Examples of the Method of Partial Fractions

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Question: How do we compute $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx$?

Answer: The key is to factor the denominator to get

$$\frac{2x+7}{x^2+x-2} = \frac{2x+7}{(x+2)(x-1)},$$

and then cleverly observe that

$$\frac{2x+7}{(x+2)(x-1)} = \frac{3}{x-1} - \frac{1}{x+2}.$$

Notice that each of the terms on the right-hand-side is easy to integrate! So we can quickly deduce that

$$\begin{aligned}\int \frac{2x+7}{x^2+x-2}\,dx &= \int \left(\frac{3}{x-1} - \frac{1}{x+2}\right) dx \\ &= 3\ln|x-1| - \ln|x+2| + \text{constant}.\end{aligned}$$

This is a nice trick, but it's not nearly as clever as it looks. Once we know that our goal is to split up the fraction into "simple fractions", finding the way in which it splits up is a fairly routine computation.

As above, we start by factoring the denominator of the given fraction:

$$\frac{2x+7}{x^2+x-2} = \frac{2x+7}{(x+2)(x-1)},$$

Now we *assume* that this fraction can indeed be split in the form

$$\frac{2x+7}{(x+2)(x-1)} = \frac{A}{x-1} + \frac{B}{x+2},$$

where A and B are constants to be determined.

Note: Since we already know that $A = 3$ and $B = -1$ will work (see above), it may seem silly to say we're "assuming" that the fraction can split in this way. But if we hadn't already seen the answer, then the existence of A and B would indeed be only a fanciful wish at this point. The fact that this dream can always be made to come true is the mathematical heart of "partial fractions".

Multiply both sides of the previous equation by $(x+2)(x-1)$ to clear fractions, giving[1]

$$2x + 7 = A(x+2) + B(x-1).$$

Collect like terms to get

$$2x + 7 = (A+B)x + (2A - B)$$

and "compare coefficients" to deduce that

$$\begin{aligned} 2 &= A + B \\ 7 &= 2A - B. \end{aligned}$$

Solving for A and B now gives $A = 3$ and $B = -1$. So we have again arrived at the suddenly-not-so-clever decomposition

$$\frac{2x+7}{(x+2)(x-1)} = \frac{3}{x-1} - \frac{1}{x+2}.$$

Example: Find $\displaystyle\int \frac{x+1}{2x^2+5x-3}\,dx$.

Solution: Factor the denominator as $(2x-1)(x+3)$ and assume the decomposition

$$\frac{x+1}{(2x-1)(x+3)} = \frac{A}{2x-1} + \frac{B}{x+3}.$$

Multiply through by $(2x-1)(x+3)$ to get

$$\begin{aligned} x + 1 &= A(x+3) + B(2x-1) \\ &= (A+2B)x + (3A - B). \end{aligned}$$

Comparing coefficients gives $\{A+2B = 1,\ 3A-B = 1\}$, and solving this system yields $A = \frac{3}{7}$ and $B = \frac{2}{7}$. Therefore:

$$\begin{aligned} \int \frac{x+1}{2x^2+5x-3}\,dx &= \int \left(\frac{3/7}{2x-1} + \frac{2/7}{x+3} \right) dx \\ &= \frac{3}{14}\ln|2x-1| + \frac{2}{7}\ln|x+3| + \text{constant}. \end{aligned}$$

[1]This is equivalent to adding $\frac{A}{x-1}$ and $\frac{B}{x+2}$ over the common denominator $(x-1)(x+2)$ to get $\frac{A(x+2)+B(x-1)}{(x+2)(x-1)} = \frac{2x+7}{(x+2)(x-1)}$, and then equating the numerators on both sides of this equation.

A Convenient Trick: In this example we first obtained $x+1 = A(x+3)+B(2x-1)$, and from here we collected like terms and compared coefficients to get a system of equations for A and B. There is another way of finding A and B from this equation that is often quite convenient. Since

$$x+1 = A(x+3)+B(2x-1),$$

is supposed to hold true for all values of x, we can substitute any values of x that we like. Notice that every value of x will produce an equation involving A and B. For instance, substituting $x = 0$ gives

$$1 = 3A - B,$$

whereas setting $x = 1$ gives

$$2 = 4A + B.$$

Solving these two equations gives $A = \frac{3}{7}$ and $B = \frac{2}{7}$, as before.

Even better, setting $x = -3$ and $x = \frac{1}{2}$ yields $-2 = -7B$ and $\frac{3}{2} = \frac{7}{2}A$, so we *instantly* get $A = \frac{3}{7}$ and $B = \frac{2}{7}$. I'm sure you can see that I didn't choose the values $x = -3$ and $x = \frac{1}{2}$ out of thin air. Where do they come from?

