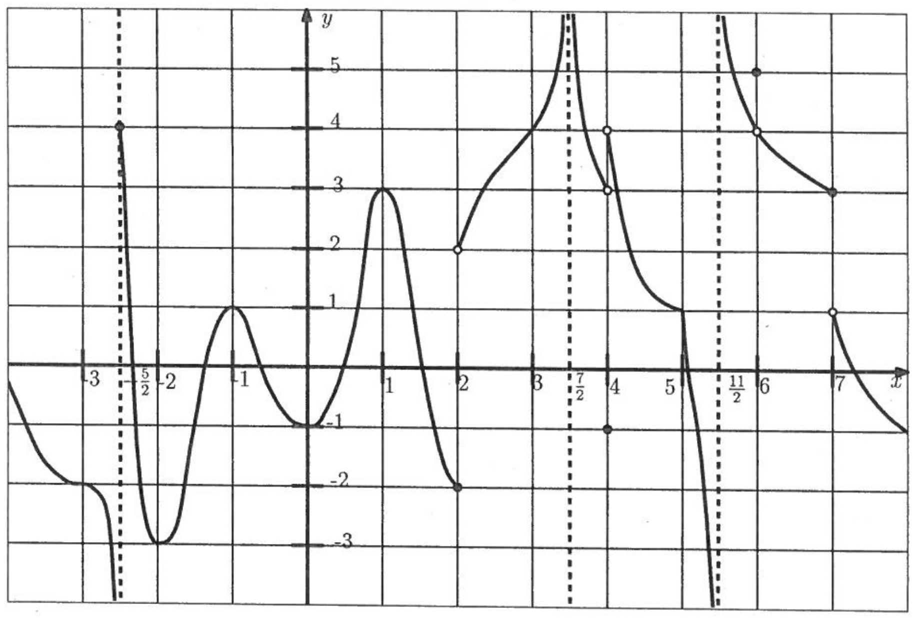
[14] 1. The graph of the function $y = f(x)$ is shown below. Fill in the blanks.



(a) f has discontinuities at: $-\frac{5}{2}, 2, \frac{7}{2}, 4, \frac{11}{2}, 6, 7$

(b) f is left-continuous, but not continuous at: $2, 7$

(c) f is continuous, but not differentiable at: 5

(d) For the following real values of a we have that $\lim_{x\to a} f(x) = \infty$: $\frac{7}{2}$

(e) $\lim_{x\to 6}(f(x)-1)^2 =$ $(4-1)^2 = 9$

(f) $\lim_{x\to 4^-} \ln(f(x)) =$ $\ln(3)$

(g) $\lim_{x\to \frac{11}{2}^+} \tan^{-1}(f(x)) =$ $\lim_{y\to\infty} \tan^{-1} y = \frac{\pi}{2}$

(h) For the following values of x we have $f'(x) = 0$: $-3, -2, -1, 0, 1$

(i) List all values of x where f has local maxima and is continuous: $-1, 1$

(j) List all values of x where f has local maxima and is not continuous: $-\frac{5}{2}, 6$

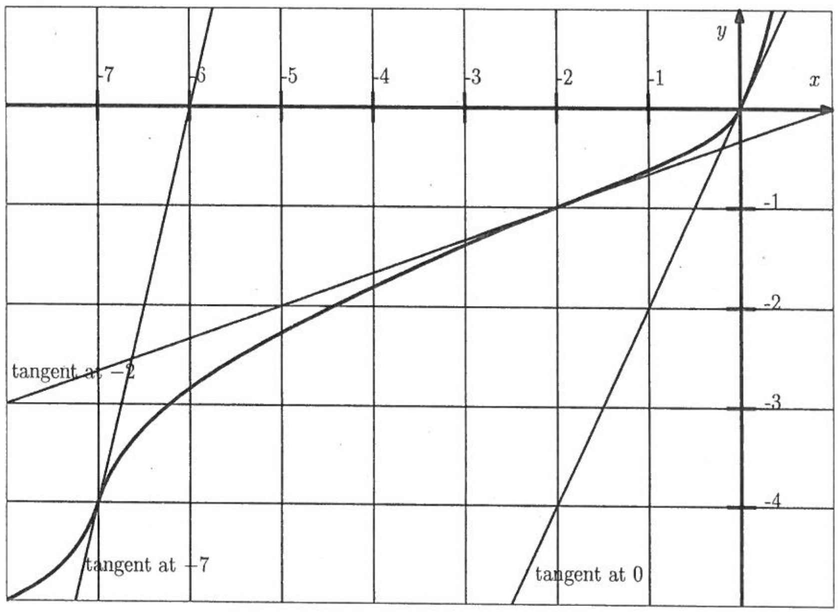
(k) The global (or absolute) maximum of f on the interval $[-\frac{5}{2}, 2]$ is 4

(l) The global (or absolute) minimum of f on the interval $[-\frac{5}{2}, 2]$ is -3

(m) The global (or absolute) minimum of f on the interval $[2, 3]$ is -2

(n) $f'(x) > 0$ on the following interval(s): $(-2,-1), (0,1), (2, \frac{7}{2})$

[14] 2. The graph of the function $y = f(x)$ is shown below.



