Name: Key	A#:	Section:

1. $\int \frac{x^2}{(x^2+9)^{3/2}}\,dx$

let $x = 3\tan\theta$

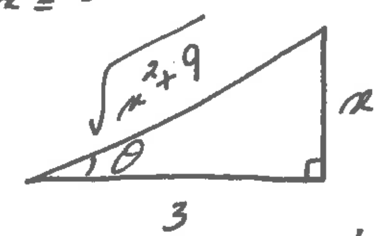
$dx = 3\sec^2\theta\, d\theta$

$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$

$= \sqrt{9\tan^2\theta+9}$

$= \sqrt{9(\tan^2\theta+1)}$

$= 3\sqrt{\sec^2\theta} = 3\sec\theta$



$= \int \frac{(3\tan\theta)^2}{(3\sec\theta)^3} \cdot 3\sec^2\theta\, d\theta$

$= \int \frac{9\tan^2\theta \cdot 3\sec^2\theta}{27\sec^3\theta}\, d\theta$

$= \int \frac{\tan^2\theta}{\sec\theta}\, d\theta$

$= \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos\theta\, d\theta$

$= \int \frac{\sin^2\theta}{\cos\theta}\, d\theta$

$= \int \frac{(1-\cos^2\theta)}{\cos\theta}\, d\theta$

$= \int \left(\frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}\right) d\theta$

$= \int \sec\theta\, d\theta - \int \cos\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| - \sin\theta + C = \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| - \frac{x}{\sqrt{x^2+9}} + C$

$\tan\theta = \frac{x}{3}$

$\sin\theta = \frac{x}{\sqrt{x^2+9}}$

$\cos\theta = \frac{3}{\sqrt{x^2+9}}$

$\sec\theta = \frac{\sqrt{x^2+9}}{3}$

$* \int \sec\theta\, d\theta$

$= \int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\, d\theta$

$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$

let $u = \sec\theta + \tan\theta$

$du = \sec\theta\tan\theta + \sec^2\theta\, d\theta$

$= \int \frac{1}{u}\, du$

$= \ln|u| + C$

$= \ln|\sec\theta + \tan\theta| + C$

2. $\int \frac{dx}{x^3\sqrt{4x^2-1}} = \int \frac{dx}{x^3\sqrt{4(x^2-1/4)}} = \frac{1}{2}\int \frac{dx}{x^3\sqrt{x^2-1/4}}$

Let $x = \frac{1}{2}\sec\theta$

$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$\sqrt{x^2 - 1/4} = \sqrt{(1/2\sec\theta)^2 - 1/4}$

$= \sqrt{1/4\sec^2\theta - 1/4}$

$= \sqrt{1/4(\sec^2\theta - 1)}$

$= \frac{1}{2}\sqrt{\tan^2\theta}$

$= \frac{1}{2}\tan\theta$

$= \frac{1}{2}\int \frac{1/2\sec\theta\tan\theta\, d\theta}{(1/2\sec\theta)^3\, 1/2\tan\theta}$

$= \frac{1}{2}\int \frac{\sec\theta\, d\theta}{1/8\sec^3\theta}$

$= 4\int \frac{1}{\sec^2\theta}\, d\theta$

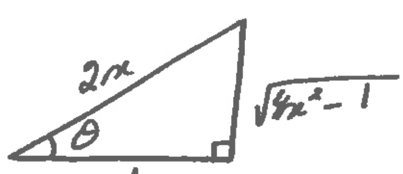
$= 4\int \cos^2\theta\, d\theta$

$= 4\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= 2\int 1\, d\theta + 2\int \cos 2\theta\, d\theta$

$= 2\theta + 2\left(\frac{1}{2}\sin 2\theta\right) + C$

$= 2\theta + 2\sin\theta\cos\theta + C$



$\sec\theta = 2x$

$\cos\theta = \frac{1}{2x}$

$\sin\theta = \frac{\sqrt{4x^2-1}}{2x}$

$= 2\theta + 2\sin\theta\cos\theta + C$

$= 2\sec^{-1}(2x) + 2\left(\frac{\sqrt{4x^2-1}}{2x}\right)\left(\frac{1}{2x}\right) + C = 2\sec^{-1}(2x) + \frac{\sqrt{4x^2-1}}{2x^2} + C$