Example: Find $\displaystyle\int \frac{x^3-3}{x^2-2x-3}dx$

Solution: Proceeding as before, let's factor the denominator as $(x-3)(x+1)$ and assume the partial fractions decomposition

$$\frac{x^3-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}.$$

Multiplying by $(x-3)(x+1)$ gives

$$x^3 - 3 = A(x+1) + B(x-3),$$

and upon setting $x = -1$ and $x = 3$ this gives $-4 = -4B$ and $24 = 4A$. So we happily conclude that $A = 6$ and $B = 1$.

However, notice that if we instead decided to set $x = 0$ and $x = 1$, we would be led to $\{-3 = A+B,\ -2 = 2A-2B\}$ and therefore $A = -2$ and $B = -1$. These values are different don't agree with our first attempt... and this makes my mathematical BS sensor start to tingle.

In fact, both attempts are incorrect. A quick check shows that $\frac{A}{x-3} + \frac{B}{x+1}$ does *not* equal $\frac{x^3-3}{(x-3)(x+1)}$ for either of our values of A and B. Looking back, we see why: The identity

$$x^3 - 3 = A(x+1) + B(x-3)$$

can't possibly hold true for *any* constants A and B. Why? Because the left-hand side is cubic and the right-hand side is linear! So our error can be traced back to our original *assumption* that the fraction could be split as $\frac{A}{x-3} + \frac{B}{x+1}$. The assumption is simply false in this case.

Take a moment to think about this, and you'll see that we'll run into exactly the same trouble whenever the *numerator* of the original fraction has degree bigger than or equal to that of the *denominator*.

Fortunately, this problem is easy to overcome. We simply reduce the size of the numerator by first performing long division.

In the present example, long division gives

$$\frac{x^3-3}{x^2-2x-3} = x+2+\frac{7x+3}{x^2-2x-3}.$$

Thus we have

$$\int \frac{x^3-3}{x^2-2x-3}dx = \frac{1}{2}x^2+2x+\int \frac{7x+3}{x^2-2x-3}\,dx.$$

Now we can focus on the right-hand integral, and this time our method will work because the numerator is smaller than the denominator. Try this for yourself. You should end up with

$$\begin{aligned}\int \frac{7x+3}{x^2-2x-3}\,dx &= \int \left(\frac{6}{x-3}+\frac{1}{x+1}\right)dx \\ &= 6\ln|x-3|+\ln|x+1|+\text{constant}.\end{aligned}$$

Example: Find $\displaystyle\int \frac{x^2+x-3}{x^3-3x-2}dx$

Solution: First we factor the denominator as $x^3-3x-2=(x+1)^2(x-2)$. If we were to blindly follow the previous examples, we might be tempted to start with

$$\frac{x^2+x-3}{(x+1)^2(x-2)} = \frac{A}{x+1}+\frac{B}{x-2}.$$

But this is doomed to failure, since adding the fractions on the right-hand side will always result in a denominator of $(x+1)(x-2)$, and *not* $(x+1)^2(x-2)$. The fact that $x=-1$ is a "double root" has thrown a wrench into our plan.

Again, it's a situation that's easy to deal with. We just assume the more general decomposition

$$\frac{x^2+x-3}{(x+1)^2(x-2)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-2},$$

for some constants A, B, and C. You may rightly be a little confused as to exactly why this is the correct form, but it's not too hard to prove to yourself with a little thought. In any case, notice that: (1) we at least get the right denominator when we add the fractions on the right-hand side, and (2) each of these fractions is easy to integrate.

Multiply the previous equation by $(x+1)^2(x-2)$ to get

$$x^2+x-3 = A(x+1)(x-2)+B(x-2)+C(x+1)^2.$$

Let $x=-1$, $x=2$ and $x=0$ (why these values?) to obtain

$$\begin{aligned} -3 &= -3B \\ 3 &= 9C \\ -3 &= -2A - 2B + C. \end{aligned}$$

From here we readily get $A=\frac{2}{3}$, $B=1$, and $C=\frac{1}{3}$. Therefore

$$\begin{aligned} \int \frac{x^2+x-3}{x^3-3x-2}dx &= \int \left(\frac{2/3}{x+1} + \frac{1}{(x+1)^2} + \frac{1/3}{x-2} \right) dx \\ &= \frac{2}{3}\ln|x+1| - \frac{1}{x+1} + \frac{1}{3}\ln|x-2| + \text{constant}. \end{aligned}$$

Example: Find $\displaystyle\int \frac{125x}{(x-2)^3(2x+1)^2}dx$

Solution: Isn't that nice? Someone already factored the denominator for us. In analogy with the previous example, this time we use the partial fractions form

$$\frac{125x}{(x-2)^3(2x+1)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{2x+1} + \frac{E}{(2x+1)^2}.$$

