Math 1211: Worksheet #4 | Winter 2016

Name: Key	A#:	Section:

1. Decide whether the following improper integrals converge or diverge. If an integral converges, find its value.

(a) $\int_0^{\pi/2} \tan x\,dx$

$= \lim_{R\to \pi/2^-} \int_0^R \tan x\,dx$

$= \lim_{R\to \pi/2^-} \left[-\ln|\cos x|\right]_0^R$

$= \lim_{R\to \pi/2^-} \left[-\ln|\cos R| + \ln|\cos 0|\right]$

$= \lim_{R\to \pi/2^-} \left[-\ln|\cos R| + \ln 1\right]$

$= -\lim_{R\to \pi/2^-} \ln|\cos R| = \infty$ ∴ Diverges.

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\left\{\begin{array}{l} \text{let } u = \cos x \\ du = -\sin x\,dx \end{array}\right.$

$= -\int \frac{1}{u}\,du$

$= -\ln|u| + C$

$= -\ln|\cos x| + C$

(b) $\int_0^1 x\ln x\,dx$

$= \lim_{R\to 0^+} \int_R^1 x\ln x\,dx$

$= \lim_{R\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_R^1$

$= \lim_{R\to 0^+} \left[\left(\frac{1}{2}\cancel{\ln 1}^{\,0} - \frac{1}{4}\right) - \left(\frac{R^2}{2}\ln R - \frac{R^2}{4}\right)\right]$

$= -\frac{1}{4} - \frac{1}{2}\lim_{R\to 0^+} R^2 \ln R + 0$

$= -\frac{1}{4}$ ∴ Converges.

$\int x\ln x\,dx$ $\left\{\begin{array}{ll} \text{let } u = \ln x & dv = x\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^2}{2} \end{array}\right.$

$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2} + C$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

* $-\frac{1}{2}\lim_{R\to 0^+} R^2 \ln R$ $\quad 0\cdot(-\infty)$

$= -\frac{1}{2}\lim_{R\to 0^+} \frac{\ln R}{1/R^2}$ $\quad -\infty/\infty$

$\overset{H}{=} -\frac{1}{2}\lim_{R\to 0^+} \frac{1/R}{-2/R^3} = \frac{1}{2}\lim_{R\to 0^+} \frac{R^2}{2} = 0$

(c) $\int_{-\infty}^{\infty} \frac{dx}{1+4x^2} = \int_{-\infty}^{0} \frac{dx}{1+4x^2} + \int_0^{\infty} \frac{dx}{1+4x^2}$

$= \lim_{R\to -\infty} \int_R^0 \frac{dx}{1+4x^2} + \lim_{R\to \infty} \int_0^R \frac{dx}{1+4x^2}$

$= \lim_{R\to -\infty} \left[\frac{1}{2}\tan^{-1}(2x)\right]_R^0 + \lim_{R\to\infty} \left[\frac{1}{2}\tan^{-1}(2x)\right]_0^R$

$= \frac{1}{2}\lim_{R\to -\infty}\left(\cancel{\tan^{-1} 0}^{\,0} - \tan^{-1}(2R)\right) + \frac{1}{2}\lim_{R\to\infty}\left(\tan^{-1}(2R) - \cancel{\tan^{-1} 0}^{\,0}\right)$

$= -\frac{1}{2}\lim_{R\to -\infty} \tan^{-1}(2R) + \frac{1}{2}\lim_{R\to\infty} \tan^{-1}(2R)$

$= -\frac{1}{2}\left(-\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ ∴ Converges.

$\int \frac{1}{1+4x^2}\,dx$ $\left\{\begin{array}{l} \text{let } u = 2x \\ du = 2\,dx \end{array}\right.$

$= \int \frac{1}{1+(2x)^2}\,dx$

$= \frac{1}{2}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{2}\tan^{-1}(u) + C$

$= \frac{1}{2}\tan^{-1}(2x) + C$

* Or $\int_{-\infty}^{\infty} \frac{1}{1+4x^2}\,dx = 2\int_0^{\infty} \frac{1}{1+4x^2}\,dx$ (since $\frac{1}{1+4x^2}$ is an even function)

2. (a) Determine $\int \frac{x^2+x}{x^4-16}\,dx$ $= \int \frac{x^2+x}{(x^2-4)(x^2+4)}\,dx = \int \frac{x^2+x}{(x-2)(x+2)(x^2+4)}\,dx$

$$\frac{x^2+x}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

$$x^2+x = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)$$

Let $x = 2$:

$6 = A(4)(8)$

$\Rightarrow 32A = 6 \Rightarrow \boxed{A = \frac{3}{16}}$

Let $x = -2$:

$2 = B(-4)(8)$

$\Rightarrow -32B = 2 \Rightarrow \boxed{B = \frac{-1}{16}}$

$A + B + C = 0 \Rightarrow C = -A - B$

$\Rightarrow C = -\frac{3}{16} + \frac{1}{16} \Rightarrow \boxed{C = \frac{-1}{8}}$