v3.1

3. $\int \frac{x^2+1}{x(x+2)^3}\,dx$

$$\frac{x^2+1}{x(x+2)^3} = \frac{A}{x} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

$$\begin{cases} A = \frac{1}{8} \\ B = -\frac{1}{8} \\ C = \frac{3}{4} \\ D = -\frac{5}{2} \end{cases}$$

$$x^2+1 = A(x+2)^3 + Bx(x+2)^2 + Cx(x+2) + Dx$$

Let $x=-2$:

$$(-2)^2+1 = -2D \Rightarrow -2D = 5 \Rightarrow \boxed{D = \frac{-5}{2}}$$

Let $x=0$:

$$1 = A(2)^3 \Rightarrow 8A = 1 \Rightarrow \boxed{A = 1/8}$$

$$\therefore x^2+1 = \frac{1}{8}(x+2)^3 + Bx(x+2)^2 + Cx(x+2) - \frac{5}{2}x$$

Let $x=1$:

$$1^2+1 = \frac{1}{8}(3)^3 + B(3)^2 + C(3) - \frac{5}{2}$$

$$2 = \frac{27}{8} + 9B + 3C - \frac{5}{2}$$

$$\therefore 9B + 3C = \frac{9}{8}$$

Let $x=-1$:

$$(-1)^2+1 = \frac{1}{8}(1)^3 - B(1)^2 - C(1) + \frac{5}{2}$$

$$2 = \frac{1}{8} - B - C + \frac{5}{2}$$

$$\therefore B + C = \frac{5}{8}$$

$$\begin{cases} 9B + 3C = \frac{9}{8} \\ B + C = \frac{5}{8} \end{cases} \Rightarrow \begin{cases} 9B + 3C = \frac{9}{8} \\ -3B - 3C = -\frac{15}{8} \end{cases}$$

$$\therefore \boxed{B = \frac{-1}{8}},\ \boxed{C = \frac{3}{4}}$$

$$\int \frac{x^2+1}{x(x+2)^3}\,dx$$

$$= \int \frac{1}{8}\left(\frac{1}{x}\right)dx + \int \frac{-1/8}{x+2}\,dx + \int \frac{3/4}{(x+2)^2}\,dx + \int \frac{-5/2}{(x+2)^3}$$

$$= \frac{1}{8}\int \frac{1}{x}\,dx - \frac{1}{8}\int \frac{1}{x+2}\,dx + \frac{3}{4}\int \frac{1}{(x+2)^2}\,dx - \frac{5}{2}\int \frac{1}{(x+2)^3}$$

$$= \frac{1}{8}\ln|x| - \frac{1}{8}\ln|x+2| + \frac{3}{4}\left(\frac{-1}{x+2}\right) - \frac{5}{2}\left(\frac{-1}{2(x+2)^2}\right) + C$$

$$= \frac{1}{8}\ln|x| - \frac{1}{8}\ln|x+2| - \frac{3}{4(x+2)} + \frac{5}{4(x+2)^2} + C$$

4. $\int \frac{dx}{x(\sqrt{x}-1)}$ [*Hint:* Try a substitution first.]

$$\begin{cases} \text{let } u = \sqrt{x} \text{ and } u^2 = x \\ du = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\,dx \\ 2\sqrt{x}\,du = dx \\ 2u\,du = dx \end{cases}$$

$$= \int \frac{2u\,du}{u^2(u-1)}$$

$$= 2\int \frac{du}{u(u-1)} = 2\int\left(\frac{-1}{u} + \frac{1}{u-1}\right)du$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$
$$= Au - A + Bu$$
$$= (A+B)u - A$$

$$A + B = 0 \Rightarrow A = -B$$

$$-A = 1 \Rightarrow A = -1$$

and $B = 1$

$$= -2\int \frac{1}{u}\,du + 2\int \frac{1}{u-1}\,du$$

$$= -2\ln|u| + 2\ln|u-1| + C$$

$$= -2\ln\sqrt{x} + 2\ln(\sqrt{x}-1) + C$$

or

$$2\ln(\sqrt{x}-1) - 2\ln\sqrt{x} + C$$

Math 1211: Worksheet #3 Winter 2016

Name: Key A#: Section:

1. $\int \frac{\sqrt{9x^2-1}}{x^2}\,dx$

Let $x = \frac{1}{3}\sec\theta$

$dx = \frac{1}{3}\sec\theta\tan\theta\, d\theta$

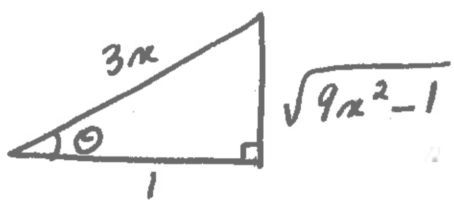
$\sqrt{x^2 - 1/9}$
$= \sqrt{\left(\frac{1}{3}\sec\theta\right)^2 - 1/9}$
$= \sqrt{\frac{1}{9}\sec^2\theta - 1/9}$
$= \frac{1}{3}\sqrt{\tan^2\theta}$
$= \frac{1}{3}\tan\theta$

$$= \int \frac{\sqrt{9(x^2 - 1/9)}}{x^2}\,dx$$

$$= 3\int \frac{\sqrt{x^2-1/9}}{x^2}\,dx$$

$$= 3\int \frac{1/3\tan\theta}{\left(\frac{1}{3}\sec\theta\right)^2}\, 1/3 \sec\theta\tan\theta\, d\theta$$

$$= 3\int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$



$$= 3\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos\theta\, d\theta$$

$$= 3\int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$\sec\theta = 3x$

$\cos\theta = \frac{1}{3x}$, $\sin\theta = \frac{\sqrt{9x^2-1}}{3x}$, $\tan\theta = \sqrt{9x^2-1}$

$$= 3\int \frac{(1-\cos^2\theta)}{\cos\theta}\,d\theta$$

$$= 3\int \frac{1}{\cos\theta}\,d\theta - 3\int \cos\theta\, d\theta = 3\int \sec\theta\, d\theta - 3\sin\theta + C$$

$$= 3\ln|\sec\theta + \tan\theta| - 3\sin\theta + C$$

$$= 3\ln\left|3x + \sqrt{9x^2-1}\right| - 3\left(\frac{\sqrt{9x^2-1}}{3x}\right) + C$$

$$= 3\ln\left|3x + \sqrt{9x^2-1}\right| - \frac{\sqrt{9x^2-1}}{x} + C$$

2. $\int \frac{x^2}{(3x^2+4)^2}\,dx$

$$= \int \frac{x^2}{\left[3\left(x^2+\frac{4}{3}\right)\right]^2}\,dx$$

$$= \frac{1}{9}\int \frac{x^2}{(x^2+4/3)^2}\,dx$$

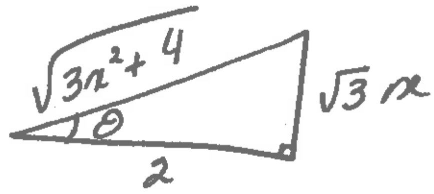
Let $x = \frac{2}{\sqrt{3}}\tan\theta$

$dx = \frac{2}{\sqrt{3}}\sec^2\theta\, d\theta$

$x^2 + \frac{4}{3} = \left(\frac{2}{\sqrt{3}}\tan\theta\right)^2 + \frac{4}{3}$
$= \frac{4}{3}\tan^2\theta + \frac{4}{3}$
$= \frac{4}{3}\sec^2\theta$

$$= \frac{1}{9}\int \frac{\left(2/\sqrt{3}\tan\theta\right)^2}{\left(4/3\sec^2\theta\right)^2}\cdot 2/\sqrt{3}\sec^2\theta\, d\theta$$

$$= \frac{1}{6\sqrt{3}}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$



$$= \frac{1}{6\sqrt{3}}\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta\, d\theta$$

$\tan\theta = \frac{\sqrt{3}}{2}x$, $\sin\theta = \frac{\sqrt{3}\,x}{\sqrt{3x^2+4}}$, $\cos\theta = \frac{2}{\sqrt{3x^2+4}}$

$$= \frac{1}{6\sqrt{3}}\int \sin^2\theta\, d\theta$$

$$= \frac{1}{6\sqrt{3}}\int \frac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{12\sqrt{3}}\int 1\, d\theta - \frac{1}{12\sqrt{3}}\int \cos 2\theta\, d\theta$$

$$= \frac{1}{12\sqrt{3}}\theta - \frac{1}{12\sqrt{3}}\cdot\frac{1}{2}\sin 2\theta + C$$

$$= \frac{1}{12\sqrt{3}}\theta - \frac{1}{12\sqrt{3}}\sin\theta\cos\theta + C$$

$$= \frac{1}{12\sqrt{3}}\theta - \frac{1}{12\sqrt{3}}\left(\frac{\sqrt{3}\,x}{\sqrt{3x^2+4}}\right)\left(\frac{2}{\sqrt{3x^2+4}}\right) + C$$

v3.2

$$= \frac{1}{12\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{x}{6(3x^2+4)} + C$$