Fill in the blanks (all numeric answers have to be in the simplest form possible):

(a) The average rate of change of f over the interval $[-7, -2]$ is $\frac{3}{5}$

(b) The equation of the tangent line to $y = f(x)$ at -2 is $y+1=\frac{1}{3}(x+2)$

(c) If g is the inverse function of f, then $g(-4) =$ -7 and $g'(-4) =$ $\frac{1}{4}$

(d) $f'(x)$ is decreasing on: $(-7,-2)$

(e) f has inflections points at the following values of x: $-7, -2$

(f) If $a(x) = f(2-x^2)$, then $a'(x) =$ $-2x\, f'(2-x^2)$, $a'(2) =$ $-\frac{4}{3}$

$-2(2)f'(-2) = 4\cdot\frac{1}{3}$

(g) If $b(x) = (f(x)-2)^{-2}$, then $b'(x) =$ $-2(f(x)-2)^{-3}f'(x)$, $b'(0) =$ $\frac{1}{2}$

$-2(0-2)^{-3}\cdot 2$

(h) If $c(x) = f(x-3)\tan^{-1}(x)$, then $c'(x) =$ $f'(x-3)\tan^{-1}(x) + \frac{f(x-3)}{1+x^2}$, $c'(1) =$ $\frac{\pi}{12}-\frac{1}{2}$

$f'(-2)\tan^{-1}(1) + \frac{f(-2)}{1+(-1)^2} = \frac{1}{3}\cdot\frac{\pi}{4} - \frac{1}{2}$

(i) If $d(x) = \dfrac{x}{f(x)}$, then $d'(x) =$ $\frac{f(x)-xf'(x)}{(f(x))^2}$, $d'(-7) =$ $\frac{3}{2}$

$\frac{-4-(-7)4}{16} = \frac{24}{16} = \frac{3}{2}$

[3] 3. State the formal (ε, δ) definition of $\lim_{x \to a} f(x) = L$ (part marks will be awarded for an informal definition):

Formal: For every $\varepsilon > 0$, there is $\delta > 0$ s.t. whenever $|x-a| < \delta$ we have $|f(x) - L| < \varepsilon$

Informal: For x close to a we have that $f(x)$ is close to L.

[2] 4. State the Extreme Value Theorem: A continuous function on a closed interval has a minimum and a maximum.

[2] 5. Let $f(x) = \dfrac{\ln(2x)}{1+e^{x+1}}$. Set up, but **do not simplify or evaluate**, $f'(2)$ as a limit (from the definition of derivative). Do not compute the derivative (no marks will be given for that).

$$f'(2) = \lim_{h \to 0} \frac{\dfrac{\ln(2(2+h))}{1+e^{2+h+1}} - \dfrac{\ln 4}{1+e^3}}{h}$$

[2] 6. List all vertical asymptotes of $y = \frac{x(x-2)^2 \ln|x^2-1|}{(x-2)(x-5)}$: $x=-1, x=1, x=5$

[2] 7. List all horizontal asymptotes of $y = 2\tan^{-1}(x)$: $y = -\pi, y = \pi$

[2] 8. Suppose that f is a function such that $f'(5) = 0$ and $f''(x) = \ln(x)$. Does f have a local minimum or a local maximum at 5? Justify your answer.

Answer: f has a local min. because $f''(5) = \ln(5) > 0$

[2] 9. $\left(\lim_{x \to 0} \dfrac{2x}{\sin x}\right) - \left(\lim_{y \to \frac{\pi}{6}} \dfrac{3y}{\sin y}\right) = 2 - \dfrac{3(\pi/6)}{1/2} = \boxed{2-\pi}$

[2] 10. $\lim_{x \to 2^+} \dfrac{2-x}{|2-x|} = \lim_{x \to 2^+} \dfrac{2-x}{-(2-x)} = -1$

[6] 11. Compute the derivative. **Do not simplify.**

$$\frac{d}{dx}\left(\sec(2x) + 2\sin^{-1}(x+2) + e^7 + \frac{3}{x^5} + \sqrt{3x+1} + \ln(e^{2x}+1)\right)$$

$$2\sec(2x)\tan(2x) + \frac{2}{\sqrt{1-(x+2)^2}} + 0 - 15x^{-6} + \frac{3}{2}(3x+1)^{-\frac{1}{2}} + \frac{2e^{2x}}{e^{2x}+1}$$

