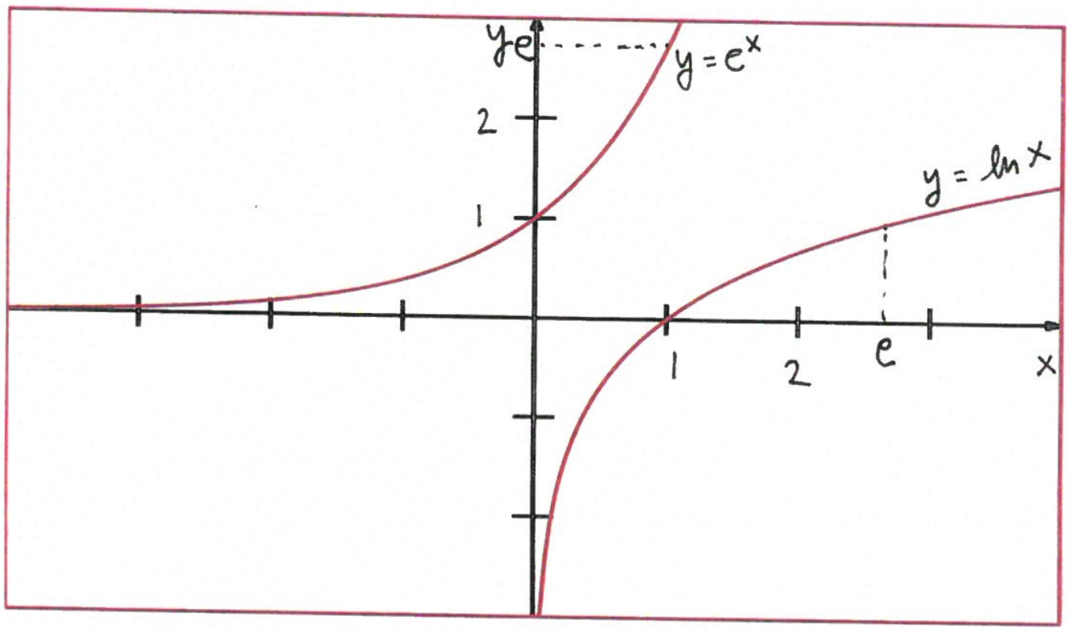
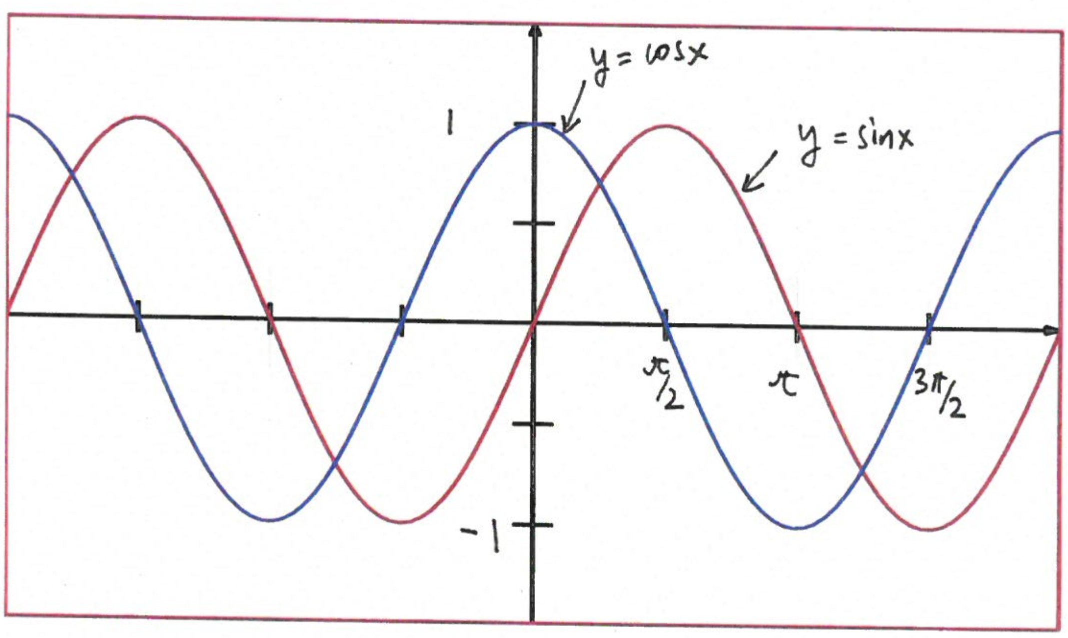
Math 1210: Worksheet #1 **Winter 2017**

Name: SOLUTION	A#:	Section: A,B

[4] 1. In the coordinate system below sketch the graphs of the functions $y = e^x$ and $y = \ln(x)$. Make sure that you clearly label the coordinate system and each curve.



[4] 2. In the coordinate system below sketch the graphs of the functions $y = \sin(x)$, $y = \cos(x)$. Make sure that you clearly label the coordinate system and each curve.



[4] 3. Find the equation of the secant line to the curve $y = e^x$ on the interval $[\ln(2), \ln(6)]$. Write the equation in the "slope, y-intercept form" and simplify as much as possible.

points : $(\ln 2, 2)$, $(\ln 6, 6)$.

slope $= \frac{6-2}{\ln 6 - \ln 2} = \frac{4}{\ln 3}$

line : $y - 2 = \frac{4}{\ln 3}(x - \ln 2)$

$$\boxed{y = \frac{4}{\ln 3}x - \frac{4 \ln 2}{\ln 3} + 2}$$

[2] 4. If $f(x) = \ln(x^2 + x + 1)$, $g(x) = e^x$ and $h(x) = x + 1$, then

$(f \circ g \circ h)(x) = f(g(h(x))) = \boxed{\ln\left(e^{2(x+1)} + e^{x+1} + 1\right)}$

[6] 5. Perform the required task and **simplify**.

(a) Write as a sum of summands of the form x^a. $\dfrac{x^2 + \sqrt[5]{x}}{x^3} = \boxed{x^{-1} + x^{-\frac{14}{5}}}$

(b) Put on common denominator. $\dfrac{1}{x-2} - \dfrac{2}{x} = \dfrac{x - 2(x-2)}{x(x-2)} = \boxed{\dfrac{4-x}{x(x-2)}}$

(c) Expand. $(x^2 + x + 3)(x^2 - 2x + 3) = x^4 - 2x^3 + 3x^2 + x^3 - 2x^2 + 3x + 3x^2 - 6x + 9$

$= \boxed{x^4 - x^3 + 4x^2 - 3x + 9}$

(d) Rationalize and cancel if possible. $\dfrac{x+2}{\sqrt{x^2+x+7} - 3} = \dfrac{(x+2)(\sqrt{x^2+x+7}+3)}{x^2+x+7-9}$

$= \dfrac{(x+2)(\sqrt{x^2+x+7}+3)}{x^2+x-2} = \dfrac{(x+2)(\sqrt{x^2+x+7}+3)}{(x+2)(x-1)} = \boxed{\dfrac{\sqrt{x^2+x+7}+3}{x-1}}$

(e) $\dfrac{(e^x)^{x-1}e^{x-1}}{e^{2-3x}} = e^{x(x-1)+(x-1)-(2-3x)} = e^{x^2-x+x-1-2+3x} = \boxed{e^{x^2+3x-3}}$