3. $\int \frac{x^3+1}{x^2(x-2)^2}\,dx$

$$\frac{x^3+1}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-2)} + \frac{D}{(x-2)^2}$$

$$x^3+1 = Ax(x-2)^2 + B(x-2)^2 + Cx^2(x-2) + Dx^2$$

Let $x = 0$:

$$1 = B(-2)^2 \Rightarrow 4B = 1 \Rightarrow \boxed{B = \frac{1}{4}}$$

Let $x = 2$:

$$2^3+1 = 4D \Rightarrow 4D = 9 \Rightarrow \boxed{D = \frac{9}{4}}$$

Now $x^3+1 = (A+C)x^3 + (B+D-4A-2C)x^2 + (4A-4B)x + 4B$

Equating the coefficients we obtain:

$$\left.\begin{array}{l} A+C=1 \\ B+D-4A-2C=0 \\ 4A-4B=0 \\ 4B=1 \end{array}\right\} \Rightarrow \begin{array}{l} A+C=1 \Rightarrow C = 1-A \Rightarrow \boxed{C=\frac{3}{4}} \\ 4A+2C=\frac{10}{4} \\ \boxed{A=\frac{1}{4}} \end{array}$$

$$\int \frac{x^3+1}{x^2(x-2)^2}\,dx = \int\left(\frac{1/4}{x} + \frac{1/4}{x^2} + \frac{3/4}{(x-2)} + \frac{9/4}{(x-2)^2}\right)dx$$

$$= \frac{1}{4}\int\frac{1}{x}\,dx + \frac{1}{4}\int\frac{1}{x^2}\,dx + \frac{3}{4}\int\frac{1}{x-2}\,dx + \frac{9}{4}\int\frac{1}{(x-2)^2}\,dx$$

$$= \frac{1}{4}\ln|x| - \frac{1}{4x} + \frac{3}{4}\ln|x-2| - \frac{9}{4(x-2)} + C$$

$$\text{or} = \frac{1}{4}\left[\ln|x| - \frac{1}{x} + 3\ln|x-2| - \frac{9}{(x-2)}\right] + C$$

4. $\int \frac{e^x}{e^{2x}-1}\,dx$ [*Hint:* Try a substitution first.]

Let $u = e^x$

$du = e^x\,dx$

$$= \int \frac{1}{u^2-1}\,du$$

$$= \int \frac{1}{(u+1)(u-1)}\,du$$

$$= \int\left(\frac{-1/2}{u+1} + \frac{1/2}{u-1}\right)du$$

$$= -\frac{1}{2}\int\frac{1}{u+1}\,du + \frac{1}{2}\int\frac{1}{u-1}\,du$$

$$= -\frac{1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1| + C$$

$$= \frac{1}{2}\ln|e^x-1| - \frac{1}{2}\ln|e^x+1| + C$$

$$\text{or } \frac{1}{2}\left[\ln|e^x-1| - \ln|e^x+1|\right] + C$$

$$\frac{1}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1}$$

$$1 = A(u-1) + B(u+1)$$

$$= Au - A + Bu + B$$

$$= (A+B)u + (B-A)$$

$$\left.\begin{array}{l} A+B=0 \\ B-A=1 \end{array}\right\} \Rightarrow \begin{cases} A=-B \\ 2B=1 \Rightarrow \boxed{B=\frac{1}{2}} \end{cases}$$

and $\boxed{A=\frac{-1}{2}}$

Math 1211: Worksheet #3 Winter 2016

Name: SOLUTIONS	A#:	Section:

1. $\int \frac{\sqrt{9x^2-1}}{x^2}\,dx$ Let $x = \frac{1}{3}\sec\theta$, so $dx = \frac{1}{3}\sec\theta\tan\theta$