Multiply through by $(x-2)^3(2x+1)^2$ to clear fractions, and then set $x = 2, -\frac{1}{2}, 0, 1,$ and -1 in the resulting equation to find that $A=\frac{4}{5}$, $B=-3$, $C=10$, $D=-\frac{8}{5}$, $E=4$. Therefore

$$\begin{aligned} \int \frac{125x}{(x-2)^3(2x+1)^2}dx &= \int \left(\frac{4/5}{x-2} - \frac{3}{(x-2)^2} + \frac{10}{(x-2)^3} - \frac{8/5}{2x+1} + \frac{4}{(2x+1)^2} \right) dx \\ &= \frac{4}{5}\ln|x-2| + \frac{3}{x-2} - \frac{5}{(x-2)^2} - \frac{4}{5}\ln|2x+1| - \frac{2}{2x+1} + \text{constant}. \end{aligned}$$

Example: Find $\displaystyle\int \frac{1+x}{x^2+4}\,dx.$

Solution: Try as we might, we can't factor the denominator x^2+4 without leaving the comfort of the real numbers and introducing the famous $i=\sqrt{-1}$. But first year calculus is no place for complex numbers![2]

[2] I can't resist mentioning that if you were so bold as to factor x^2+4 as $(x+2i)(x-2i)$, then you *could* write down a partial fractions decomposition $\frac{A}{x+2i} + \frac{B}{x-2i}$, then determine *complex* constants A and B as usual, and integrate the result to get $\int \frac{1+x}{x^2+4}\,dx = A\ln|x+2i| + B\ln|x-2i|$. This looks very strange, but Euler's master equation $e^{i\theta} = \cos\theta + i\sin\theta$, which links the exponential function with the trig functions, can be used to show that $\ln|x\pm 2i| = \frac{1}{2}\ln(x^2+4) \mp i\tan^{-1}\frac{x}{2}$. You will then find that, with the correct values of A and B, all the *imaginary* stuff cancels and we are left with a *real* answer. If you think this is cool you should take more math courses.

Our fancy new technique does not help us here. Instead we do it the old-fashioned way, by substitution: First split up the numerator to get

$$\int \frac{1+x}{x^2+4}\,dx = \int \frac{1}{x^2+4}\,dx + \int \frac{x}{x^2+4}\,dx.$$

Now use the substitution $x = 2\tan\theta$ for the integral on the left, and $u = x^2+4$ on the right. You should find that

$$\int \frac{1+x}{x^2+4}dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\ln(x^2+4) + \text{constant}.$$

Example: Find $\displaystyle\int \frac{3x+7}{4-4x+x^2-x^3}\,dx$.

Solution: Here the denominator *does* factor, as $(1-x)(4+x^2)$. The naive partial fractions approach would begin with

$$\frac{3x+7}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{B}{4+x^2},$$

from which we get

$$\begin{aligned} 3x+7 &= A(4+x^2) + B(1-x) \\ &= Ax^2 - Bx + (4A+B). \end{aligned}$$

Comparing coefficients of x^2 and x on the left- and right-hand sides gives $A = 0$ and $B = -3$. But clearly these values of A and B give mismatched constant terms on the left and right. So (as usual!) something has gone wrong.

Giving our little failure some careful thought will leave you unsurprised by how we avoid this latest trouble. We instead begin with the partial fractions decomposition

$$\frac{3x+7}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{B+Cx}{4+x^2}.$$

This leads to

$$\begin{aligned} 3x+7 &= A(4+x^2) + (B+Cx)(1-x) \\ &= (A-C)x^2 + (C-B)x + (4A+B). \end{aligned}$$

and comparing coefficients now gives $\{A-C=0, C-B=3, 4A+B=7\}$, which leads to $A=2$, $B=-1$, $C=2$. Therefore

$$\begin{aligned} \int \frac{3x+7}{4-4x+x^2-x^3}\,dx &= \int \left(\frac{2}{1-x} + \frac{-1+2x}{4+x^2}\right)dx \\ &= 2\int \frac{dx}{1-x} - \int \frac{dx}{4+x^2} + \int \frac{2x}{4+x^2}dx \\ &= -2\ln|1-x| - \frac{1}{2}\tan^{-1}\frac{x}{2} + \ln|4+x^2| + \text{constant}. \end{aligned}$$

Example: Find $\int \frac{x^3}{(x^2+4x+5)^2}\,dx$.

