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Math 1211: Worksheet #2 Winter 2016

Name: Key	A#:	Section:

1. $\int \frac{\ln x}{x^2}\,dx$

Let $u = \ln x$, $du = \frac{1}{x}dx$ ✓ $V = -x^{-1} = \frac{-1}{x}$, $dV = \frac{1}{x^2}dx$

$$\int \frac{\ln x}{x^2}dx = -\frac{1}{x}\ln x - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= -\frac{1}{x}\ln x + \int \frac{1}{x^2}dx \quad ✓$$

$$= -\frac{1}{x}\ln x - \frac{1}{x} + C$$

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$$\text{or} = \frac{-1}{x}\left(\ln x + 1\right) + C \quad ✓$$

2. $\int \sin\sqrt{x}\,dx$

Let $w = \sqrt{x}$, $dw = \frac{1}{2}x^{-1/2}dx$

$dw = \frac{1}{2\sqrt{x}}dx$ ✓

$2\sqrt{x}\,dw = dx$

$$\int \sin\sqrt{x}\,dx = \int \sin w \cdot 2w\,dw$$

$$= 2\int w\sin w\,dw \quad ✓$$

Let $u = w$ ✓ $dV = \sin w\,dw$

$du = dw$ $V = -\cos w$

$$= 2\left[-w\cos w - \int -\cos w\,dw\right] \quad ✓$$

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$$= -2w\cos w + 2\sin w + C \quad ✓$$

$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

3. $\int \csc^3 x\,dx$

$$= \int \csc x \csc^2 x\,dx$$

Let $u = \csc x$ ✓ $dV = \csc^2 x\,dx$

$du = -\csc x\cot x\,dx$ $V = -\cot x$

$$= -\csc x\cot x - \int \csc x\cot^2 x\,dx \quad ✓$$

$$= -\csc x\cot x - \int \csc x\cot^2 x\,dx$$

$$= -\csc x\cot x - \int \csc x(\csc^2 x - 1)\,dx \quad ✓$$

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$$= -\csc x\cot x - \int(\csc^3 x - \csc x)\,dx$$

$$= -\csc x\cot x - \int \csc^3 x\,dx + \int \csc x\,dx \quad ✓$$

$$= -\csc x\cot x - \int \csc^3 x\,dx + \ln|\csc x - \cot x| + C$$

* $\int \csc x\,dx$

$$= \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x}dx$$

$$= \int \frac{\csc^2 x - \csc x\cot x}{\csc x - \cot x}dx$$

Let $u = \csc x - \cot x$

$du = -\csc x\cot x + \csc^2 x$

$$\rightsquigarrow = \int \frac{1}{u}du$$

$$= \ln|u| + C$$

$$= \ln|\csc x - \cot x| + C$$

$$\therefore\ 2\int \csc^3 x\,dx = -\csc x\cot x + \ln|\csc x - \cot x| + C$$

Hence, $\int \csc^3 x\,dx = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$ ✓

v2.1

4. $\int \sin^3 x \cos^5 x\, dx$

Power of sine is odd:

$$\int \sin^3 x \cos^5 x\, dx$$
$$\int \sin^2 x \cos^5 x \sin x\, dx$$
$$\int (1-\cos^2 x)\cos^5 x \sin x\, dx$$
$$-\int (1-u^2)u^5\, du \qquad \left\{\begin{array}{l}\text{let } u = \cos x \\ du = -\sin x\, dx\end{array}\right.$$
$$-\int (u^5 - u^7)\, du$$
$$\int (u^7 - u^5)\, du$$
$$\frac{u^8}{8} - \frac{u^6}{6} + c$$
$$= \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$$

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Or

Power of cosine is odd:

$$\int \sin^3 x \cos^5 x\, dx$$
$$= \int \sin^3 x \cos^4 x \cos x\, dx$$
$$= \int \sin^3 x (1-\sin^2 x)^2 \cos x\, dx$$
$$= \int u^3 (1-u^2)^2\, du \qquad \left\{\begin{array}{l}\text{let } u = \sin x \\ du = \cos x\, dx\end{array}\right.$$
$$= \int u^3 (1 + u^4 - 2u^2)\, du$$
$$= \int (u^3 + u^7 - 2u^5)\, du$$
$$= \frac{u^4}{4} + \frac{u^8}{8} - \frac{2u^6}{6}$$
$$= \frac{\sin^4 x}{4} + \frac{\sin^8 x}{8} - \frac{\sin^6 x}{3} + C$$