$= \int \frac{\tan\theta}{\frac{1}{9}\sec^2\theta} \cdot \frac{1}{3}\sec\theta\tan\theta\, d\theta$ and $\sqrt{9x^2-1} = \tan\theta$

$= 3\int \frac{\tan^2\theta}{\sec\theta}\, d\theta$

$= 3\int \frac{\sec^2\theta - 1}{\sec\theta}\, d\theta$

$= 3\int \sec\theta\, d\theta - 3\int \cos\theta\, d\theta$

$= 3\ln|\sec\theta + \tan\theta| - 3\sin\theta + C$

$= \boxed{3\ln\left|3x + \sqrt{9x^2-1}\right| - \frac{\sqrt{9x^2-1}}{x} + C}$

$\sec\theta = 3x$

2. $\int \frac{\sqrt{4-3x^2}}{x}\,dx$ Let $x = \frac{2}{\sqrt{3}}\sin\theta$ so $dx = \frac{2}{\sqrt{3}}\cos\theta\, d\theta$

$= \int \frac{2\cos\theta}{\frac{2}{\sqrt{3}}\sin\theta} \cdot \frac{2}{\sqrt{3}}\cos\theta\, d\theta$ and $\sqrt{4-3x^2} = 2\cos\theta$

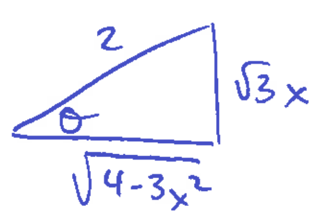
$= 2\int \frac{\cos^2\theta}{\sin\theta}\, d\theta$

$= 2\int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$

$= 2\int \csc\theta\, d\theta - 2\int \sin\theta\, d\theta$

$\sin\theta = \frac{\sqrt{3}x}{2}$

$= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C$



$= \boxed{2\ln\left|\frac{2}{\sqrt{3}x} - \frac{\sqrt{4-3x^2}}{\sqrt{3}x}\right| + \sqrt{4-3x^2} + C}$

$\left(= 2\ln\left|\frac{2-\sqrt{4-3x^2}}{x}\right| + \sqrt{4-3x^2} + C'\right)$

v3.3

3. $\int \frac{x^3+1}{x^2(x-2)^2}\,dx$

Set $\frac{x^3+1}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$

$\Rightarrow \quad x^3+1 = Ax(x-2)^2 + B(x-2)^2 + Cx^2(x-2) + Dx^2$

Let $x=0$ to get $1 = 4B \Rightarrow B = 1/4$

Let $x=2$ to get $9 = 4D \Rightarrow D = 9/4$

Compare coeff of x^3 to get $1 = A + C \Rightarrow C = 1-A$

Let $x=1$ to get $2 = A + B - C + D$

$\Rightarrow 2 = A + 1/4 - (1-A) + 9/4$

$\Rightarrow A = 1/4$

$\Rightarrow C = 1 - A = 3/4$

So $\int \frac{x^3+1}{x^2(x-2)^2}\,dx = \int \left(\frac{1}{4}\cdot\frac{1}{x} + \frac{1}{4}\cdot\frac{1}{x^2} + \frac{3}{4}\cdot\frac{1}{x-2} + \frac{9}{4}\cdot\frac{1}{(x-2)^2} \right) dx$

$= \boxed{\frac{1}{4}\ln|x| - \frac{1}{4x} + \frac{3}{4}\ln|x-2| - \frac{9}{4(x-2)} + C}$

4. $\int \frac{e^x}{e^{2x}-1}\,dx$ [*Hint:* Try a substitution first.]

$= \int \frac{du}{u^2-1}$ Let $u = e^x$ so $du = e^x\,dx$ and $e^{2x} = u^2$

$= \int \frac{du}{(u-1)(u+1)}$

Let $\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1} \Rightarrow 1 = A(u+1) + B(u-1)$

Set $u=1$ to get $2A = 1 \Rightarrow A = 1/2$

Set $u=-1$ to get $-2B = 1 \Rightarrow B = -1/2$

So $\int \frac{du}{(u-1)(u+1)} = \int \left(\frac{1}{2}\cdot\frac{1}{u-1} - \frac{1}{2}\cdot\frac{1}{u+1} \right) du$

$= \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$

$= \boxed{\frac{1}{2}\ln|e^x-1| - \frac{1}{2}\ln|e^x+1| + C}$

v3.3