(f) $\ln(3e^x/5) + \ln(5e^x) - \ln(3e^{x^3}) = \ln\left(\dfrac{\frac{3e^x}{5} \cdot 5e^x}{3e^{x^3}}\right) = \ln\left(e^{2x-x^3}\right)$

$= \boxed{2x - x^3}$

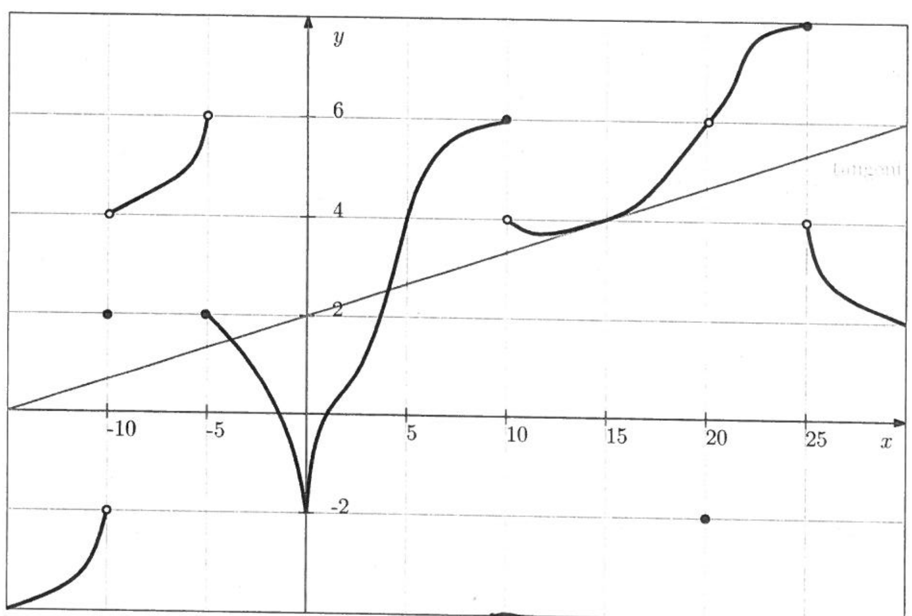
Math 1210: Worksheet #2 **Fall 2017**

Name: SOLUTION	A#:	Section: A, B

[2] 1. State the formal (ε, δ) definition of $\lim_{x\to a} f(x) = L$. Part marks will be given for an informal definition.

For every $\varepsilon > 0$ there is $\delta > 0$ such that for $|x-a| < \delta$ we have $|f(x) - L| < \varepsilon$

[5] 2. Let f be a function whose graph of $y = f(x)$ is given below.



(a) $\lim_{x\to 15} \dfrac{f(x)}{x^2 - 10x} = \dfrac{4}{(15)^2 - (10)(15)} = \boxed{\dfrac{4}{75}}$

(b) $\lim_{x\to 10^+} (2\ln(x) - 2f(x)) = (2\ln(10)) - 2(4) = \boxed{2(\ln 10) - 8}$

(c) List all values of a in the interval $(-15, 30)$ for which limit $\lim_{x\to a} f(x)$ does not exist:

$-10, -5, 10, 25$

(d) The average rate of change of $f(x)$ over the interval $[-5, 10]$ is $\dfrac{4}{15}$

(e) The instantaneous rate of change of $f(x)$ when $x = 15$ is $\dfrac{2}{15}$

[3] 3. Let $f(x) = \frac{1}{x}$ and let $a \neq 0$. Find the instantaneous rate of change of $f(x)$ when $x = a$, as a function (call it g) of a (you are not allowed to use any theory of derivatives).

$$g(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = \lim_{x\to a} \frac{\frac{a-x}{ax}}{x-a}$$

$$= \lim_{x\to a} \frac{(a-x)}{(x-a)\cdot ax} = \lim_{x\to a}\left(-\frac{1}{ax}\right) = \boxed{-\frac{1}{a^2}}$$

4. Compute the limit or show that it does not exist.

[3] (a) $\lim_{x\to 2} \dfrac{\sqrt{x^2+5}-\sqrt{5x-1}}{x^2+3x-10} = \lim_{x\to 2} \dfrac{(\sqrt{x^2+5}-\sqrt{5x-1})(\sqrt{x^2+5}+\sqrt{5x-1})}{(x-2)(x+5)(\sqrt{x^2+5}+\sqrt{5x-1})}$

can skip

$= \lim_{x\to 2} \dfrac{(x^2+5)-(5x-1)}{(x-2)(x+5)(\sqrt{x^2+5}+\sqrt{5x-1})} = \lim_{x\to 2} \dfrac{x^2-5x+6}{(x-2)(x+5)(\sqrt{x^2+5}+\sqrt{5x-1})}$

$= \lim_{x\to 2} \dfrac{(x-2)(x-3)}{(x-2)(x+5)(\sqrt{x^2+5}+\sqrt{5x-1})} = \lim_{x\to 2} \dfrac{x-3}{(x+5)(\sqrt{x^2+5}+\sqrt{5x-1})}$

can. skip

$= \dfrac{2-3}{(2+5)(\sqrt{2^2+5}+\sqrt{5(2)-1})} = \dfrac{-1}{(7)(6)} = \boxed{-\dfrac{1}{42}}$

[3] (b) $\lim_{x\to 2^-} \left(\ln(2-x) - \ln(6-x-x^2)\right) = \lim_{x\to 2^-} \ln\left(\dfrac{2-x}{6-x-x^2}\right) = \lim_{x\to 2^-} \ln\left(\dfrac{2-x}{(2-x)(3+x)}\right)$

$= \lim_{x\to 2^-} \ln\left(\dfrac{1}{3+x}\right) = \ln\left(\dfrac{1}{3+2}\right) = \ln\left(\dfrac{1}{5}\right) = \boxed{-\ln 5}$

[4] (c) $\lim_{x\to 2} \dfrac{|4-x^2|}{x^2-x-2}$ d.n.e.

$\lim_{x\to 2^-} \dfrac{|4-x^2|}{x^2-x-2} = \lim_{x\to 2^-} \dfrac{4-x^2}{x^2-x-2} = \lim_{x\to 2^-} \dfrac{(2-x)(2+x)}{(x-2)(x+1)} = \lim_{x\to 2^-} \dfrac{-(2+x)}{x+1} = \boxed{-\dfrac{4}{3}}$

$\lim_{x\to 2^+} \dfrac{|4-x^2|}{x^2-x-2} = \lim_{x\to 2^+} \dfrac{-(4-x^2)}{x^2-x-2} = \lim_{x\to 2^+} \dfrac{-(2-x)(2+x)}{(x-2)(x+1)} = \lim_{x\to 2^+} \dfrac{+(2+x)}{x+1} = \boxed{+\dfrac{4}{3}}$

Math 1210: Worksheet #3 **Fall 2017**

Name: SOLUTION	A#:	Section: A, B

[2] 1.

(a) List vertical asymptotes of

$$y = \frac{x^2+3x-10}{x^2-4} + \ln|x^2-x| = \frac{(x-2)(x+5)}{(x-2)(x+2)} + \ln|x(x-1)|$$

V.a.: $x=-2,\ x=0,\ x=1$

(b) List horizontal asymptotes of $y = \tan^{-1}(x)$.

