MAT 1211 : Quiz #6A

Solutions

PROBLEM 1: $\displaystyle\int \frac{x^2}{x^2+2x-3}\,dx$

SOLUTION: First divide to get $\dfrac{x^2}{x^2+2x-3} = 1 + \dfrac{3-2x}{x^2+2x-3}$.

Then let $\dfrac{3-2x}{x^2+2x-3} = \dfrac{3-2x}{(x+3)(x-1)} = \dfrac{A}{x+3} + \dfrac{B}{x-1}$

This gives $3-2x = A(x-1) + B(x+3)$. Set $x=1$ to get $1 = 4B \Rightarrow B = 1/4$
Set $x=-3$ to get $9 = -4A \Rightarrow A = -9/4$.

Thus $\displaystyle\int \frac{x^2}{x^2+2x-3}\,dx = \int \left(1 - \frac{9}{4}\cdot\frac{1}{x+3} + \frac{1}{4}\cdot\frac{1}{x-1}\right)dx$

$$= x - \frac{9}{4}\ln|x+3| + \frac{1}{4}\ln|x-1| + C.$$

PROBLEM 2: $\displaystyle\int \frac{2x-1}{x^2(x+2)}\,dx$

SOLUTION: Let $\dfrac{2x-1}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$, so $2x-1 = Ax(x+2) + B(x+2) + Cx^2$.

Set $x=0$ to get $-1 = 2B \Rightarrow B = -1/2$
Set $x=-2$ to get $-5 = 4C \Rightarrow C = -5/4$
Compare coefficients of x^2 to get $0 = A + C \Rightarrow A = 5/4$

Thus $\displaystyle\int \frac{2x-1}{x^2(x+2)}\,dx = \int \left(\frac{5}{4x} - \frac{1}{2x^2} - \frac{5}{4}\cdot\frac{1}{x+2}\right)dx$

$$= \frac{5}{4}\ln|x| + \frac{1}{2x} - \frac{5}{4}\ln|x+2| + C$$

PROBLEM 3: $\displaystyle\int \frac{2x^2-3x+5}{(x-1)(x^2-2x+5)}\,dx$

SOLUTION: Let $\dfrac{2x^2-3x+5}{(x-1)(x^2-2x+5)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2-2x+5}$

Then $2x^2-3x+5 = A(x^2-2x+5) + (Bx+C)(x-1)$.

So $x=1 \Rightarrow 4=4A \Rightarrow A=1$, $x=0 \Rightarrow 5=5A-C \Rightarrow C=0$

Compare coefficients of x^2 to get $2=A+B \Rightarrow B=1$.

Thus $\displaystyle\int \frac{2x^2-3x+5}{(x-1)(x^2-2x+5)}\,dx = \int \left(\frac{1}{x-1} + \frac{x}{x^2-2x+5}\right)dx$

$$= \ln|x-1| + \int \frac{x}{(x-1)^2+4}\,dx$$

$$= \ln|x-1| + \int \left(\frac{x-1}{(x-1)^2+4} + \frac{1}{(x-1)^2+4}\right)dx$$

$$= \ln|x-1| + \tfrac{1}{2}\ln\left((x-1)^2+4\right) + \tfrac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

$$= \ln|x-1| + \tfrac{1}{2}\ln(x^2-2x+5) + \tfrac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C.$$

PROBLEM 4: $\displaystyle\int \sin\sqrt{x}\,dx$

SOLUTION: Set $z=\sqrt{x}$, so $dz = \dfrac{1}{2\sqrt{x}}\,dx$, and thus $dx = 2\sqrt{x}\,dz = 2z\,dz$.

Then $\displaystyle\int \sin\sqrt{x}\,dx = 2\int z\sin z\,dz$.

Use integration by parts, with $\left\{\begin{matrix} u=z & dv=\sin z\,dz \\ du=dz, & v=-\cos z\,dz \end{matrix}\right\}$

This gives $\displaystyle\int z\sin z\,dz = -z\cos z + \int \cos z\,dz = -z\cos z + \sin z + C$.

Hence $\displaystyle\int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$.