Math 1211: Taylor Series Practice — Winter 2016

1. Find the Taylor series for the given function $f(x)$ centred at the given point c:

   (a) $f(x) = \dfrac{1}{x}, \quad c = -3$

   (b) $f(x) = e^{2x}, \quad c = 1$

   (c) $f(x) = \sin x, \quad c = \dfrac{\pi}{2}$

   (d) $f(x) = x^4 + 1, \quad c = -1$

2. The following table lists some common Maclaurin series and their radii of convergence:

$f(x)$	Maclaurin series	R
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	∞
$\sin x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	∞
$\cos x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	∞
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$	1
$\ln(1+x)$	$\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	1
$\tan^{-1} x$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	1

   Use this table to find the Maclaurin series for the following functions, along with their radii of convergence.

   (a) $f(x) = e^{3x}$

   (b) $f(x) = (1 + 5x)^{-1}$

   (c) $f(x) = \sin(x^2)$

   (d) $f(x) = \tan^{-1} \dfrac{x}{2}$

   (e) $f(x) = \cos^2 x \qquad$ [*Hint:* $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$]

   (f) $f(x) = \ln \dfrac{1+x}{1-x} \qquad$ [*Hint:* $\ln \frac{a}{b} = \ln a - \ln b$]

   (g) $f(x) = \dfrac{1}{5-x} \qquad$ [*Hint:* $\frac{1}{5-x} = \frac{1}{5} \cdot \frac{1}{1-(x/5)}$]

3. Use the table in Question #2 to find the terms up to **fourth degree** of the Maclaurin series of the following functions:

   (a) $f(x) = \cos x + \sin x$

   (b) $f(x) = \dfrac{x^2}{1 - 2x}$

   (c) $f(x) = e^x \tan^{-1} x$

   (d) $f(x) = \dfrac{\ln(1 + x)}{1 + x^2}$

4. Express the following as infinite series, valid for the given values of x:

   (a) $(1 - 3x)^{-3}$, $\quad |x| < 1 \quad$ [*Hint:* This is almost the second derivative of $\frac{1}{1-3x}$]

   (b) $\displaystyle\int_0^x \ln(1 + t^2)\, dt$, $\quad |x| < 1$

   (c) $\displaystyle\int_0^x \frac{1 - \cos t}{t}\, dt$, $\quad$ all x

Answers

1. (a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{-1}{3^{n+1}}\right)(x+3)^n = -\frac{1}{3}-\frac{1}{9}(x+3)-\frac{1}{27}(x+3)^2-\cdots-\frac{1}{3^{n+1}}(x+3)^n-\cdots$

   (b) $\displaystyle\sum_{n=0}^{\infty}\frac{e^2 2^n}{n!}(x-1)^n = e^2+2e^2(x-1)+\frac{4e^2}{2!}(x-1)^2+\cdots+\frac{2^n e^2}{n!}(x-1)^n+\cdots$

   (c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{2n}}{(2n)!}(x-\tfrac{\pi}{2})^{2n} = 1-\frac{1}{2!}(x-\tfrac{\pi}{2})^2+\frac{1}{4!}(x-\tfrac{\pi}{2})^4+\cdots+\frac{(-1)^{2n}}{(2n)!}(x-\tfrac{\pi}{2})^{2n}+\cdots$

   (d) $2-4(x+1)+6(x+1)^2-4(x+1)^3+(x+1)^4$

2. (a) $\displaystyle\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}, \qquad R=\infty$

   (b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n 5^n x^n, \qquad R=\tfrac{1}{5}$

   (c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}, \qquad R=\infty$

   (d) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}(2n+1)}, \qquad R=2$

   (e) $\displaystyle 1+\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!}, \qquad R=\infty$

   (f) $\displaystyle 2\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1}, \qquad R=1$

   (g) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}, \qquad R=5$

3. (a) $1+x-\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4$

   (b) $x^2+2x^3+4x^4$

   (c) $x+x^2+\frac{1}{6}x^3-\frac{1}{6}x^4$

   (d) $x-\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4$

4. Express the following as infinite series, valid for the given values of x:

   (a) $\displaystyle\sum_{n=2}^{\infty}\frac{n(n-1)}{2}3^{n-2}x^{n-2} \qquad \left(\text{equivalently, } \frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)3^n x^n\right)$

   (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{n(2n+1)}$

   (c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{2n\cdot(2n)!}$