H.a: $y=\frac{\pi}{2},\ y=-\frac{\pi}{2}$

[6] 2. Find horizontal asymptotes of $y = \dfrac{1-3x}{\sqrt{x^2-x+3}}$

$$\lim_{x\to\infty} \frac{1-3x}{\sqrt{x^2-x+3}} = \lim_{x\to\infty} \frac{\frac{1}{x}-3}{\sqrt{1-\frac{1}{x}+\frac{3}{x^2}}} = \frac{0-3}{\sqrt{1-0+0}} = \boxed{-3}$$

$$\lim_{x\to-\infty} \frac{1-3x}{\sqrt{x^2-x+3}} = \lim_{x\to-\infty} \frac{\frac{1}{x}-3}{-\sqrt{1-\frac{1}{x}+\frac{3}{x^2}}} = \boxed{3}$$

H.a.: $\boxed{y=-3}$ and $\boxed{y=3}$

[12] 3. Compute the limits (as a number, as ∞ , as $-\infty$, or as 'does not exist'; whichever is most precise).

(a) $\lim\limits_{x\to 1^+} \frac{x-2}{x^2+2x-3} = \lim\limits_{x\to 1^+} \frac{\overset{\to -1}{(x-2)}}{\underset{\downarrow\, 4}{(x+3)}\underset{\downarrow\, 0^+}{(x-1)}} = \lim\limits_{x\to 1^+} \frac{-1}{4(x-1)} = \boxed{-\infty}$

(b) $\lim\limits_{x\to 0} \frac{2\sin x}{x} + \lim\limits_{t\to -\infty} \frac{\sin t}{2t} + \lim\limits_{s\to \frac{\pi}{6}} \frac{\sin s}{s} = 2\cdot 1 + 0 + \frac{\sin(\pi/6)}{\frac{\pi}{6}} = \boxed{2+\frac{3}{\pi}}$

(c) $\lim\limits_{x\to\infty} \left(e^x - \sqrt{e^{2x}-e^x+1}\right) = \lim\limits_{x\to\infty} \frac{(e^x-\sqrt{e^{2x}-e^x+1})(e^x+\sqrt{e^{2x}-e^x+1})}{e^x+\sqrt{e^{2x}-e^x+1}}$

$= \lim\limits_{x\to\infty} \frac{e^{2x}-(e^{2x}-e^x+1)}{e^x+\sqrt{e^{2x}-e^x+1}} = \lim\limits_{x\to\infty} \frac{e^x-1}{e^x+\sqrt{e^{2x}-e^x+1}} = \lim\limits_{x\to\infty} \frac{1-e^{-x}}{1+\sqrt{1-e^{-x}+e^{-2x}}}$

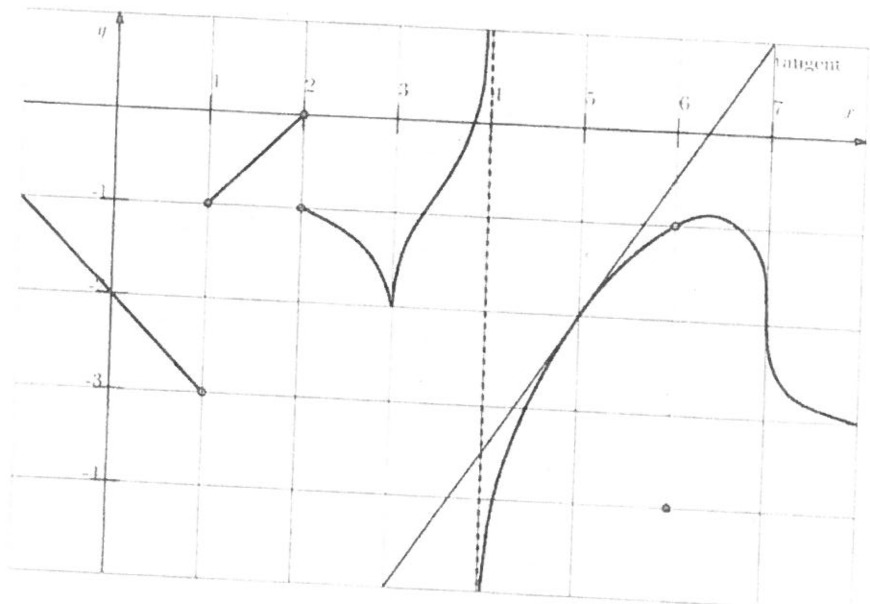
$= \frac{1-0}{1+\sqrt{1-0+0}} = \boxed{\frac{1}{2}}$

(d) $\lim\limits_{x\to -\infty} \left(x^3 - 3x + 2e^{+x}\right) = \lim\limits_{x\to -\infty} x^3\left(1 - \overset{\to 0}{\frac{3}{x^2}} + \overset{\to 0}{\frac{2e^{+x}}{x^3}}\right) = \boxed{-\infty}$

Math 1210: Worksheet #4 Fall 2017

Name: SOLUTION	A#:	Section: A,B

[8] 1. Let f be a function whose graph of $y = f(x)$ is given below.



Fill in the following.

(a) List all x where f is not continuous: 1, 2, 4, 6

(b) List all x where f is left-continuous, but f is not right-continuous: 2

(c) List all x where f is continuous, but not differentiable: 3, 7

(d) $\lim_{t\to 4^+} e^{f(t)} = \lim_{y\to -\infty} e^y = \boxed{0}$

(e) If $g(x) = \dfrac{f(2x+1)}{x^2+1}$, then $g'(x) = \dfrac{2f'(\overset{2x+1}{\text{[redacted]}})(x^2+1) - 2x\, f(\overset{2x+1}{\text{[redacted]}})}{(x^2+1)^2}$ and $g'(2) = \dfrac{23}{25}$

(f) If $h(x) = e^x f(x)$, then $h'(x) = e^x f(x) + e^x f'(x)$ and the equation of the tangent line to the curve $y = h(x)$ at $x = 5$ is $y + 2e^5 = -\frac{1}{2}e^5(x-5)$

$h(5) = e^5 f(5) = e^5(-2) = -2e^5$

$h'(5) = e^5 f(5) + e^5 f'(5) = e^5(-2) + e^5\left(\frac{3}{2}\right) = -\frac{1}{2}e^5$

$g'(2) = \dfrac{2f'(5)\cdot 5 - 2(2)f(5)}{5^2} = \dfrac{2\cdot\frac{3}{2}\cdot 5 - 2\cdot 2\cdot(-2)}{25} = \dfrac{23}{25}$

[2] 2. Find a function f and a number a such that

$$f'(a) = \lim_{h\to 0} \frac{(3+h)^3 + \tan^{-1}(3(3+h)+2) - 27 - \tan^{-1}(11)}{h}$$

$a =$ 3

$f(x) = x^3 + \tan^{-1}(3x+2)$

[6] 3. Find values of a and b such that the function

$$f(x) = \begin{cases} \tan^{-1}\left(\frac{1}{1-x}\right) & , \text{ for } x < 1 \\ a & , \text{ for } x = 1 \\ b\sin^{-1}\left(\frac{x-1}{x^2-1}\right) & , \text{ for } x > 1 \end{cases}$$

is continuous ~~everywhere~~ everywhere.

