Saint Mary's University
DEPARTMENT OF MATHEMATICS
AND COMPUTING SCIENCE

Name: SOLUTIONS

Signature: ____________________

ID: ____________________

Math 1211: Introductory Calculus II

Midterm Test

March 3, 2016 (7:00–9:00pm)

Instructor: J. Irving

Instructions:

- *No electronic devices, or aids of any kind, are permitted.*
- *There are 6 pages plus this cover page. Check that your test paper is complete.*
- *There are a total of 75 points. The value of each question is indicated in the margin.*
- *Answer in the spaces provided, using backs of pages for additional space if necessary.*
- *Show all your work. Insufficient justification will result in a loss of points.*

Page	Maximum	Your Score
1	14	
2	16	
3	8	
4	12	
5	15	
6	10	
Total	**75**	

1. Evaluate the following. Simplify your answers as much as possible:

[3] (a) $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx$

$\begin{cases} u = 1 + e^{2x} \\ du = 2e^{2x}\,dx \end{cases}$

$= \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= \frac{1}{2}\cdot 2\sqrt{u} + C$

$= \boxed{\sqrt{1+e^{2x}} + C}$

[5] (b) $\displaystyle\int \cos^3 x \sin^3 x\,dx$

$= \int \cos^2 x \sin^3 x \cdot \sin x\,dx$ $\begin{cases} u = \sin x \\ du = \cos x \end{cases}$

$= \int (1-u^2)u^3\,du$

$= \int (u^3 - u^5)\,du$

$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \boxed{\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C}$

[6] (c) $\displaystyle\int x^2(\ln x)^2\,dx$

$\begin{cases} u = (\ln x)^2 \\ du = \frac{2\ln x}{x}\,dx \end{cases}$ $dv = x^2\,dx$, $v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int x^2 \ln x\,dx$ $\begin{cases} u = \ln x & dv = x^2\,dx \\ du = \frac{1}{x}\,dx & v = \frac{1}{3}x^3 \end{cases}$

$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\left[\frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2\,dx\right]$

$= \boxed{\frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C}$

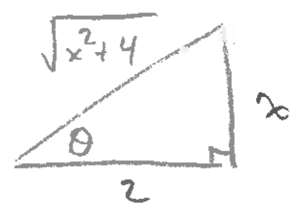
[8] (d) $\displaystyle\int \frac{x^2}{(x^2+4)^2}\,dx$ [*Hint:* Trigonometric substitution.]

$$= \int \frac{4\tan^2\theta}{(4\sec^2\theta)^2}\, 2\sec^2\theta \qquad \begin{cases} x = 2\tan\theta \\ dx = 2\sec^2\theta\, d\theta \end{cases}$$

$$= \frac{1}{2}\int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{2}\int \sin^2\theta\, d\theta$$

$$= \frac{1}{4}\int (1-\cos 2\theta)\, d\theta$$



$$= \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{4}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{1}{4}\left(\tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}}\right) + C = \boxed{\frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{2(x^2+4)} + C}$$

[8] (e) $\displaystyle\int \frac{x+1}{(x^2+9)(x-1)}\,dx$

$$\frac{x+1}{(x^2+9)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$\Rightarrow \quad x+1 = A(x^2+9) + (Bx+c)(x-1)$$

Set $x=1$ to get $2 = 10A \Rightarrow A = 1/5$

Set $x=0$ to get $1 = 9A - C \Rightarrow C = 4/5$

Compare coeff. of x^2 to get $0 = A+B \Rightarrow B = -1/5$

So $\displaystyle\int \frac{x+1}{(x^2+9)(x-1)}\,dx = \frac{1}{5}\int\frac{dx}{x-1} + \frac{1}{5}\int\frac{-x+4}{x^2+9}\,dx$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{5}\int\frac{x}{x^2+9}\,dx + \frac{4}{5}\int\frac{dx}{x^2+9}$$

$$= \boxed{\frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+9| + \frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

2. Determine whether the following improper integrals converge or diverge. If an integral converges, find its value.

[3] (a) $\int_1^2 \frac{dt}{(1-t)^3}$

$= \lim_{R \to 1^+} \int_R^2 \frac{dt}{(1-t)^3}$

$= \lim_{R \to 1^+} \frac{1}{2}(1-t)^{-2} \Big]_R^2$

$= \lim_{R \to 1^+} \left(\frac{1}{2} - \frac{1}{2(1-R)^2} \right)$ ∞

$= -\infty$ $\boxed{\therefore \text{ Diverges}}$

OR

$\int_1^2 \frac{dt}{(1-t)^3} = -\int_0^1 \frac{du}{u^3}$ $[u = t-1]$

and this diverges as it is a p-integral with $p = 3 > 1$

[3] (b) $\int_1^\infty e^{-2x}\, dx$

$= \lim_{R \to \infty} \int_1^R e^{-2x}\, dx$

$= \lim_{R \to \infty} -\frac{1}{2} e^{-2x} \Big]_1^R$

$= \lim_{R \to \infty} \left(-\frac{1}{2} e^{-2R} + \frac{1}{2} e^{-2} \right)$ 0