$8A - 8B - 4D = 0$

$\Rightarrow -4D = 8B - 8A$

$\Rightarrow 4D = 8\left(\frac{-1}{16}\right) - 8\left(\frac{3}{16}\right)$

$\Rightarrow \boxed{D = +\frac{1}{2}}$

$$\int \frac{x^2+x}{x^4-16}\,dx = \int \left(\frac{3/16}{x-2} - \frac{1/16}{x+2} + \frac{-\frac{1}{8}x + \frac{1}{2}}{x^2+4} \right) dx$$

$$= \frac{3}{16}\int \frac{1}{x-2}\,dx - \frac{1}{16}\int \frac{1}{x+2}\,dx - \frac{1}{8}\int \frac{x}{x^2+4}\,dx + \frac{1}{2}\int \frac{1}{x^2+4}\,dx$$

$$= \frac{3}{16}\ln|x-2| - \frac{1}{16}\ln|x+2| - \frac{1}{16}\ln|x^2+4| + \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

(b) Use (a) to compute $\int_3^\infty \frac{x^2+x}{x^4-16}\,dx.$ $= \lim_{R\to\infty} \int_3^R \frac{x^2+x}{x^4-16}\,dx$

$$= \lim_{R\to\infty} \left[\frac{3}{16}\ln|x-2| - \frac{1}{16}\ln|x+2| - \frac{1}{16}\ln|x^2+4| + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) \right]_3^R$$

$$= \lim_{R\to\infty} \Big[\frac{3}{16}\ln|R-2| - \frac{1}{16}\ln|R+2| - \frac{1}{16}\ln|R^2+4| + \frac{1}{4}\tan^{-1}\left(\frac{R}{2}\right)$$
$$- \cancel{\frac{3}{16}\ln 1}^{\,0} + \frac{1}{16}\ln 5 + \frac{1}{16}\ln 13 - \frac{1}{4}\tan^{-1}\left(\frac{3}{2}\right) \Big]$$

$$= \lim_{R\to\infty} \left[\ln(R-2)^{3/16} - \ln(R+2)^{1/16} - \ln(R^2+4)^{1/16} + \frac{1}{4}\tan^{-1}(R/2) + \frac{1}{16}\ln 65 - \frac{1}{4}\tan^{-1}\left(\frac{3}{2}\right) \right]$$

$$= \lim_{R\to\infty} \frac{1}{4}\tan^{-1}\left(\frac{R}{2}\right) + \lim_{R\to\infty} \left(\ln \frac{(R-2)^{3/16}}{(R+2)^{1/16}} - \ln(R^2+4)^{1/16} \right) + \frac{1}{16}\ln 65 - \frac{1}{4}\tan^{-1}(3/2)$$

$$= \frac{1}{4}\lim_{R\to\infty} \tan^{-1}(R/2) + \lim_{R\to\infty} \ln \frac{(R-2)^{3/16}}{(R+2)^{1/16}(R^2+4)^{1/16}} + \frac{1}{16}\ln 65 - \frac{1}{4}\tan^{-1}(3/2)$$

$$= \frac{\pi}{8} + \frac{1}{16}\ln 65 - \frac{1}{4}\tan^{-1}(3/2) \quad \text{or} \quad \frac{1}{16}\left(2\pi + \ln 65 - 4\tan^{-1}(3/2)\right)$$

3. Use comparison to decide whether $\int_1^\infty \frac{dx}{\sqrt{x}(x+1)}$ converges or diverges.

$$\int_1^\infty \frac{dx}{\sqrt{x}(x+1)} = \int_1^\infty \frac{dx}{x^{3/2} + x^{1/2}}$$

$$x^{3/2} + x^{1/2} \geq x^{3/2}$$

$$\frac{1}{x^{3/2} + x^{1/2}} \leq \frac{1}{x^{3/2}}$$

$\int_1^\infty \frac{1}{x^{3/2}}\,dx \rightsquigarrow p = \frac{3}{2} > 1 \;\therefore$ converges by the P-integral theorem.

$\therefore \int_1^\infty \frac{dx}{\sqrt{x}(x+1)}$ also converges by the Comparison test

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Name: SOLUTIONS	A#:	Section:

1. Decide whether the following improper integrals converge or diverge. If an integral converges, find its value.

(a) $\int_0^{\pi/2} \sec x \, dx$

$= \lim_{R \to \pi/2^-} \int_0^R \sec x \, dx$

$= \lim_{R \to \pi/2^-} \ln|\sec x + \tan x| \Big]_0^R$

$= \lim_{R \to \pi/2^-} \left(\ln|\sec R + \tan R| - \ln|1+0| \right)$

$= \boxed{\infty}$, since $\sec R \to \infty$ and $\tan R \to \infty$ as $R \to \pi/2^-$.