$$\lim_{x \to 1^-} f(x) = \lim_{x\to 1^-} \tan^{-1}\left(\frac{1}{1-x}\right) = \lim_{y \to \infty} \tan^{-1}(y) = \frac{\pi}{2}.$$

$f(1) = a.$

$$\lim_{x\to 1^+} f(x) = \lim_{x \to 1^+} b\sin^{-1}\left(\frac{\cancel{x-1}}{(\cancel{x-1})(x+1)}\right) = b\sin^{-1}\left(\frac{1}{1+1}\right) = b\sin^{-1}\left(\frac{1}{2}\right) = b\cdot\frac{\pi}{6}.$$

For function to be continuous at 1 we need $a = \frac{\pi}{2} = b \cdot \frac{\pi}{6}$, this happens if and only if $\boxed{a = \frac{\pi}{2} \text{ and } b = 3}$

[4] 4. Compute the derivative. **Do not simplify.**

$$\frac{d}{dt}\left(\frac{\sqrt[3]{t}\cos(t)}{1+\sec(2t+1)}\right) =$$

$$= \frac{\left(\frac{1}{3}t^{-2/3}\cos t - \sqrt[3]{t}\sin(t)\right)(1+\sec(2t+1)) - \sqrt[3]{t}\cdot\cos t \cdot 2 \cdot \sec(2t+1)\tan(2t+1)}{(1+\sec(2t+1))^2}$$

Math 1210: Worksheet #5 Fall 2017

Name: SOLUTION	A#:	Section:

[12] 1. Compute the derivative. **Do not simplify.**

(a) $\frac{d}{du}\left(\sec(e^{\sqrt{u}})\right) = \sec(e^{\sqrt{u}})\cdot\tan(e^{\sqrt{u}})\cdot e^{\sqrt{u}}\cdot\frac{1}{2\sqrt{u}}$

(b) $\frac{d}{dx}\left(x\ln(e^x+1)\tan^{-1}(x^2)\right) = \ln(e^x+1)\tan^{-1}(x^2) + x\frac{e^x}{e^x+1}\tan^{-1}(x^2)$

$+ x\ln(e^x+1)\frac{2x}{1+x^4}$

or $(x^2)^2$.

(c) $\frac{d}{dx}\left((\sin^{-1}(x))^{\sin(x)}\right) = \frac{d}{dx}e^{\sin x\ln(\sin^{-1}x)} = e^{\sin x\ln(\sin^{-1}x)}\cdot$

$\cdot\left((\cos x)\ln(\sin^{-1}x) + \sin x\cdot\frac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1}x}\right)$

(d) $\frac{d}{dt}\frac{\ln(t^4+1)}{t^4+1} = \frac{\frac{4t^3}{t^4+1}(t^4+1) - (\ln(t^4+1))4t^3}{(t^4+1)^2}$.

[4] 2. Find the equation of the tangent line to the curve

$$x + y - 1 = x\cos(y)$$

at the point $(0, 1)$.

$$1 + y' = \cos y + x(-\sin y)\,y'$$

$$y'(1 + x\sin y) = \cos y - 1 .$$

$$y' = \frac{\cos y - 1}{1 + x\sin y}$$

$$y'\Big|_{\substack{x=0\\ y=1}} = \frac{\cos(1) - 1}{1 + 0\cdot\sin(1)} = \cos(1) - 1 .$$

$$\boxed{y - 1 = (\cos(1) - 1)\,x}$$

[4] 3. Find the equation of the tangent line to the curve

$$3(x^2 + y^2)^2 = 25(x^2 - y^2)$$

at the point $(2, 1)$. (the curve is called lemniscate)

$$6(x^2+y^2)(2x + 2yy') = 25(2x - 2yy') .$$

$$y'(12y(x^2+y^2) + 50y) = 50x - 12x(x^2+y^2) .$$

$$y' = \frac{50x - 12x(x^2+y^2)}{12y(x^2+y^2) + 50y}$$

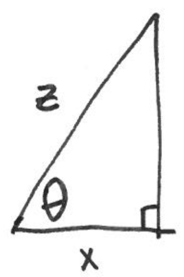
$$y'\Big|_{\substack{x=2\\ y=1}} = \frac{50(2) - 12(2)(2^2+1^2)}{12(1)(2^2+1^2) + 50(1)} = \frac{-20}{110} = -\frac{2}{11}$$

Tangent: $\boxed{y - 1 = -\frac{2}{11}(x - 2)}$

Math 1210: Worksheet #6 Fall 2017

Name: SOLUTION	A#:	Section:

[6] 1. Consider a right-angled triangle with catheti (adjacent sides) x and y and hypothenuse (opposite side) z. When $x = 5$ and $y = 12$ we have that x is growing at the rate of $7cm/s$ and the angle θ between x and z is growing at the rate of $\frac{1}{4}rad/s$, (the right angle $\frac{\pi}{2}$ between x and y is fixed throughout the process). What is the rate of change of z at that point?



$$\frac{x}{z} = \cos\theta.$$

$$x = z \cdot \cos\theta$$

$$\frac{dx}{dt} = \frac{dz}{dt}\cos\theta + z(-\sin\theta)\frac{d\theta}{dt}$$

$$\frac{dz}{dt} = \frac{\frac{dx}{dt} + z\sin\theta\frac{d\theta}{dt}}{\cos\theta}$$

When $x=5, y=12$, we have $z = \sqrt{x^2+y^2} = 13$, $\sin\theta = \frac{12}{13}$ and $\cos\theta = \frac{5}{13}$

$$\left.\frac{dz}{dt}\right|_{\substack{x=5\\y=12}} = \frac{7 + 13 \cdot \frac{12}{13}\cdot\frac{1}{4}}{\frac{5}{13}} = \frac{10 \cdot 13}{5} = \boxed{26}$$

[3] 2. Let x and y be functions of t related by $x^2 + y^2 = xy + 7$. When $x = 3$ and $y = 2$ we have that $\frac{dx}{dt} = -2$. Find the value of $\frac{dy}{dt}$ at that point.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$

$$\frac{dy}{dt}(2y - x) = \frac{dx}{dt}(y - 2x).$$

$$\frac{dy}{dt} = \frac{dx}{dt}\,\frac{y-2x}{2y-x}.$$

$$\left.\frac{dy}{dt}\right|_{\substack{x=3\\y=2}} = (-2)\,\frac{2-2(3)}{2(2)-3} = \boxed{8}$$

[3] 3. Let f be a function whose **derivative** is

$$f'(x) = \frac{(x+2)^4 \ln(x^2) e^x}{\sqrt[3]{x+9}}.$$

List the intervals where f is increasing and list as well as classify all critical ~~values~~ points.

loc min, neither, loc. max, neither, loc. min.

f': $-$ | $+$ | $+$ | $-$ | $-$ | $+$ at -9, -2, -1, 0, 1

this is sufficient for classification

Critical points: -9 (loc. min), -2 (not. ext.), -1 (loc. max), 0 (not ext.), 1 (loc. min)

Increasing: $(-9,-2), (-2,-1), (1,\infty)$

[8] 4. Find the global maximum and the global minimum of $f(x) = (x^2 - 2x)e^x$ on the interval $[0, 3]$. For **5 bonus marks** find the global maximum and minimum of f on $(-\infty,$ [redacted]$]$ or explain why they do not exist.