$\boxed{= \frac{1}{2e^2}}$ (CONVERGES)

[2] (c) $\int_0^\infty \frac{x}{1+x^6}\, dx$ [*Hint:* Use comparison.] DO NOT EVALUATE

Note that for $x > 0$ we have $\frac{x}{1+x^6} < \frac{x}{x^6} = \frac{1}{x^5}$

Since $\int_1^\infty \frac{dx}{x^5}$ converges (p-integral, $p = 5 > 1$), we conclude by comparison that $\int_0^\infty \frac{x}{1+x^6}\, dx$ also $\boxed{\text{converges}}$

3. Let $\mathcal{R}$ be the region bounded between the curves $x = y^2 - 1$ and $x + y = 1$.

[3] (a) Sketch $\mathcal{R}$ and label all points of intersection of its bounding curves.

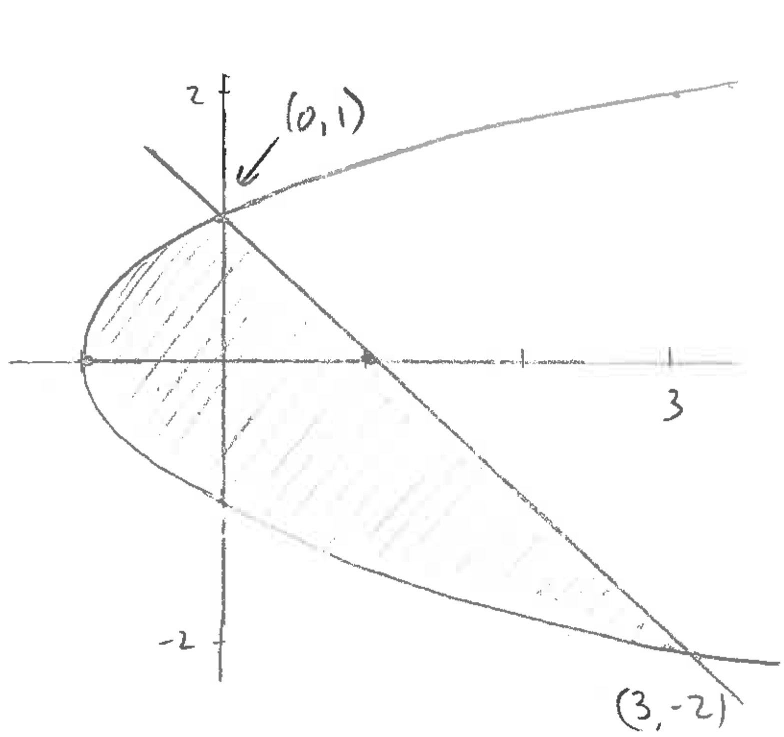
Solve: $y^2 - 1 = 1 - y$

$\iff y^2 + y - 2 = 0$

$\iff (y+2)(y-1) = 0$

$\iff y = -2$ or $y = 1$

So $(x,y) = (3,-2)$ and $(0,1)$ are intersection pts



[4] (b) Find the area of $\mathcal{R}$.

$$\text{Area} = \int_{-2}^{1} \left((1-y) - (y^2 - 1) \right) dy$$

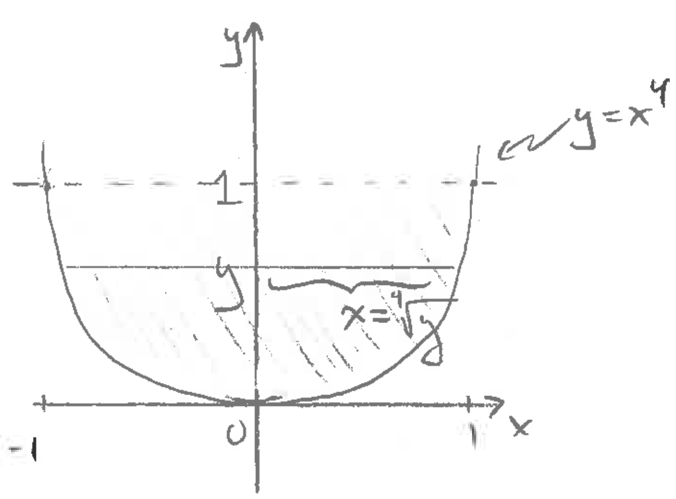
$$= \int_{-2}^{1} (2 - y - y^2)\, dy$$

$$= \left(2y - \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 \right) \Big]_{-2}^{1}$$

$$= \left(2 - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(-4 - 2 + \tfrac{8}{3}\right)$$

$$= \boxed{\tfrac{9}{2}}$$

[5] 4. The base of a solid is the region enclosed by the curve $y = x^4$ and the line $y = 1$. The cross sections perpendicular to the y-axis are squares. Find the volume of this solid.