5. $\int \sin^2 x \cos^2 x\, dx$

$$= \int (\sin x \cos x)^2\, dx$$
$$= \int \left(\frac{\sin 2x}{2}\right)^2 dx$$
$$= \frac{1}{4}\int \sin^2 2x\, dx$$
$$= \frac{1}{4}\int \frac{1}{2}(1 - \cos 2(2x))\, dx$$
$$= \frac{1}{8}\int (1 - \cos 4x)\, dx$$
$$= \frac{1}{8}x - \frac{1}{8}\cdot\frac{1}{4}\sin 4x + c$$
$$= \frac{x}{8} - \frac{1}{32}\sin 4x + C$$

Or

$$\int \sin^2 x \cos^2 x\, dx = \int \sin^2 x (1 - \sin^2 x)\, dx$$
$$= \int (\sin^2 x - \sin^4 x)\, dx$$
$$= \int \sin^2 x\, dx - \int \sin^4 x\, dx$$
$$= \int \frac{1}{2}(1 - \cos 2x)\, dx - \int \left(\frac{1}{2}(1 - \cos 2x)\right)^2 dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] - \frac{1}{4}\int (1 + \cos^2 2x - 2\cos 2x)\, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] - \frac{1}{4}x - \frac{1}{4}\int \frac{1}{2}(1 + \cos 4x)\, dx + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + C$$
$$= \frac{x}{2} - \frac{\sin 2x}{4} - \frac{x}{4} - \frac{x}{8} - \frac{1}{8}\cdot\frac{1}{4}\sin 4x + \frac{\sin 2x}{4} + C$$
$$= \frac{x}{8} - \frac{1}{32}\sin 4x + C$$

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Or

$$\int \sin^2 x \cos^2 x\, dx$$
$$= \int (1 - \cos^2 x)\cos^2 x\, dx$$
$$= \int (\cos^2 x - \cos^4 x)\, dx$$
$$= \int \cos^2 x\, dx - \int \cos^4 x\, dx$$
$$= \int \frac{1}{2}(1 + \cos 2x)\, dx - \int \left(\frac{1}{2}(1 + \cos 2x)\right)^2 dx$$
$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] - \frac{1}{4}\int (1 + \cos^2 2x + 2\cos 2x)\, dx$$
$$= \frac{x}{2} + \frac{\sin 2x}{4} - \frac{x}{4} - \frac{1}{4}\int \frac{1}{2}(1 + \cos 4x)\, dx - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + C$$
$$= \frac{x}{2} + \frac{\sin 2x}{4} - \frac{x}{4} - \frac{x}{8} - \frac{1}{8}\cdot\frac{1}{4}\sin 4x - \frac{\sin 2x}{4} + C$$
$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

6. $\int \tan^3 x \sec^5 x\, dx$

$$= \int \tan^2 x \sec^4 x \tan x \sec x\, dx$$
$$= \int (\sec^2 x - 1)\sec^4 x \tan x \sec x\, dx \qquad \left\{\begin{array}{l}\text{let } u = \sec x \\ du = \sec x \tan x\, dx\end{array}\right.$$
$$= \int (u^2 - 1)u^4\, du$$
$$= \int (u^6 - u^4)\, du$$
$$= \frac{u^7}{7} - \frac{u^5}{5} + c$$
$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

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v2.1

Math 1211: Worksheet #2 Winter 2016

Name: Key	A#:	Section:

1. $\int x(\ln x)^2\,dx$ Let $u=(\ln x)^2$ $\quad dv = x\,dx$

$du = 2(\ln x)\cdot\frac{1}{x}dx \quad v=\frac{x^2}{2}$

$$\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \int \frac{x^2}{2}\cdot 2\ln x\cdot\frac{1}{x}dx$$

$$= \frac{x^2}{2}(\ln x)^2 - \int x\ln x\,dx \quad \left\{ \begin{array}{ll} \text{Let } U=\ln x & dV = x\,dx \\ dU=\frac{1}{x}dx & V=\frac{x^2}{2}\end{array}\right.$$

$$= \frac{x^2}{2}(\ln x)^2 - \left(\frac{x^2}{2}\ln x - \int\frac{x^2}{2}\cdot\frac{1}{x}dx\right)$$

$$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{1}{2}\frac{x^2}{2} = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4}$$

$$\text{or } \frac{x^2}{4}\left(2(\ln x)^2 - 2\ln x + 1\right).$$

2. $\int x^3e^{x^2}\,dx$

$$\left.\begin{array}{l}\text{Let } w=x^2 \\ dw = 2x\,dx \\ \frac{dw}{2} = x\,dx\end{array}\right\} \int x^3e^{x^2}dx = \int x^2e^{x^2}x\,dx$$

