

Saint Mary's University
DEPARTMENT OF MATHEMATICS
AND COMPUTING SCIENCE

Name: SOLUTIONS

Signature: _____

ID: _____

Math 1211: Fall 2011
Midterm Test #1

October 6, 2011

Recitation Section:

- Section A: 10:00–11:15R
- Section B: 11:30–12:45R

Instructions:

- No electronic devices or aids of any kind are to be in your immediate possession during the test. Possession of such items will be construed as an act of academic dishonesty.
- There are 4 pages plus this cover page. Check that your test paper is complete.
- There are a total of 45 marks. The value of each question is indicated in the margin.
- Answer in the spaces provided, using backs of pages for additional space if necessary.
- Show all your work. Insufficient justification will result in a loss of marks.

Page	Maximum	Your Score
1	13	
2	12	
3	12	
4	8	
Total	45	

- [9] 1. Find the area bounded between the curves $y = x^2 + 2x$ and $y = x + 2$. A proper sketch is required for full credit.

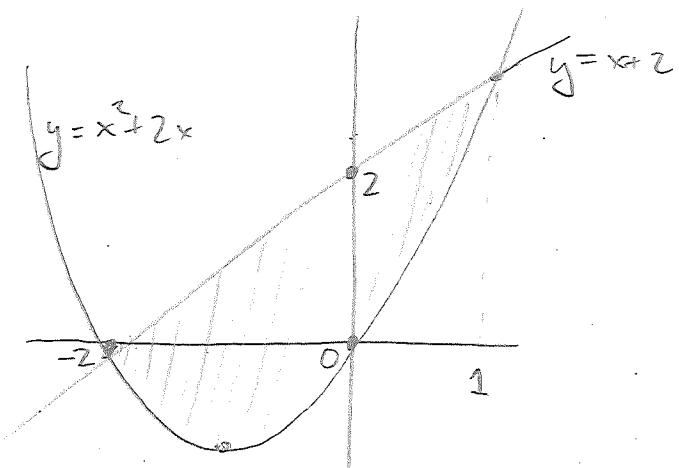
Find intersection points:

$$x^2 + 2x = x + 2$$

$$\Leftrightarrow x^2 + x - 2 = 0$$

$$\Leftrightarrow (x+2)(x-1) = 0$$

$$\Leftrightarrow x = -2 \text{ and } x = 1$$



$$\text{Area} = \int_{-2}^1 ((x+2) - (x^2 + 2x)) dx$$

$$= \int_{-2}^1 (2 - x - x^2) dx$$

$$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-2}^1$$

$$= \frac{7}{6} - \left(-\frac{10}{3} \right)$$

$$= \frac{27}{6}$$

$$= \boxed{\frac{9}{2}}$$

- [4] 2. Find the average value of the function $f(x) = \frac{1}{1+4x^2}$ over the interval $[-\frac{1}{2}, \frac{1}{2}]$.

$$\text{Avg Value} = \frac{1}{\frac{1}{2} - \left(-\frac{1}{2}\right)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{1+4x^2}$$

$$= \frac{1}{2} \tan^{-1}(2x) \Big|_{x=-\frac{1}{2}}^{x=\frac{1}{2}}$$

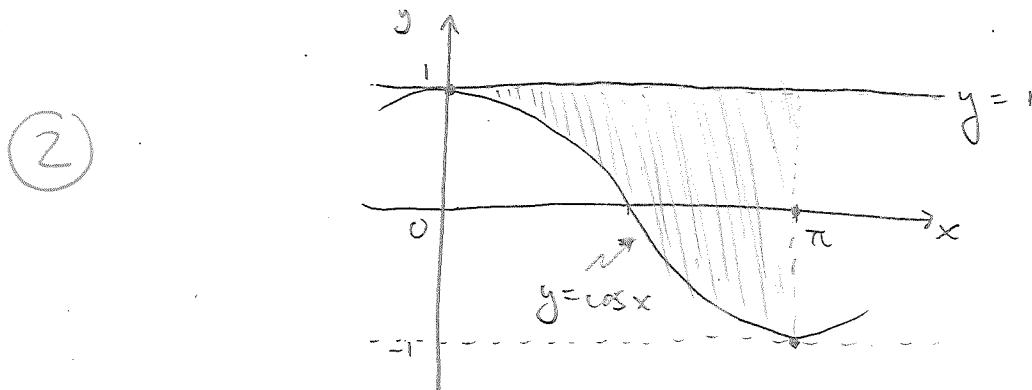
$$= \frac{1}{2} \tan^{-1}(1) - \frac{1}{2} \tan^{-1}(-1)$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \left(-\frac{\pi}{4} \right)$$

$$= \boxed{\frac{\pi}{4}}$$

- [12] 3. Let \mathcal{R} be the region bounded between the curves $y = \cos x$ and $y = 1$ on the interval $0 \leq x \leq \pi$.

(a) Sketch the region \mathcal{R} . Label all relevant points and curves.



- (b) Find the volume of the solid obtained by revolving \mathcal{R} around the y -axis.