$f'(x) = (2x-2)e^x + (x^2-2x)e^x = (x^2-2)e^x.$

Critical points: $-\sqrt{2}, \sqrt{2}$

$-\sqrt{2}$ not in the interval.

x	$f(x)$	
0	0	
$\sqrt{2}$	$(2-2\sqrt{2})e^{\sqrt{2}}$	min.
3	$3e^3$	max

Bonus:

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x^2-2x)e^x = 0$

$f(-\sqrt{2}) = ((\sqrt{2})^2 - 2(-\sqrt{2}))e^{-\sqrt{2}} = \boxed{(2\sqrt{2}+2)e^{-\sqrt{2}} \quad \text{max}}$

$f(-1) = 3e^{-1}.$

$f(-1) < f(-\sqrt{2})$ as f is decreasing on the interval $(-\sqrt{2}, \sqrt{2})$, as $f'(x) < 0$ there.

$\boxed{\text{Min. does not exist}}$

v6.AB

Math 1210: Worksheet #7 Fall 2017

Name: Solutions (Micah)	A#:	Section:

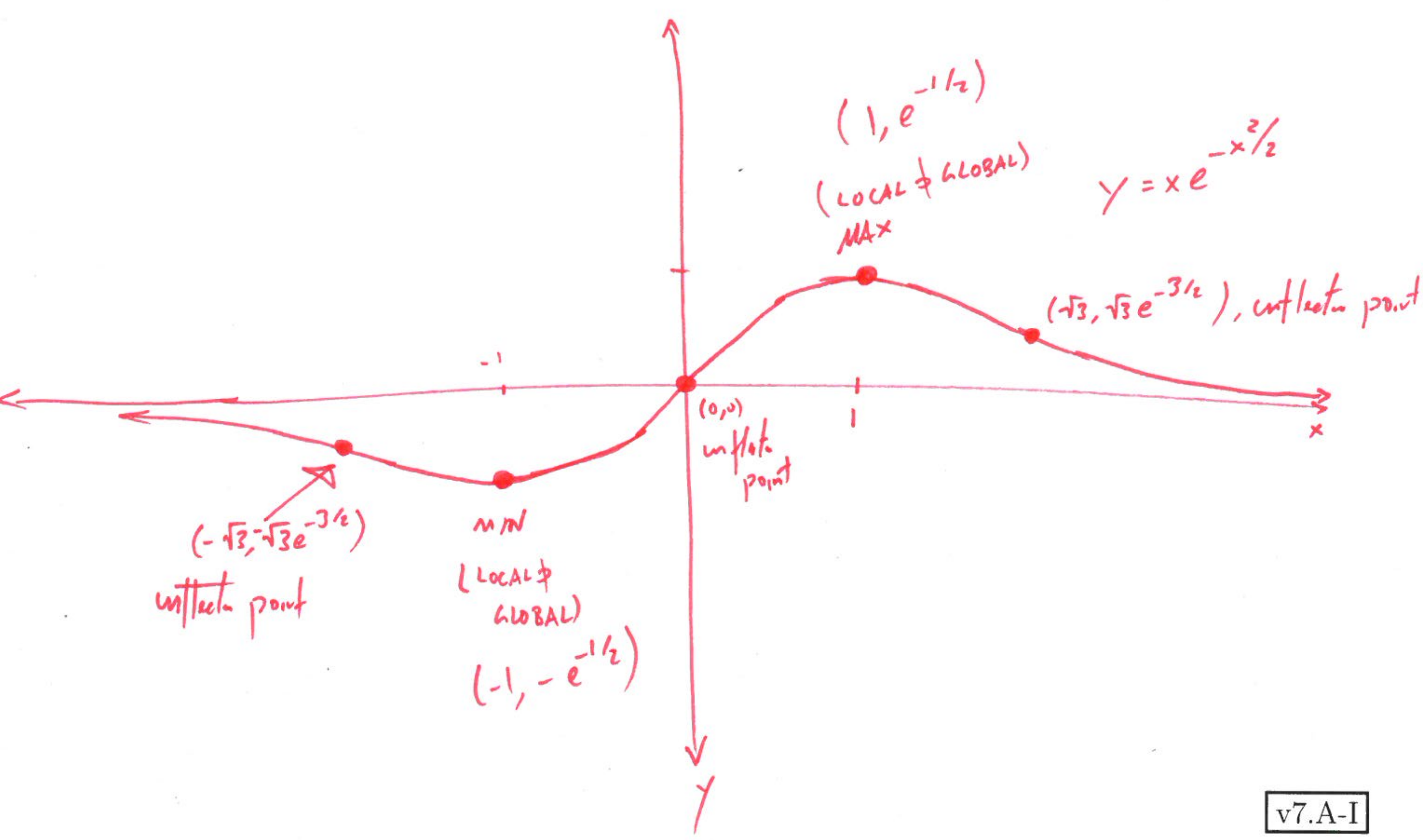
[8] 1. Sketch the graph of $y = xe^{-x^2/2}$. You can use $y' = (1-x^2)e^{-x^2/2}$ and $y'' = x(x^2-3)e^{-x^2/2}$. Make sure that your graph clearly shows any x and y intercepts, horizontal asymptotes, vertical asymptotes, local extrema, intervals where the function is increasing and where it is decreasing, concavity, and inflection points.

$\lim_{x\to+\infty} xe^{-x^2/2} = 0$, $\lim_{x\to-\infty} xe^{-x^2/2} = 0$, these are horizontal asymptotes.

when $x=0$, $y=0$, this is the only time $y=0$. The function is odd.