[4] 12. $\lim\limits_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} = \lim\limits_{x\to 2} \frac{(\sqrt{x^2+5}-3)(\sqrt{x^2+5}+3)}{(x^2-4)(\sqrt{x^2+5}+3)} = \lim\limits_{x\to 2} \frac{x^2+5-9}{(x^2-4)(\sqrt{x^2+5}+3)} = \lim\limits_{x\to 2} \frac{x^2-4}{(x^2-4)(\sqrt{x^2+5}+3)}$

$$= \frac{1}{\sqrt{2^2+5}+3} = \boxed{\frac{1}{6}}$$

[6] 13. Find horizontal asymptotes of $y = \dfrac{e^x + 2x + 3\sin x}{2e^x + 3x}$.

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1 + 2\frac{x}{e^x} + 3\frac{\sin x}{e^x}}{2 + 3\frac{x}{e^x}} = \frac{1+0+0}{2+0} = \boxed{\frac{1}{2}}$$

$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{\frac{e^x}{x} + 2 + 3\frac{\sin x}{x}}{2\frac{e^x}{x} + 3} = \frac{0+2+0}{0+3} = \boxed{\frac{2}{3}}$$

H.a.: $y = \frac{1}{2}$ and $y = \frac{2}{3}$

[4] 14. Compute the derivative. **Do not simplify.**

$$\frac{d}{dt}\left(t^2+1\right)^{\sin t} = \frac{d}{dt} e^{\sin t \ln(t^2+1)} = e^{\sin t \ln(t^2+1)} \left((\cos t) \ln(t^2+1) + (\sin t) \frac{2t}{t^2+1} \right).$$

[7] 15. For which values of constants a and b will the function

$$f(x) = \begin{cases} \frac{x^2-1}{x^2+x-2} & , \text{ for } x < 1 \\ b & , \text{ for } x = 1 \\ a + e^{1/(1-x)} & , \text{ for } x > 1 \end{cases}$$

be continuous ~~everywhere~~ at 1? For 3 **bonus marks** also compute $\lim_{x\to\infty} f(x)$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-1}{x^2+x-2} = \lim_{x\to 1^-} \frac{(x-1)(x+1)}{(x-1)(x+2)} = \boxed{\frac{2}{3}}$$

$$f(1) = b$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(a + e^{\frac{1}{1-x}}\right) \underset{y=\frac{1}{1-x}}{=} a + \lim_{y\to -\infty} e^{y} = a + 0 = a.$$

Function will be cont. at 1 if and only if $a = b = \frac{2}{3}$.

Bonus $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(a + e^{\frac{1}{1-x}}\right) = a + e^0 = \boxed{a+1}$

important!

[7] 16. Find the equation of the tangent line to the curve

$$xy^3 = 2x^2 + y^2 + 2$$

at the point $(3, 2)$.

$y^3 + 3xy^2y' = 4x + 2yy' \rightarrow y'(3xy^2 - 2y) = 4x - y^3 \rightarrow y' = \frac{4x - y^3}{3xy^2 - 2y}$

$y'\big|_{\substack{x=3\\y=2}} = \frac{4(3) - 2^3}{3(3)2^2 - 2(2)} = \frac{4}{32} = \boxed{\frac{1}{8}}$

Tangent: $\boxed{y - 2 = \frac{1}{8}(x - 3)}$

[6] 17. Let x, y be functions of t related by

$$\sin(2x - y^3) = x^2 - y^4.$$

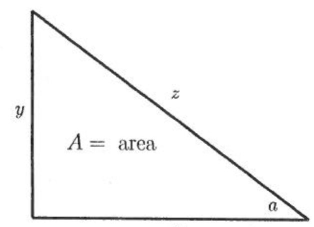
When $x = 4$ and $y = 2$ we have that $\frac{dx}{dt} = -2$. Compute $\frac{dy}{dt}$ at that point.

$(\cos(2x - y^3)) \cdot (2\frac{dx}{dt} - 3y^2\frac{dy}{dt}) = 2x\frac{dx}{dt} - 4y^3\frac{dy}{dt} \rightarrow \frac{dy}{dt}(4y^3 - 3y^2\cos(2x - y^3)) = \frac{dx}{dt}(2x - 2\cos(2x - y^3))$.