Solution: The quadratic x^2+4x+5 does not factor any further. By this point you won't be surprised that, because two copies of it appear in the denominator, we start with the partial fraction form

$$\frac{x^3}{(x^2+4x+5)^2} = \frac{A+Bx}{x^2+4x+5} + \frac{C+Dx}{(x^2+4x+5)^2}.$$

Clear fractions as usual, by multiplying through by $(x^2+4x+5)^2$, to get

$$x^3 = (A+Bx)(x^2+4x+5) + (C+Dx).$$

We could substitute in 4 values of x to obtain 4 equations in the unknowns A, B, C, D. But in this case there are no values of x that make the substitution particular "simple", so we're just as well off to compare coefficients. Either way, check that you end up with $A = -4$, $B = 1$, $C = 20$, $D = 11$. Therefore

$$\int \frac{x^3}{(x^2+4x+5)^2}\,dx = \int \left(\frac{x-4}{x^2+4x+5} + \frac{20+11x}{(x^2+4x+5)^2} \right) dx.$$

Unfortunately, we still have a lot of work to do. The first step is to "complete the square" and write

$$x^2+4x+5 = (x+2)^2+1.$$

Then we can substitute $u = x+2$ integrals to get

$$\begin{aligned}
\int \frac{x^3}{(x^2+4x+5)^2}\,dx &= \int \left(\frac{u-6}{u^2+1} + \frac{11u-2}{(u^2+1)^2} \right) du \\
&= \int \frac{u}{u^2+1} du - 6\int \frac{du}{u^2+1} + 11\int \frac{u}{(u^2+1)^2} du - 2\int \frac{du}{(u^2+1)^2} \\
&= \frac{1}{2}\ln(u^2+1) - 6\tan^{-1} u - \frac{11}{2(u^2+1)} - 2\left(\frac{u}{2(u^2+1)} + \frac{1}{2}\tan^{-1} u \right),
\end{aligned}$$

where the last line follows by performing the substitution $z = u^2+1$ on the first and third integrals, and $u = \tan\theta$ on the second and fourth.

Collecting like terms and converting back from u to x gives the final result:

$$\int \frac{x^3}{(x^2+4x+5)^2}\,dx = \frac{1}{2}\ln|x^2+4x+5| - 7\tan^{-1}(x+2) - \frac{2x+15}{2(x^2+4x+5)} + \text{constant}.$$

Example: Find $\displaystyle\int \frac{2x^6+9x^4+x^3}{x^4+6x^2+9}dx$

Solution: We start with long division (because the numerator is bigger than the denominator) and then factor the denominator. Putting these steps together, we have:

$$\begin{aligned}\int \frac{2x^6+9x^4+x^3}{x^4+6x^2+9}dx &= \int \left(2x^2-3+\frac{27+x^3}{x^4+6x^2+9}\right)dx\\ &= \frac{2}{3}x^3-3x+\int \frac{27+x^3}{(x^2+3)^2}\,dx.\end{aligned}$$

Now we follow the usual partial fractions route for the remaining integral:

$$\begin{aligned} & \frac{27+x^3}{(x^2+3)^2} = \frac{A+Bx}{x^2+3}+\frac{C+Dx}{(x^2+3)^2}\\ \implies\quad & 27+x^3 = (A+Bx)(x^2+3)+C+Dx\\ \implies\quad & 27+x^3 = Bx^3+Ax^2+(3B+D)x+(3A+C)\\ \implies\quad & \{B=1, A=0, 3B+D=0, 3A+C=27\}\\ \implies\quad & A=0,\ B=1,\ C=27,\ D=-3.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{27+x^3}{(x^2+3)^2}dx &= \int \left(\frac{x}{x^2+3}+\frac{27-3x}{(x^2+3)^2}\right)dx\\ &= \int \frac{x}{x^2+3}\,dx + 27\int \frac{dx}{(x^2+3)^2} - 3\int \frac{x}{(x^2+3)^2}dx\\ &= \frac{1}{2}\ln(x^2+3)+27\left(\frac{x}{6(x^2+3)}+\frac{\sqrt{3}}{18}\tan^{-1}(x\sqrt{3})\right)-\frac{3}{2(x^2+3)}+\text{constant}\\ &= \frac{1}{2}\ln(x^2+3)+\frac{9x-3}{2(x^2+3)}+\frac{3\sqrt{3}}{2}\tan^{-1}(x\sqrt{3})+\text{constant}\end{aligned}$$

where we tackled the integrals via the substitutions $u=x^2+3$ and $x=\sqrt{3}\tan\theta$.

Finally, collecting everything together gives the final answer, namely:

$$\int \frac{2x^6+9x^4+x^3}{x^4+6x^2+9}dx = \frac{2x^3}{3}-3x+\frac{\ln(x^2+3)}{2}+\frac{9x-3}{2(x^2+3)}+\frac{3\sqrt{3}}{2}\tan^{-1}(x\sqrt{3})+\text{constant}.$$

Phew!