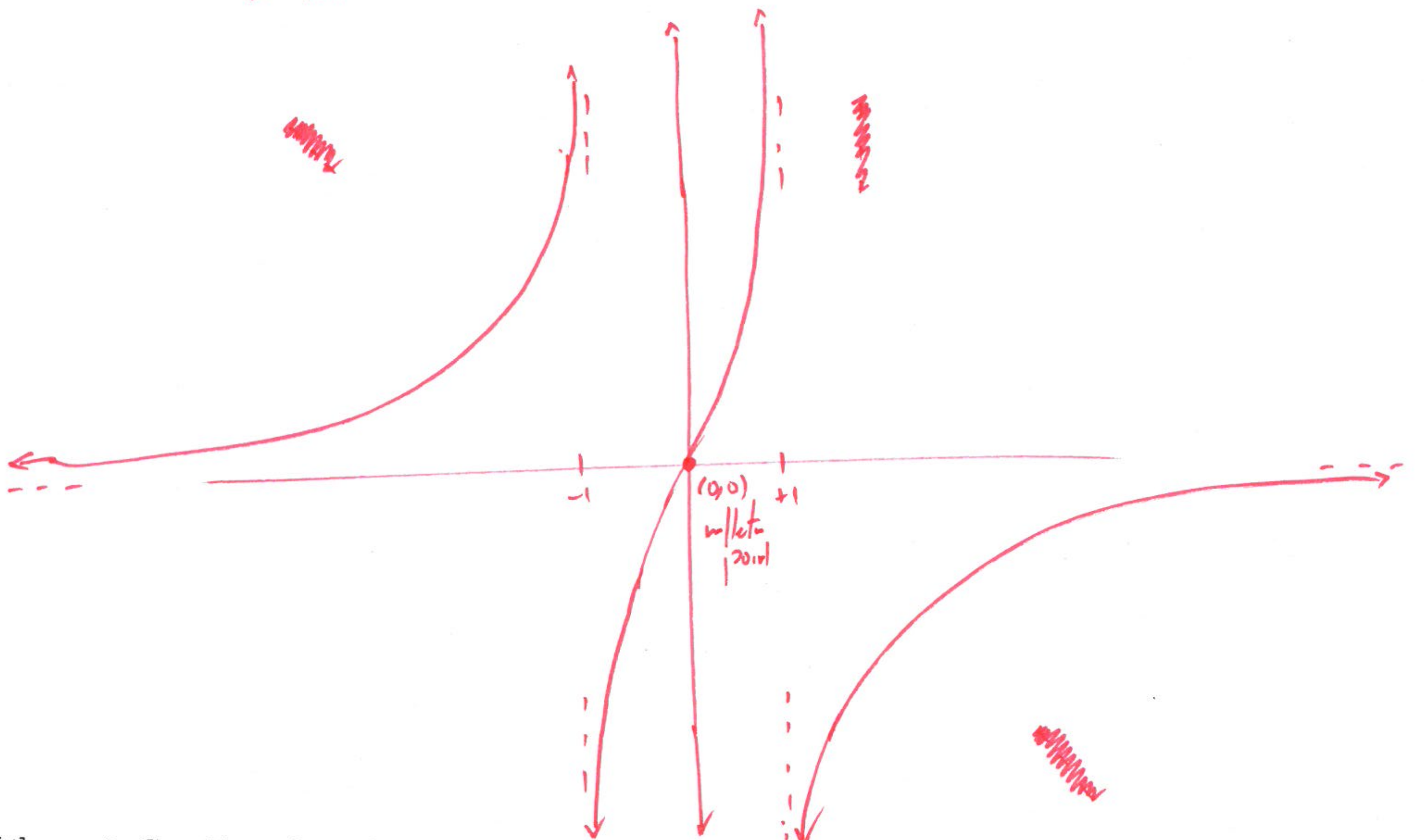
y' is everywhere defined, hence critical points occur when $y'=0$, that is, $x=\pm 1$.

y' is positive when $x \in (-1,+1)$ and negative on $(-\infty,-1)$ and $(1,\infty)$, hence y is increasing on $(-1,+1)$ and decreasing on $(-\infty,-1)$ and $(1,\infty)$. Candidates for inflection points are when $y''=0$, that is, when $x = 0, \pm\sqrt{3}$. Since $y''>0$ when $x\in(-\sqrt{3},0)$ and $x \in (\sqrt{3},\infty)$, hence y is concave up there; similarly y is concave down on $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$ since $y''<0$ on those intervals. Hence we see that $0, \pm\sqrt{3}$ are inflection points and that $+1$ is a max and that -1 is a min.



[8] 2. Sketch the graph of $y = \frac{x}{1-x^2}$. You can use $y' = \frac{1+x^2}{(1-x^2)^2}$ and $y'' = \frac{2x(x^2+3)}{(1-x^2)^3}$. Make sure that your graph clearly shows any x and y intercepts, horizontal asymptotes, vertical asymptotes, local extrema, intervals where the function is increasing and where it is decreasing, concavity, and inflection points.

As $x \to +\infty$, $y \to 0$, and as $x \to -\infty$, $y \to 0$. When $x = 0$, $y = 0$, this is the only place where $y = 0$. y is undefined at $x = \pm 1$ and has vertical asymptotes there. Since $y' > 0$ for all x except ± 1 where it is undefined, it is increasing on $(-\infty, -1)$ and $(-1, +1)$ and $(1, \infty)$ and has critical points at $x = \pm 1$. $y'' = 0$ has roots $x = 0$ so this is the only candidate for an inflection point, since $y'' > 0$ when $x \in (-\infty, -1)$ and $(0, 1)$, y is concave up on these intervals, similarly it is concave down on $(-1, 0)$ and $(1, +\infty)$ since $y'' < 0$ on those intervals. We see then that $x = 0$ is an inflection point.



[4] 3. Consider a hypothetical function f defined everywhere with the property that f has a discontinuity at 3 and critical points at $-5, -3, 1, 7, 9$. Find the global maximum and global minimum, or state that they do not exists, of f on $I = [0, \infty)$ for the following cases.

(a)

$\lim_{x\to-\infty} f(x)$	$f(-5)$	$f(-3)$	$f(0)$	$f(1)$	$\lim_{x\to3^-} f(x)$	$f(3)$	$\lim_{x\to3^+} f(x)$	$f(7)$	$f(9)$	$\lim_{x\to\infty} f(x)$
∞	-42	15	7	0	-7	-5	9	10	4	3

Maximum $x = 7$, $f(7) = 10$ **Minimum** DOES NOT EXIST

(b)

$\lim_{x\to-\infty} f(x)$	$f(-5)$	$f(-3)$	$f(0)$	$f(1)$	$\lim_{x\to3^-} f(x)$	$f(3)$	$\lim_{x\to3^+} f(x)$	$f(7)$	$f(9)$	$\lim_{x\to\infty} f(x)$
$-\infty$	-42	5	5	7	5	9	9	10	14	∞

Maximum DOES NOT EXIST **Minimum** $x = 0$, $f(0) = 5$

Name: SOLUTION	A#:	Section:

[5] 1. Find the linearisation $L(x)$ of $f(x) = \tan^{-1}(x)$ centred at 1.

$f'(x) = \dfrac{1}{1+x^2}$, $f'(1) = \frac{1}{2}$, $f(1) = \tan^{-1}(1) = \frac{\pi}{4}$

$$\boxed{L(x) = \frac{\pi}{4} + \frac{1}{2}(x-1)}$$

[5] 2. Estimate $\sqrt{23}$. Is your estimate larger or smaller then $\sqrt{23}$? Justify your answer.

$f(x) = \sqrt{x}$, $a = 25$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $f(25) = 5$, $f'(25) = \frac{1}{10}$

$$\boxed{L(x) = 5 + \frac{1}{10}(x-25)}$$

$$\sqrt{23} \approx L(23) = 5 + \frac{1}{10}(23-25) = 5 - \frac{2}{10} = \boxed{\frac{24}{5}}$$

$f''(x) = \dfrac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = -\frac{1}{4}x^{-3/2}$. This is negative on $[23,25]$,

so $\boxed{\frac{24}{5} > \sqrt{23}}$

[5] 3. Find the point(s) on the curve $y = \sqrt{x+1}$, $x \geq -1$, that is closest to the point $(0, 0)$.
(Hint: minimize the square of the distance)

$f(x) = \text{Distance}^2 = x^2 + y^2 = x^2 + x + 1$.

$f'(x) = 2x + 1$. Critical point $= -\frac{1}{2}$

f': $-$ on the left of $-\frac{1}{2}$, $+$ on the right.

$f(-1) = 1$, $f(-\frac{1}{2}) = \frac{3}{4}$, $\lim_{x \to \infty} f(x) = \infty$

(or f has a local min at $-\frac{1}{2}$ & $-\frac{1}{2}$ is the only crit. point).