$$= \int we^w\frac{dw}{2}$$

$$= \frac{1}{2}\int we^w dw$$

Let $u=w \quad dv=e^w$

$du=dw \quad v=e^w$

$$= \frac{1}{2}\left[we^w - \int e^w dw\right]$$

$$= \frac{1}{2}we^w - \frac{1}{2}e^w + C$$

$$= \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

$$\text{or } \frac{1}{2}e^{x^2}(x^2-1) + C$$

3. $\int e^{3x}\cos x\,dx$

Let $u=\cos x \quad dv = e^{3x}dx$

$du = -\sin x\,dx \quad v=\frac{1}{3}e^{3x}$

$$\int e^{3x}\cos x\,dx = \frac{1}{3}e^{3x}\cos x - \int\frac{1}{3}e^{3x}(-\sin x)dx$$

$$= \frac{1}{3}e^{3x}\cos x + \frac{1}{3}\int e^{3x}\sin x\,dx \quad \left\{\begin{array}{ll}\text{Let } U=\sin x & dV=e^{3x}dx \\ dU=\cos x\,dx & V=\frac{1}{3}e^{3x}\end{array}\right.$$

$$= \frac{1}{3}e^{3x}\cos x + \frac{1}{3}\left[\frac{1}{3}e^{3x}\sin x - \int\frac{1}{3}e^{3x}\cos x\,dx\right]$$

$$= \frac{1}{3}e^{3x}\cos x + \frac{1}{9}e^{3x}\sin x - \frac{1}{9}\int e^{3x}\cos x\,dx$$

$$\therefore \frac{10}{9}\int e^{3x}\cos x\,dx = \frac{1}{3}e^{3x}\cos x + \frac{1}{9}e^{3x}\sin x + C$$

Hence, $\int e^{3x}\cos x\,dx = \frac{9}{10}\left[\frac{1}{3}e^{3x}\cos x + \frac{1}{9}e^{3x}\sin x\right] + C$

$$\text{or } = \frac{3}{10}e^{3x}\cos x + \frac{1}{10}e^{3x}\sin x + C$$

$$\text{or } \frac{1}{10}e^{3x}(3\cos x + \sin x) + C$$

v2.2

Q.3: $\int e^{3x}\cos x\,dx$ $\left\{\begin{array}{ll} \text{let } u = e^{3x} & dv = \cos x\,dx \\ du = 3e^{3x}dx & v = \sin x \end{array}\right.$

$= e^{3x}\sin x - \int 3e^{3x}\sin x\,dx$

$= e^{3x}\sin x - 3\int e^{3x}\sin x\,dx$ $\left\{\begin{array}{ll} \text{let } U = e^{3x} & dV = \sin x\,dx \\ dU = 3e^{3x}dx & V = -\cos x \end{array}\right.$

$= e^{3x}\sin x - 3\left[-e^{3x}\cos x - \int 3e^{3x}(-\cos x)\,dx\right]$

$= e^{3x}\sin x + 3e^{3x}\cos x - 9\int e^{3x}\cos x\,dx$

$\therefore\ 10\int e^{3x}\cos x\,dx = e^{3x}\sin x + 3e^{3x}\cos x + C$

Hence $\int e^{3x}\cos x\,dx = \frac{1}{10}e^{3x}\sin x + \frac{3}{10}e^{3x}\cos x + C$

4. $\int \sin^2 x \cos^3 x \, dx$ Power of cosine is odd

$= \int \sin^2 x \cos^2 x \cos x \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$\left\{ \begin{array}{l} \text{let } u = \sin x \\ du = \cos x \, dx \end{array} \right.$

$= \int u^2 (1 - u^2) \, du$

$= \int (u^2 - u^4) \, du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

5. $\int \sin^2 x \cos^2 x \, dx$

$* \sin 2x = 2 \sin x \cos x$

$\therefore \frac{\sin 2x}{2} = \sin x \cos x$

$= \int (\sin x \cos x)^2 \, dx$

$= \int \left(\frac{\sin 2x}{2}\right)^2 dx$

$= \frac{1}{4} \int \sin^2 2x \, dx$

$= \frac{1}{4} \int \frac{1}{2} (1 - \cos 4x) \, dx$

$= \frac{1}{8} x - \frac{1}{8} \cdot \frac{1}{4} \sin 4x + C$

$= \frac{x}{8} - \frac{1}{32} \sin 4x + C$

6. $\int \sec^6 x \, dx$ Power of secant is even

$= \int \sec^4 x \sec^2 x \, dx$

$= \int (\sec^2 x)^2 \sec^2 x \, dx$

$\left\{ \begin{array}{l} \text{let } u = \tan x \\ du = \sec^2 x \, dx \end{array} \right.$

