Math 1210: Introductory Problems

Winter 2012

Imagine a particle travelling on straight line that passes through a point P and extends a long way (infinitely!) to the east and west. Let $d(t)$ be the displacement of the particle from a fixed point P at time t, where time is measured in seconds and distance in metres.

Assume that a positive value of $d(t)$ indicates that the particle is to the east of P at time t, and a negative sign indicates that it is to the west. Notice that the time "$t = 0$" is arbitrary; so we allow negative values of t and interpret such a time like $t = -3$ to mean 3 seconds before the $t = 0$ mark.

1. Suppose the particle's motion is governed by the rule $d(t) = t^3$. Answer the following questions:

    (i) Where is the particle at time $t = 1$?

    (ii) What is the average velocity between $t = 1$ and $t = 4$?

    (iii) What is the instantaneous velocity at time $t = 1$?

    (iv) What is the instantaneous velocity $v(t)$ at time t?

    (v) At what times t is the particle stationary (i.e. has velocity 0)?

    (vi) At what times t is the particle moving westward?

2. Repeat Question #1 for the following displacement functions:

    (a) $d(t) = \dfrac{5}{t^2}$

    (b) $d(t) = \dfrac{1}{t+2}$

    (c) $d(t) = -\sqrt{2t}$

    (d) $d(t) = 3t^2 + 4$

    (e) $d(t) = \dfrac{3}{\sqrt{t+5}}$

    (f) $d(t) = t^2 - 4t$

    (g) $d(t) = \dfrac{2}{t} - t^2$

    (h) $d(t) = t^3 + 6t^2 + 5$

Answers

1. (i) At $t = 1$ we have $d(t) = d(1) = 1^1 = 1$. So the particle is 1 metre east of P after 1 seconds.

   (ii) The average velocity between $t = 1$ and $t = 4$ is

$$v_{\text{avg}} = \frac{d(4) - d(1)}{4 - 1} = \frac{4^3 - 1^3}{3} = \frac{63}{3} = 21\,\text{m/s}.$$

   (iii) The instantaneous velocity at $t = 1$ is given by

$$\begin{aligned} v(1) &= \lim_{t\to 1} \frac{d(t) - d(1)}{t - 1} \\ &= \lim_{t\to 1} \frac{t^3 - 1}{t - 1} \\ &= \lim_{t\to 1} \frac{(t-1)(t^2 + t + 1)}{t - 1} \\ &= \lim_{t\to 1} (t^2 + t + 1) \\ &= 3. \end{aligned}$$

   So the particle is moving at 1 m/s in the eastward direction at the instant when $t = 1$.

   Note: Alternatively, we could have used

$$v(1) = \lim_{h\to 0} \frac{d(1+h) - d(1)}{h}$$

   to obtain the same result. See (iv), below, for a more general calculation along these lines.

   (iv) The instantaneous velocity at time t is given by

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{d(t+h) - d(t)}{h} \\ &= \lim_{h\to 0} \frac{(t+h)^3 - t^3}{h} \\ &= \lim_{h\to 0} \frac{t^3 + 3t^2h + 3th^2 + h^3 - t^3}{h} \\ &= \lim_{h\to 0} \frac{3t^2h + 3th^2 + h^3}{h} \\ &= \lim_{h\to 0} \frac{3t^2 + 3th + h^2}{h} \\ &= 3t^2. \end{aligned}$$

   Note: Since we now know $v(t) = 3t^2$, we can simply set $t = 1$ to get $v(1) = 3$, which of course agrees with what we found in part (iii).

(v) The particle is stationary when $v(t) = 0$. Using part (iv), we have

$$v(t) = 0 \quad \Longleftrightarrow \quad 3t^2 = 0 \quad \Longleftrightarrow \quad t = 0.$$

So the particle is stationary only at time $t = 0$.

(vi) The particle is moving westward when $v(t)$ is negative. But we know $v(t) = 3t^2$, and this quantity cannot be negative. So the particle is never moving westward.

2. (a) (i) 5 m east of P (ii) $-\frac{25}{16}$ m/s (iii) -10 m/s (iv) $v(t) = -10/t^3$ (v) no stationary points (vi) at all times $t > 0$

(b) (i) $\frac{1}{3}$ m east of P (ii) $-\frac{1}{18}$ m/s (iii) $-\frac{1}{9}$ m/s (iv) $v(t) = -1/(1+t)^2$ (v) no stationary points (vi) at all times $t \neq -2$

(c) (i) $\sqrt{2}$ m west of P (ii) $-\frac{\sqrt{2}}{3}$ m/s (iii) $-\frac{1}{\sqrt{2}}$ m/s (iv) $v(t) = -1/\sqrt{2t}$ (v) no stationary points (vi) at all times $t > 0$

(d) (i) 7 m east of P (ii) 15 m/s (iii) 6 m/s (iv) $v(t) = 6t$ (v) $t = 0$ (vi) at all times $t < 0$

(e) (i) $\frac{\sqrt{6}}{2}$ m east of P (ii) $\frac{2-\sqrt{6}}{6}$ m/s (iii) $-\frac{\sqrt{6}}{24}$ m/s (iv) $v(t) = -\frac{3}{2}(t+5)^{3/2}$ (v) no stationary points (vi) at all times t

(f) (i) 3 m west of P (ii) 1 m/s (iii) -2 m/s (iv) $v(t) = 2t - 4$ (v) $t = 2$ (vi) at all times $t < 2$

(g) (i) 1 m east of P (ii) $-\frac{11}{2}$ m/s (iii) -4 m/s (iv) $v(t) = -\frac{2(1+t^3)}{t^2}$ (v) $t = -1$ (vi) all times $t > -1$ except $t = 0$

(h) (i) 12 m east of P (ii) 51 m/s (iii) 15 m/s (iv) $v(t) = 3t^2 + 12t$ (v) $t = 0$ and $t = -4$ (vi) $-4 < t < 0$