So $\boxed{\left(-\frac{1}{2}, \sqrt{\frac{1}{2}}\right)}$ is the closest point.

[5] 4. Find the point(s) (x, y) on the curve $y = -x^3 + 8x^2 - 10x$ with largest value of $P = xy$.

$P = xy = x(-x^3 + 8x^2 - 10x) = -x^4 + 8x^3 - 10x^2$

$P' = -4x^3 + 24x^2 - 20x = -4x(x^2 - 6x + 5) = -4x(x-5)(x-1)$.

Critical points : $0, 1, 5$

$\lim_{x \to -\infty} P(x) = -\infty$, $P(0) = 0$, $P(1) = -3$, $P(5) = 125$, $\lim_{x \to \infty} P(x) = -\infty$.

Maximum is at $x = 5$, the point is $\boxed{(5, 25)}$.

Math 1210: Worksheet #Fall Break Fall 2017

Name: SOLUTION	A#:	Section:

1. Compute the limit and explain why L'Hôpital's rule does not apply.

   (a) $\lim_{x\to\frac{\pi}{4}} \frac{\sin x}{x} = \frac{\sin(\pi/4)}{\pi/4} = \frac{2\sqrt{2}}{\pi}$

   L'Hôpital's rule does not apply because we do not have an indeterminate form.

   (b) $\lim_{x\to\infty} \frac{\ln x + 3x + \cos x}{3\ln x + 3x + 3\cos x} = \lim_{x\to\infty} \frac{\frac{\ln x}{x} + 3 + \frac{\cos x}{x}}{3\frac{\ln x}{x} + 3 + \frac{3\cos x}{x}} = \frac{0+3+0}{0+3+0} = 1.$

   $\lim_{x\to\infty} \frac{\frac{1}{x} + 3 - \sin x}{\frac{3}{x} + 3 - 3\sin x} = \lim_{x\to\infty} \frac{3-\sin x}{3-3\sin x}$ does not exist; so L'Hôpital's rule does not apply because it produces a limit that does not exist.

2. Compute the limit.

   (a) $\lim_{x\to\frac{\pi}{2}} (2x-\pi)\sec x = \lim_{x\to\pi/2} \frac{2x-\pi}{\cos x} \overset{\text{L'Hôpital "}\frac{0}{0}\text{"}}{=} \lim_{x\to\pi/2} \frac{2}{-\sin x} = \frac{2}{-1} = \boxed{-2}$

   (b) $\lim_{x\to 1} \frac{x - \ln x - 1}{x^2 - 2x + 1} \overset{\text{L'Hôpital "}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{1-\frac{1}{x}}{2x-2} = \lim_{x\to 1} \frac{x-1}{2x(x-1)} = \lim_{x\to 1} \frac{1}{2x} = \boxed{\frac{1}{2}}$

(c) $\lim_{x\to 0^+} (\sin x)^x = e^{\lim_{x\to 0^+} \ln((\sin x)^x)} = e^0 = \boxed{1}$

$\ln((\sin x)^x) = x \ln(\sin x)$

$\lim_{x\to 0^+} x \ln(\sin x) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cdot \cos x}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -\frac{x^2 \cos x}{\sin x}$

L'Hôpital "$\frac{-\infty}{\infty}$"

$= \lim_{x\to 0^+} (-x)\cdot\frac{\sin x}{x}\cdot \cos x$

$= 0 \cdot 1 \cdot 1 = \boxed{0}$

(d) $\lim_{x\to 0} \frac{e^{2x} - 1 - 2x - 2x^2}{x\cos x - x}$ L'Hôpital "$\frac{0}{0}$" $= \lim_{x\to 0} \frac{2e^{2x} - 2 - 4x}{\cos x - x\sin x - 1}$ L'Hôpital "$\frac{0}{0}$" $= \lim_{x\to 0} \frac{4e^{2x} - 4}{-\sin x - \sin x - x\cos x}$

L'Hôpital "$\frac{0}{0}$" $= \lim_{x\to 0} \frac{8e^{2x}}{-2\cos x - \cos x + x\sin x} = \frac{8e^0}{-3\cos(0) + 0\cdot\sin(0)} = \boxed{-\frac{8}{3}}$

3. Prove that the functions $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$ differ by a constant. (Hint: compute the derivatives.)

$$\left.\begin{aligned} f'(x) &= 2\sec x \cdot \sec x \cdot \tan x = 2\sec^2 x \tan x \\ g'(x) &= 2\tan x \cdot \sec^2 x \end{aligned}\right\} \therefore f'(x) = g'(x).$$

4. Compute the indefinite integrals.

(a) $\int \left(e^{3x} + \frac{3}{1+(2x+1)^2} - 2\sec^2(5x) + 2\sin(\pi x) - \frac{4}{\sqrt{1-(x-1)^2}} \right) dx$

$= \frac{1}{3} e^{3x} + \frac{3}{2}\tan^{-1}(2x+1) - \frac{2}{5}\tan(5x) - \frac{2}{\pi}\cos(\pi x) - 4\sin^{-1}(x-1) + C$

(b) $\int \frac{1 + 2\sqrt{x} + x + x^3}{x^3} dx = \int (x^{-3} + 2x^{-\frac{5}{2}} + x^{-2} + 1)\, dx$

$= \frac{x^{-2}}{-2} + 2\cdot\frac{2}{3} x^{-\frac{3}{2}} + \frac{x^{-1}}{-1} + x + C.$

Math 1210: Worksheet #9 **Fall 2017**

Name: SOLUTION	A#:	Section:

[2] 1. Suppose that $F(x) = x^x$ is an antiderivative of $f(x)$. Find $f(x)$.

$f(x) = F'(x) = \frac{d}{dx} e^{x\ln x} = e^{x\ln x}(\ln x + x\cdot\frac{1}{x}) = x^x(\ln x + 1)$.

[6] 2. Solve the initial value problem $y'' = \frac{x^2+\sqrt{x}}{x}$, $y(1) = 1$, $y'(1) = 0$.

$y' = \int y\,dx = \int(x + x^{-\frac{1}{2}})dx = \frac{1}{2}x^2 + 2x^{\frac{1}{2}} + C$.

$0 = y'(1) = \frac{1}{2} + 2 + C \quad \therefore \quad C = -\frac{5}{2}$.

$y' = \frac{1}{2}x^2 + 2x^{\frac{1}{2}} - \frac{5}{2}$.

$y = \int y'\,dx = \int(\frac{1}{2}x^2 + 2x^{\frac{1}{2}} - \frac{5}{2})\,dx = \frac{1}{6}x^3 + \frac{4}{3}x^{\frac{3}{2}} - \frac{5}{2}x + D$.

$1 = y(1) = \frac{1}{6} + \frac{4}{3} - \frac{5}{2} + D = -1 + D \quad \therefore \quad D = 2$.

$$\boxed{y = \frac{1}{6}x^3 + \frac{4}{3}x^{3/2} - \frac{5}{2}x + 2}$$

[3] 3. If $F(x) = \int_{-\ln x}^{\sin x} e^{t^2}\,dt$, then compute $F'(x)$.

$F'(x) = -(-\frac{1}{x})e^{(-\ln x)^2} + \cos x\, e^{(\sin x)^2} = \frac{1}{x}e^{(\ln x)^2} + \cos x\, e^{\sin^2 x}$

[9] 4. Compute the integral.

