x = -\frac{2}{3}$.

(b) The derivative is

$$y' = \frac{1 \cdot (1+2x^3) - 6x^2 \cdot x}{(1+2x^3)^2} = \frac{1-4x^3}{(1+2x^3)^2}.$$

Therefore $y' = 0$ only when $1-4x^3 = 0$, which occurs when $x = \sqrt[3]{\frac{1}{4}}$. So the tangent is horizontal only at $x = 1/\sqrt[3]{4}$.

(c) The derivative works out to $y' = 2e^{2x} - e^x = e^x(2e^x - 1)$. This is zero when either $e^x = 0$ or $2e^x - 1 = 0$. Note that $e^x = 0$ is impossible. On the other hand, $2e^x - 1 = 0 \iff e^x = \frac{1}{2} \iff x = \ln \frac{1}{2}$. So the tangent is horizontal only at $x = \ln \frac{1}{2} = -\ln 2$.

(d) The derivative is

$$y' = 3 \sin^2(2x) \cdot \cos(2x) \cdot 2 = 6 \sin^2(2x) \cos(2x).$$

Thus $y' = 0$ if and only if $\sin 2x = 0$ or $\cos 2x = 0$. The first condition occurs when $2x$ is a multiple of π , and the second occurs when $2x$ is an odd multiple of $\frac{\pi}{2}$. Thus $y' = 0$ precisely when $x = k\pi/2$ for some integer k .