(b) $\int_e^{\infty} \frac{\ln x}{x^2} \, dx$

Use parts $u = \ln x$, $du = 1/x \, dx$, $dv = 1/x^2 \, dx$, $v = -1/x$ to get $\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx$

$= -\frac{\ln x}{x} - \frac{1}{x} + C$

Then: $\int_e^{\infty} \frac{\ln x}{x^2} dx = \lim_{R \to \infty} \left(-\frac{\ln x}{x} - \frac{1}{x} \right) \Big|_e^R$

$= \lim_{R \to \infty} \left(-\frac{\ln R}{R} - \frac{1}{R} + \frac{1}{e} + \frac{1}{e} \right)$

$= \boxed{\frac{2}{e}}$, since $\lim_{R \to \infty} \frac{\ln R}{R} = \lim_{R \to \infty} \frac{1/R}{1} = 0$

by l'Hopital's Rule.

(c) $\int_{-\infty}^{\infty} \frac{dx}{1+4x^2}$

$\int_0^{\infty} \frac{dx}{1+4x^2} = \lim_{R \to \infty} \int_0^R \frac{dx}{1+(2x)^2} = \lim_{R \to \infty} \frac{1}{2} \tan^{-1}(2x) \Big]_0^R$

$= \lim_{R \to \infty} \left(\frac{1}{2} \tan^{-1}(2R) - 0 \right) = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$

And $\int_{-\infty}^0 \frac{dx}{1+4x^2} = \lim_{R \to -\infty} \frac{1}{2} \tan^{-1}(2x) \Big]_R^0 = \lim_{R \to -\infty} \left(0 - \frac{1}{2} \tan^{-1}(2R) \right)$

$= -\frac{1}{2}\left(-\frac{\pi}{2}\right) = \pi/4$.

So $\int_{-\infty}^{\infty} \frac{dx}{1+4x^2}$ converges to $\frac{\pi}{4} + \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$

2. (a) Determine $\int \frac{x^2+x}{x^4-16}\,dx$

Write $\frac{x^2+x}{x^4-16} = \frac{x^2+x}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$

Then $x^2+x = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)$

Set $x=2$: $6 = 32A \Rightarrow \boxed{A = 3/16}$

Set $x=-2$: $2 = -32B \Rightarrow \boxed{B = -1/16}$

Set $x=0$: $0 = 8A - 8B - 4D \Rightarrow D = 2(A-B) \Rightarrow \boxed{D = 1/2}$

Compare coeff of x^3: $0 = A+B+C \Rightarrow C = -A-B \Rightarrow \boxed{C = -1/8}$

So $\int \frac{x^2+x}{x^4-16}\,dx = \int \left(\frac{3}{16}\frac{1}{x-2} - \frac{1}{16}\cdot\frac{1}{x+2} - \frac{1}{8}\cdot\frac{x}{x^2+4} + \frac{1}{2}\cdot\frac{1}{x^2+4}\right)dx$

$= \boxed{\frac{3}{16}\ln|x-2| - \frac{1}{16}\ln|x+2| - \frac{1}{16}\ln(x^2+4) + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + C}$

(b) Use (a) to compute $\int_3^\infty \frac{x^2+x}{x^4-16}\,dx$. [*Hint:* First combine all logarithmic terms.]

From (a) get $\int_3^\infty \frac{x^2+x}{x^4-16}\,dx = \lim_{R\to\infty} \left(\frac{1}{16}\ln\left|\frac{(x-2)^3}{(x+2)(x^2+4)}\right| + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right)\right)\Big|_3^R$

$= \lim_{R\to\infty}\left(\underbrace{\frac{1}{16}\ln\frac{(R-2)^3}{(R+2)(R^2+4)}}_{\to \ln(1)=0} + \underbrace{\frac{1}{4}\tan^{-1}\left(\frac{R}{2}\right)}_{\to \pi/2} - \frac{1}{16}\ln\left(\frac{1}{65}\right) - \frac{1}{4}\tan^{-1}\left(\frac{3}{2}\right)\right)$

$= \boxed{\frac{\pi}{8} + \frac{\ln 65}{16} - \frac{1}{4}\tan^{-1}\left(\frac{3}{2}\right)}$

3. Use comparison to decide whether $\int_0^{1/2} \frac{dx}{x(1-x)}$ converges or diverges.

Note that $x(1-x) = x - x^2 \geq x$ for $x \geq 0$.

So $\frac{1}{x(1-x)} \leq \frac{1}{x}$

Since $\int_0^{1/2} \frac{dx}{x}$ diverges (p-integral, $p=1$)

we know by comparison that $\int_0^{1/2} \frac{dx}{x(1-x)}$ $\boxed{\text{diverges}}$

v4.2