$\frac{dy}{dt} = \frac{dx}{dt} \cdot \frac{2x - 2\cos(2x - y^3)}{4y^3 - 3y^2\cos(2x - y^3)}$, $\frac{dy}{dt}\Big|_{\substack{x=4\\y=2}} = -2 \cdot \frac{2(4) - 2(1)}{4(2)^3 - 3 \cdot 2^2 \cdot 1} = -2 \cdot \frac{6}{32 - 12} = \frac{-12}{20} = \boxed{\frac{-3}{5}}$

$\cos 0 = 1.$

[7] 18. Consider a right-angled-triangle with catheti (adjacent sides) x and y, hypothenuse (opposite side) z, and angle a between x and z. See the picture below. At some point in time we have that x is 3 centimetres, y is 4 centimetres, x is decreasing at the rate of 2 centimetres per second, and a is increasing at the rate of $\frac{2}{5}$ radians per second (the angle $\frac{\pi}{2}$ between x and y is not changing). Find the rate of change of the area A of the triangle at that time.



$$A = \frac{1}{2}xy = \frac{1}{2}x\,(x\tan a) = \frac{1}{2}x^2\tan a$$

$$\frac{dA}{dt} = x(\tan a)\cdot\frac{dx}{dt} + \frac{1}{2}x^2(\sec^2 a)\frac{da}{dt} = x\cdot\frac{y}{x}\cdot\frac{dx}{dt} + \frac{1}{2}x^2\cdot\frac{x^2+y^2}{x^2}\frac{da}{dt}$$

$$= y\frac{dx}{dt} + \frac{1}{2}(x^2+y^2)\frac{da}{dt}.$$

$$\left.\frac{dA}{dt}\right|_{\substack{x=3\\y=4}} = 4(-2) + \frac{1}{2}\cdot 25\cdot\frac{2}{5} = -8+5 = \boxed{-3\ \frac{cm^2}{s}}$$

[3] 19. List and classify (i.e., local max./min./neither) critical points of a function f whose **derivative** is

$$f'(x) = \frac{(x+3)^2 \ln|x|}{\sqrt[5]{x-5}}.$$

f': $-$ | $-$ | $+$ | $+$ | $-$ | $+$

f: -3 neither; -1 loc. min; 0 neither; 1 loc. max; 5 loc. min.

[3] 20. Find the critical points of $f(x) = x^2e^{-x^2}$.

$f'(x) = 2x e^{-x^2} + x^2(-2x)e^{-x^2} = 2x(1-x^2)e^{-x^2} = 2x(1-x)(1+x)e^{-x^2}$

Critical points: $\boxed{-1, 0, 1.}$

[9] 21. Find the global (absolute) maximum and the global minimum of $f(x) = 4x^3 - 9x^2$ on the interval $[1, 2]$. For 4 **bonus marks** also solve the problem on the interval $(-\infty, 1)$.

$f'(x) = 12x^2 - 18x = 6x(2x-3)$

Critical points: $0, \frac{3}{2}$.

not in [1,2]

$4 \cdot \frac{27}{8} - 9 \cdot \frac{9}{4}$

$= \frac{27}{2} - \frac{81}{4} = \frac{-27}{4}$

x	$f(x)$	
1	-5	
$\frac{3}{2}$	$-\frac{27}{4}$	min
2	-4	max

Bonus:

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty}(4x^3 - 9x^2) = \lim_{x \to -\infty} x^3\left(4 - \frac{9}{x}\right) = -\infty$

$f(0) = 0$

$\lim_{x \to 1^-} f(x) = -5$.

$\boxed{\text{Maximum} = 0, \text{ minimum d.n.e.}}$

On this page there are two bonus questions. Most students are not expected to have time for both. You should not spend a lot of time on them unless you are already finished with the rest of the exam. Part marks will only be awarded for significant progress.

[8] B1. Let f be the function given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$$

Compute $f'(0)$ and show that the function f' is not continuous at 0. Fully justify your answer.

$f'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = \lim_{h \to 0} h \sin(\frac{1}{h}) = 0$ ← 0 × bounded

$x \neq 0$, $f'(x) = 2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x}) \cdot (-\frac{1}{x^2}) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$.

$\lim_{x\to 0} f'(x) = \lim_{x \to 0} (2x\sin(\frac{1}{x}) - (\cos(\frac{1}{x}))) = \lim_{x\to 0} -\cos\frac{1}{x}$ d.n.e.

[8] B2. Which is larger e^π or π^e? Fully justify your answer. Hint: First determine which is larger π or $e\ln(\pi)$. To determine this, consider the function $f(x) = x - e\ln(x)$ on the interval $[e, \pi]$. There are also other approaches to this question.

$f'(x) = 1 - \frac{e}{x}$. On $[e, \pi]$ we have $x > e$, so $\frac{e}{x} < 1$, so $f'(x) > 0$, So f is ~~decreasing~~ increasing on $[e, \pi]$ and hence $f(e) < f(\pi)$. Now $f(e) = 0$ and $f(\pi) = \pi - e\ln\pi$. So $0 < \pi - e\ln\pi$, hence $e\ln\pi < \pi$. Therefore $\pi^e = e^{e\ln\pi} < e^{\pi}$.