Math 1210: Notes for the Week of January 16

Important Note: *I made a big mistake in posting the homework for this week. Please check your email for a message from me pertaining to the week's homework and the upcoming recitation test.*

Last week we defined the **derivative** of a function $f(x)$ to be the function $f'(x)$ given by the rule
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$
We then calculated $f'(x)$ by evaluating this limit for a handful of simple functions, such as $f(x) = x^2$ and $f(x) = \sqrt{x}$ and $f(x) = 1/x$.

However, finding the derivatives of complicated functions in this manner would be very tedious. Instead, we want to develop a set of general rules for computing derivatives.

We started off with the following simple rules:

(1) **Constant functions:** The derivative of any constant function is 0.

For example, if $f(x) = 7\pi$ then $f'(x) = 0$.

Note that this rule is obvious from a graphical point of view. The derivative measures the slope of a function (or, more precisely, the slope of its tangent lines) and the graph of a constant function is a horizontal line, which has slope 0.

(2) **Constant scaling:** If a function is scaled by a constant, then its derivative is scaled by the same constant.

For example, since the derivative of x^2 is $2x$, the derivative of $9x^2$ is $18x$ (scaling by 9) and the derivative of $-x^2$ is $-2x$ (scaling by -1).

We did not prove these rules formally in class but you should take some time do so on your own. (See the exercises at the bottom of this page.)

We then discovered the following more general rules:

(3) **Powers:** If $f(x) = x^a$ then $f'(x) = ax^{a-1}$.

For example, if $f(x) = x^7$ then $f'(x) = 7x^6$. If $g(x) = \sqrt[4]{x}$ then $g'(x) = \frac{1}{4}x^{-3/4}$.

Remember that this rule only applies when the exponent a is constant. In particular, it tells us nothing about the derivative of 7^x.

(4) **Sums and differences:** If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$. That is, the derivative of a sum or difference of functions is the sum or difference of their derivatives.

For example, if $f(x) = x^3 + x^4$, then $f'(x) = 3x^2 + 4x^3$. If $g(x) = x^2 - x^3$ then $g'(x) = 2x - 3x^2$.

Notice that the rule for differences is just a combination of the rule for sums along with the rule for scaling by -1.

(5) **Products:** If $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$.
For example, if $f(x) = (x^3 + x)(\sqrt{x} + 1)$ then

$$f'(x) = (3x^2 + 1)(\sqrt{x} + 1) + (x^3 + x) \cdot \frac{1}{2}x^{-1/2}$$

This rule can be iterated to find the derivative of a product of any number of functions. The essential thing to remember is that every function gets its turn. For instance, the derivative of the product fgh is $f'gh + fg'h + fgh'$.

(6) **Compositions:** If $f(x) = g(h(x))$, then $f'(x) = g'(h(x)) \cdot h'(x)$.
For example, if $f(x) = (1 + x^3)^5$, then $f'(x) = 5(1 + x^3)^4 \cdot 3x^2$.
The rule can be iterated to apply to larger chains of functions. For instance, if $f(x) = \sqrt[3]{1 + (1 + x^2)^7}$ then

$$f'(x) = \tfrac{1}{3}(1 - (1 + x^2)^7)^{-2/3} \cdot 7(1 + x^2)^6 \cdot 2x.$$

(7) **Quotients:** If $f(x) = \dfrac{g(x)}{h(x)}$ then $f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$.
For example, if $f(x) = \dfrac{x^3}{1 + x^2}$ then

$$f'(x) = \frac{3x^2(1 + x^2) - x^3(2x)}{(1 + x^2)^2}.$$

This rule can be viewed as a consequence of rules (2), (5) and (6). In particular, the quotient $g(x)/h(x)$ is the product of the functions $g(x)$ and $(h(x))^{-1}$, while $(h(x))^{-1}$ itself is the composition of the power function x^{-1} with $h(x)$. That's why the rule looks a little bit complicated.

Knowing only these rules, you are now able to "easily" find the derivative of any function that can be created by adding, subtracting, multiplying, dividing, exponentiating, or composing power functions! For instance, you can find the derivative of

$$f(x) = \left(\frac{(1 + x^3)^8 (x^{4/5} + \sqrt[3]{x^2 - 7})}{\sqrt{x - \sqrt{x^4}} + (x^4 - \sqrt{x})^2} \right)^{10}.$$

Of course, the answer will be horrendous. But getting to the answer is an entirely mechanical process: Just follow the rules above.

However, your powers are still limited. For instance, the rules above cannot be applied[1] to find the derivative of very common functions such as $f(x) = 2^x$ or $f(x) = \sin x$.

[1]This is a bit of a lie, and mathematically it's quite interesting why it's a lie. But if you're not interested, then just take it as the truth!

Next week we will learn new rules for dealing with exponential and trigonometric functions. You will notice that I have been very careful to avoid such functions in the assigned problems for this week. But as you are reading through the suggested sections of the textbook, you will encounter some exponential functions. You can either read this material and store it away for later, or you can safely ignore it for this week's homework.