We use the SHELLS method to get

$$\begin{aligned} \text{Volume} &= 2\pi \int_0^\pi x(1 - \cos x) dx \\ &= 2\pi \left[\int_0^\pi x dx - \int_0^\pi x \cos x dx \right] \end{aligned}$$

$$\text{But } \int_0^\pi x dx = \frac{1}{2}x^2 \Big|_0^\pi = \frac{\pi^2}{2}$$

$$\begin{aligned} \text{and } \int_0^\pi x \cos x dx &= x \sin x \Big|_0^\pi - \int_0^\pi \sin x dx \quad [\text{By PARTS}] \\ &= 0 + \cos x \Big|_0^\pi \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{So Volume} &= 2\pi \left(\frac{\pi^2}{2} + 2 \right) \\ &= \boxed{\pi^3 + 4\pi} \end{aligned}$$

- (c) Give an expression, in terms of a definite integral, for the solid obtained by revolving \mathcal{R} around the line $y = 2$.

Use the WASHERS method:

$$\text{Volume} = \pi \int_0^\pi ((2 - \cos x)^2 - 1) dx$$

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4. Evaluate the following integrals:

$$(a) \int_0^1 \frac{x^2}{\sqrt[3]{1+x^3}} dx$$

$$\begin{aligned} &= \frac{1}{2} (1+x^3)^{\frac{2}{3}} \Big|_0^1 \\ &= \frac{1}{2} \cdot 2^{\frac{2}{3}} - \frac{1}{2} \quad (4) \\ &= \boxed{\frac{2^{\frac{2}{3}} - 1}{2}} \end{aligned}$$

By inspection, or use substitution $u = 1+x^3$ to get $\frac{1}{3} \int_1^2 \frac{du}{\sqrt[3]{u}}$

$$(b) \int \frac{e^{3t}}{1-e^{3t}} dt$$

$$= -\frac{1}{3} \ln(1-e^{3t}) + C$$

By inspection, or substitute $u = 1-e^{3t}$ to get $-\frac{1}{3} \int \frac{du}{u}$

(3)

$$(c) \int x\sqrt{1+2x} dx$$

Let $u = 1+2x$, so $du = 2dx$ and $x = \frac{u-1}{2}$.

$$\begin{aligned} \text{Then } \int x\sqrt{1+2x} dx &= \int \frac{u-1}{2} \sqrt{u} \frac{du}{2} \\ &= \frac{1}{4} \int (u^{3/2} - u^{1/2}) du \\ &= \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \boxed{\frac{1}{10} (1+2x)^{5/2} - \frac{1}{6} (1+2x)^{3/2} + C} \end{aligned}$$

You can also do this by parts!

$$u = x \quad du = dx$$

$$dv = \sqrt{1+2x} \quad v = \frac{1}{3} (1+2x)^{3/2}$$

Note: This actually simplifies

$$\rightarrow \frac{1}{15} (3x-1)(1+2x)^{3/2} + C$$

[20]

$$(d) \int \sin^{-1} 2x \, dx$$

Use integration by parts: $u = \sin^{-1} 2x \quad \begin{cases} du = \frac{2 \, dx}{\sqrt{1-4x^2}} \\ dv = dx \end{cases}$ $v = x$

$$\text{Then } \int \sin^{-1} 2x \, dx = x \sin^{-1} 2x - 2 \int \frac{x}{\sqrt{1-4x^2}} \, dx$$

$$\textcircled{(4)} \quad = x \sin^{-1} 2x + \frac{1}{2} \sqrt{1-4x^2} + C$$

$$(e) \int_1^4 \frac{\ln x}{\sqrt{x}} \, dx$$

Integration by parts: $u = \ln x \quad du = \frac{1}{x} \, dx$
 $dv = x^{-1/2} \, dx \quad v = 2x^{1/2}$

④

$$\begin{aligned} \int_1^4 \frac{\ln x}{\sqrt{x}} \, dx &= 2\sqrt{x} \ln x \Big|_1^4 - 2 \int_1^4 x^{-1/2} \, dx \\ &= 4 \ln 4 - 4x^{1/2} \Big|_1^4 \\ &= 4 \ln 4 - 4 \end{aligned}$$

5. For extra credit, evaluate the integral you obtained in Question 3(c).

$$\begin{aligned} \pi \int_0^\pi ((2-\cos x)^2 - 1) \, dx &= \pi \int_0^\pi (3 - 4\cos x + \cos^2 x) \, dx \\ &= \pi \left[\int_0^\pi 3 \, dx - 4 \int_0^\pi \cos x \, dx + \int_0^\pi \cos^2 x \, dx \right] \\ &= \pi \left[3\pi + 0 + \int_0^\pi \left(\frac{1+\cos 2x}{2} \right) \, dx \right] \\ &= 3\pi^2 + \frac{\pi}{2} \int_0^\pi (1 + \cos 2x) \, dx \end{aligned}$$

UP TO +3

$$\begin{aligned} &= 3\pi^2 + \frac{\pi^2}{2}, \quad \text{since } \int_0^\pi \cos 2x \, dx = 0 \quad \text{by symmetry} \\ &= \boxed{\frac{7\pi^2}{2}} \end{aligned}$$