$= \int (1 + \tan^2 x)^2 \sec^2 x \, dx$

$= \int (1 + u^2)^2 \, du$

$= \int (1 + u^4 + 2u^2) \, du$

$= u + \frac{u^5}{5} + \frac{2u^3}{3} + C$

$= \tan x + \frac{\tan^5 x}{5} + \frac{2 \tan^3 x}{3} + C$

Math 1211: Worksheet #2 Winter 2016

Name: SOLUTIONS	A#:	Section:

1. $\int (\ln x)^2\, dx$

$= x(\ln x)^2 - 2\int \ln x\, dx$

PARTS: $u = (\ln x)^2$ $dv = dx$
$du = 2\ln x \cdot \frac{1}{x}dx$ $v = x$

$= x(\ln x)^2 - 2\left(x\ln x - \int 1\, dx\right)$

PARTS: $u = \ln x$ $dv = dx$
$du = \frac{1}{x}dx$ $v = x$

$= \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}$

2. $\int e^{2x}\sin x\, dx$

$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\, dx$

$u = e^{2x}$ $dv = \sin x\, dx$
$du = 2e^{2x}dx$ $v = -\cos x$

$= -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2\int e^{2x}\sin x\, dx\right)$

$u = e^{2x}$ $dv = \cos x\, dx$
$du = 2e^{2x}dx$ $v = \sin x$

$\Rightarrow 5\int e^{2x}\sin x\, dx = -e^{2x}\cos x + 2e^{2x}\sin x$

$\Rightarrow \int e^{2x}\sin x\, dx = \boxed{\frac{1}{5}\left(-e^{2x}\cos x + 2e^{2x}\sin x\right) + C}$

3. $\int \tan^{-1}\sqrt{x}\, dx$

$= 2\int y\tan^{-1}y\, dy$

SUB $y = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}}dx$
$\Rightarrow dx = 2\sqrt{x}\, dy$
$\Rightarrow dx = 2y\, dy$

$= y^2\tan^{-1}y - \int \frac{y^2}{1+y^2}dy$

PARTS: $u = \tan^{-1}y$ $dv = y\,dy$
$du = \frac{1}{1+y^2}dy$ $v = \frac{1}{2}y^2$

$= y^2\tan^{-1}y - \int\left(1 - \frac{1}{1+y^2}\right)dy$

SINCE $\frac{y^2}{1+y^2} = \frac{(1+y^2)-1}{1+y^2}$
$= 1 - \frac{1}{1+y^2}$

$= y^2\tan^{-1}y - y + \tan^{-1}y + C$

$= \boxed{x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C}$

v2.3

4. $\int \sin^2 x \cos^5 x \, dx$

$$= \int \sin^2 x \cos^4 x \cdot \cos x \, dx \qquad u = \sin x, \; du = \cos x \, dx$$

$$= \int u^2 (1-u^2)^2 \, du$$

$$= \int (u^2 - 2u^4 + u^6) \, du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}$$

5. $\int \sin^2 x \cos^2 x \, dx$

$$= \int \frac{1}{2}(1-\cos 2x) \cdot \frac{1}{2}(1+\cos 2x) \, dx$$

$$= \frac{1}{4}\int (1 - \cos^2 2x) \, dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \boxed{\frac{1}{8}x - \frac{1}{32}\sin 4x + C}$$

OR write

$$\sin^2 x \cos^2 x = \left(\frac{\sin 2x}{2}\right)^2$$

$$= \frac{1}{4} \cdot \frac{1}{2}(1 - \cos 4x)$$

OR

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \int \sin^2 x (1 - \sin^2 x) \, dx$$

$$= \int \sin^2 x \, dx - \int \sin^4 x \, dx$$

$$= \ldots$$

6. $\int \tan^5 x \sec x \, dx$

$$= \int \tan^4 x \cdot \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1)^2 \cdot \tan x \sec x \, dx \qquad u = \sec x, \; du = \sec x \tan x \, dx$$

$$= \int (u^2 - 1)^2 \, du$$

$$= \int (u^4 - 2u^2 + 1) \, du$$

$$= \frac{1}{5}u^5 - \frac{2}{3}u^3 + u + C$$

$$= \boxed{\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C}$$