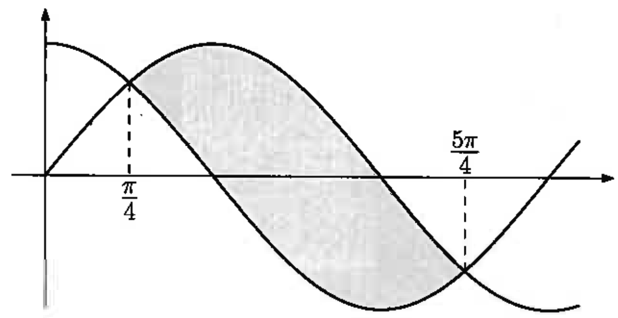
Cross sectional area through y is $A(y) = (2\sqrt[4]{y})^2 = 4\sqrt{y}$

So volume $= \int_0^1 4\sqrt{y}\, dy$

$$= 4 \cdot \tfrac{2}{3} y^{3/2} \Big]_0^1$$

$$= \boxed{\tfrac{8}{3}}$$

[15] 5. The shaded region $\mathcal{R}$ shown below is bounded between the curves $y = \cos x$ and $y = \sin x$.



Give expressions, in terms of definite integrals, for each of the following quantities.
Do not simplify or evaluate your expressions!

(a) The **volume** of the solid obtained by revolving $\mathcal{R}$ about the y axis.

$$2\pi \int_{\pi/4}^{5\pi/4} x(\sin x - \cos x)\, dx$$

(b) The **volume** of the solid obtained by revolving $\mathcal{R}$ about the line $x = 5$.

$$2\pi \int_{\pi/4}^{5\pi/4} (5-x)(\sin x - \cos x)\, dx$$

(c) The **volume** of the solid obtained by revolving $\mathcal{R}$ about the line $y = 2$.

$$\pi \int_{\pi/4}^{5\pi/4} \left((2-\cos x)^2 - (2-\sin x)^2 \right) dx$$

(d) The **area** of $\mathcal{R}$.

$$\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$$

(e) The **perimeter** of $\mathcal{R}$. [*Hint:* Use the formula for arc length.]

$$\int_{\pi/4}^{5\pi/4} \sqrt{1+\cos^2 x}\, dx + \int_{\pi/4}^{5\pi/4} \sqrt{1+(-\sin x)^2}\, dx$$

(f) The **surface area** of the solid obtained by revolving $\mathcal{R}$ about the line $y = -2$.

$$2\pi \int_{\pi/4}^{5\pi/4} (2+\sin x)\sqrt{1+\cos^2 x}\, dx + 2\pi \int_{\pi/4}^{5\pi/4} (2+\cos x)\sqrt{1+(-\sin x)^2}\, dx$$

[10] 6. Consider the parametric curve C given by $(x, y) = (2\cos t, 1 + 2\sin t)$, for $0 \le t \le 2\pi$.

(a) Eliminate the parameter t to find the Cartesian equation of C.

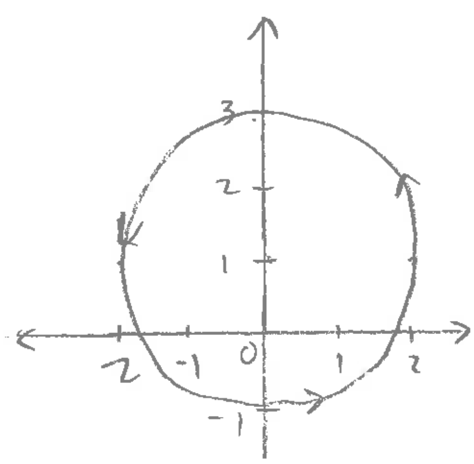
Since $x = 2\cos t$ and $y - 1 = 2\sin t$

get
$$x^2 + (y-1)^2 = 4\cos^2 t + 4\sin^2 t$$
$$= 4(\cos^2 t + \sin^2 t)$$
$$= 4$$

Cartesian eqn is $\boxed{x^2 + (y-1)^2 = 4}$

(b) Sketch C, being sure to indicated the direction of travel.

From (a), we see that C is a circle of radius 2 centred at $(0, 1)$.



(c) Find the equation of the tangent line to C at the point where $t = \frac{5\pi}{6}$.

$\frac{dx}{dt} = -2\sin t$ and $\frac{dy}{dt} = 2\cos t$

So $\left.\frac{dy}{dx}\right|_{t=5\pi/6} = \frac{2\cos(5\pi/6)}{-2\sin(5\pi/6)} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

And at $t = 5\pi/6$ get $(x, y) = (-\sqrt{3}, 2)$

So tangent line is $\boxed{y - 2 = \sqrt{3}(x + \sqrt{3})}$

(OR $y = \sqrt{3}x + 5$)