(a) $\int \frac{1}{\sqrt{2x-x^2}}dx$ (Hint: complete the square)

$= \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \sin^{-1}(x-1) + C$

(b) $\int_0^1 \frac{e^{2x}-e^{-2x}}{e^{x+1}}dx = \int_0^1 (e^{x-1} - e^{-3x-1})dx = \left(e^{x-1} + \frac{1}{3}e^{-3x-1}\right)\Big|_0^1$

$= \left(e^0 + \frac{1}{3}e^{-4}\right) - \left(e^{-1} + \frac{1}{3}e^{-1}\right) = \boxed{1 - \frac{4}{3}e^{-1} + \frac{1}{3}e^{-4}}$

(c) $\int_1^{\sqrt{3}} \frac{1}{3+x^2}dx = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\Big|_1^{\sqrt{3}} = \frac{1}{\sqrt{3}}\tan^{-1}(1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \boxed{\frac{\pi}{12\sqrt{3}}}$

by inspection, or $\int \frac{1}{3+x^2}dx = \frac{1}{3}\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}dx = \frac{1}{3}\cdot\left(\frac{1}{\frac{1}{\sqrt{3}}}\right)\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C.$

Math 1210: Worksheet #10 Fall 2017

Name: solution	A#:	Section:

[3] 1. $\int_0^{\frac{\pi}{4}} \sin(x)\sec^6(x)dx$

$$= \int_0^{\pi/4} \tan(x)\sec^5(x)\,dx \qquad \begin{cases} \text{let } u = \sec x \\ du = \sec x \tan x\, dx \end{cases} \qquad \begin{cases} x=0 \to u = \sec(0) = 1 \\ x = \frac{\pi}{4} \to u = \sec(\pi/4) = \sqrt{2} \end{cases}$$

$$= \int_0^{\pi/4} \sec^4(x)\sec(x)\tan(x)\,dx$$

$$= \int_1^{\sqrt{2}} u^4\,du$$

$$= \frac{u^5}{5}\Big]_1^{\sqrt{2}}$$

$$= \frac{1}{5}\left((\sqrt{2})^5 - 1\right) = \frac{1}{5}\left(4\sqrt{2}-1\right)$$

[3] 2. $\int_0^{\ln(2)} \frac{e^{2x}}{e^{2x}+1}dx = \int_0^{\ln(2)} \frac{e^{2x}}{e^{2x}+1}dx$ $\qquad \begin{cases} \text{let } u = e^{2x}+1 \\ du = 2e^{2x}dx \rightsquigarrow \frac{du}{2} = e^{2x}dx \end{cases}$

$$\begin{cases} x = 0 \to u = e^0 + 1 = 2 \\ x = \ln 2 \to u = e^{2\ln 2} + 1 = 5 \end{cases}$$

$$= \int_2^5 \frac{1}{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\int_2^5 \frac{1}{u}\,du$$

$$= \frac{1}{2}\left[\ln|u|\right]_2^5$$

$$= \frac{1}{2}\left[\ln 5 - \ln 2\right] = \boxed{\frac{1}{2}\ln\left(5/2\right)}$$

[3] 3. $\int_0^{\frac{1}{2}\ln(3)} \frac{e^x}{e^{2x}+1}dx = \int_0^{\frac{1}{2}\ln(3)} \frac{e^x}{(e^x)^2+1}dx$ $\qquad \begin{cases} \text{let } u = e^x \\ du = e^x dx \end{cases}$

$$\begin{cases} x = 0 \to u = e^0 = 1 \\ x = \frac{1}{2}\ln(3) \to u = e^{\frac{1}{2}\ln(3)} = \sqrt{3} \end{cases}$$

$$= \int_1^{\sqrt{3}} \frac{1}{u^2+1}\,du$$

$$= \left[\tan^{-1}(u)\right]_1^{\sqrt{3}}$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \boxed{\frac{\pi}{12}}$$

or $u = x+1$ also works!

[3] 4. $\int_0^1 \frac{x^2-1}{\sqrt{1+x}}dx = \int_1^{\sqrt{2}} [(u^2-1)^2-1]\,2\,du$

$$\left\{\begin{array}{l} \text{let } u=\sqrt{1+x} \rightsquigarrow u^2 = 1+x \Rightarrow x = (u^2-1) \\ du = \frac{1}{2\sqrt{1+x}}dx \\ 2du = \frac{1}{\sqrt{1+x}}dx \end{array}\right.$$

$= 2\int_1^{\sqrt{2}} [u^4+1-2u^2-1]\,du$

$= 2\int_1^{\sqrt{2}} (u^4-2u^2)\,du$

$= 2\left[\frac{u^5}{5} - \frac{2u^3}{3}\right]_1^{\sqrt{2}}$

$$\left\{\begin{array}{l} x=0 \to u=1 \\ x=1 \to u=\sqrt{2} \end{array}\right.$$

$= 2\left[\left(\frac{(\sqrt{2})^5}{5} - \frac{2}{3}(\sqrt{2})^3\right) - \left(\frac{1}{5}-\frac{2}{3}\right)\right]$

$= 2 \cdot \frac{4\sqrt{2}}{5} - \frac{4}{3}\cdot 2\sqrt{2} - \frac{2}{5} + \frac{4}{3} = \frac{8\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} - \frac{2}{5} + \frac{4}{3} = \frac{14-16\sqrt{2}}{15}$

[4] 5. $\int \frac{x+2}{x^2+4}dx = \int \left(\frac{x}{x^2+4} + \frac{2}{x^2+4}\right)dx$

$$\left\{\begin{array}{l} \text{let } u = x^2+4 \\ du = 2x\,dx \rightsquigarrow \frac{du}{2} = x\,dx \end{array}\right.$$

$= \int \frac{x}{x^2+4}dx + 2\int \frac{1}{x^2+4}dx$

can also skip this

$= \frac{1}{2}\int \frac{1}{u}du + 2\int \frac{1}{4\left(\frac{x^2}{4}+1\right)}dx$

$$\left\{\begin{array}{l} \text{let } u = \frac{x}{2} \\ du = \frac{1}{2}dx \rightsquigarrow 2du = dx \end{array}\right.$$

$= \frac{1}{2}\ln|u| + \frac{1}{2}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}dx$

$= \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\int \frac{1}{u^2+1}\,2du$

$= \frac{1}{2}\ln(x^2+4) + \tan^{-1}(u) + C = \frac{1}{2}\ln(x^2+4) + \tan^{-1}\left(\frac{x}{2}\right) + C$

[4] 6. $\int \sqrt{1+\sqrt{x}}\,dx$

$$\left\{\begin{array}{l} \text{let } u = 1+\sqrt{x} \rightsquigarrow \sqrt{x} = u-1 \\ du = \frac{1}{2\sqrt{x}}dx \rightsquigarrow 2\sqrt{x}\,du = dx \\ \qquad 2(u-1)\,du = dx \end{array}\right.$$

$= \int \sqrt{u}\cdot 2(u-1)\,du$

$= 2\int u^{1/2}(u-1)\,du$

$= 2\int (u^{3/2} - u^{1/2})\,du$

$= 2\frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2} + C$

$= \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + C$