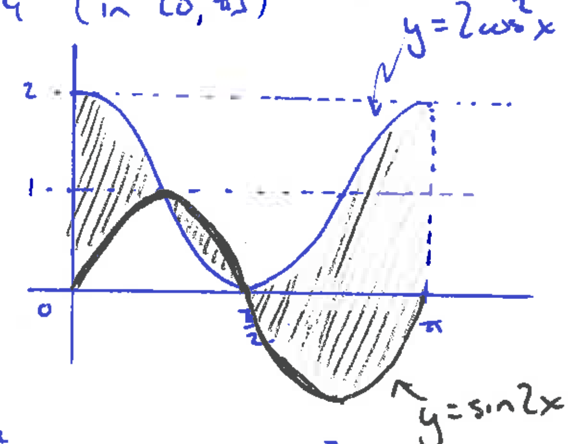
Name: SOLUTIONS	**A#:**	**Section:**

1. Find the area bounded between $y = \sin 2x$ and $y = 2\cos^2 x$ on the interval $[0, \pi]$.

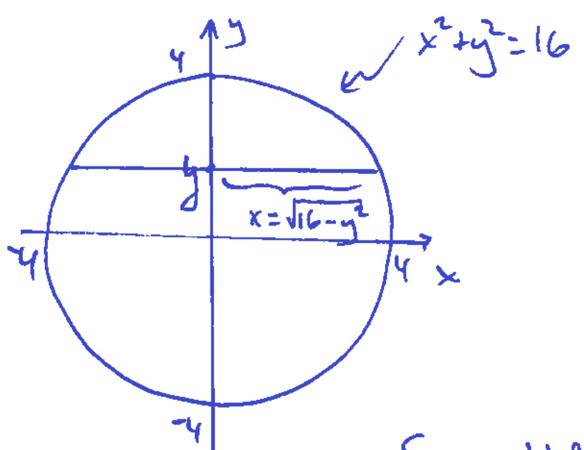
$$\sin 2x = 2\cos^2 x \iff 2\sin x\cos x = 2\cos^2 x$$
$$\iff 2\cos x(\sin x - \cos x) = 0$$
$$\iff \cos x = 0 \text{ OR } \sin x = \cos x$$
$$\iff x = \pi/2 \text{ OR } x = \pi/4 \ (\text{in } [0,\pi])$$



Let
$$I = \int (2\cos^2 x - \sin 2x)\,dx$$
$$= \int (1 + \cos 2x - \sin 2x)\,dx$$
$$= x + \tfrac{1}{2}\sin 2x - \tfrac{1}{2}\cos 2x + C$$

Then:
$$\text{Area} = \int_0^{\pi/4} (2\cos^2 x - \sin 2x)\,dx + \int_{\pi/4}^{\pi/2} (\sin 2x - 2\cos^2 x)\,dx + \int_{\pi/2}^{\pi} (2\cos^2 x - \sin 2x)\,dx$$
$$= I\Big]_0^{\pi/4} - I\Big]_{\pi/4}^{\pi/2} + I\Big]_{\pi/2}^{\pi}$$
$$= \left(\tfrac{\pi}{4} + \tfrac{1}{2} - \tfrac{1}{2}\right) - \left(\tfrac{\pi}{2} - \tfrac{1}{2} - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\right)\right) + \left(\pi + \tfrac{1}{2} - \left(\tfrac{\pi}{2} - \tfrac{1}{2}\right)\right)$$
$$= \boxed{\frac{\pi}{2} + 2}$$

2. Find the volume of the solid whose base is the circle $x^2 + y^2 = 16$, with cross sections perpendicular to the y-axis being squares.



The cross section through y is a square with side length $2\sqrt{16-y^2}$

So
$$\text{Volume} = \int_{-4}^{4} \left(2\sqrt{16-y^2}\right)^2 dy$$
$$= 4\int_{-4}^{4} (16 - y^2)\,dy$$
$$= 4\left(16y - \frac{y^3}{3}\right)\Big|_{y=-4}^{4}$$
$$= 4\left(\left(64 - \frac{64}{3}\right) - \left(-64 + \frac{64}{3}\right)\right)$$
$$= \boxed{\frac{1024}{3}}$$

3. Let $\mathcal{R}$ be the region bounded between the curves $y^2 = x - 2$ and $y + x = 4$.

(a) Sketch $\mathcal{R}$, noting all points of intersection of its bounding curves.

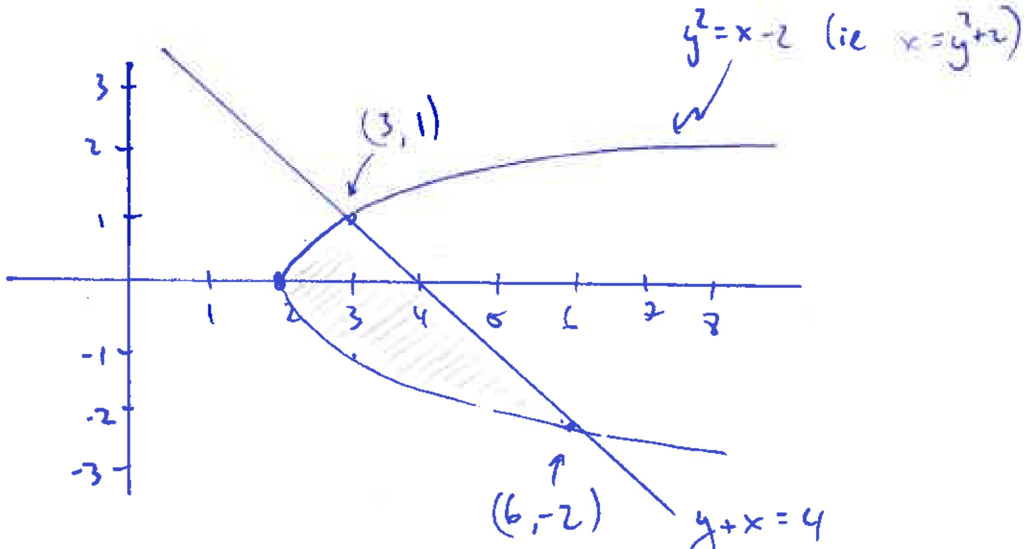
$2 + y^2 = 4 - y$

$\Leftrightarrow y^2 + y - 2 = 0$

$\Leftrightarrow (y+2)(y-1) = 0$

$\Leftrightarrow y = 1$ or $y = -2$

So intersections at $(3,1)$ and $(6,-2)$



(b) Give expressions, in terms of definite integrals, for the following. **Do not evaluate your integrals!**

i. The area of $\mathcal{R}$.

$$\boxed{\int_{-2}^{1} \left((4-y) - (y^2+2) \right) dy} \quad \left[= \int_{-2}^{1} (2 - y - y^2)\, dy \right]$$

$$\left(\text{OR: } \int_2^3 2\sqrt{x-2}\, dx + \int_3^6 (4 - x + \sqrt{x-2})\, dx \right)$$

ii. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = -1$.

$$\pi \int_{-2}^{1} \left((1+4-y)^2 - (1+y^2+2)^2 \right) dy$$

$$\left[= \pi \int_{-2}^{1} \left((5-y)^2 - (3+y^2)^2 \right) dy \right.$$

iii. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = 8$.

$$\pi \int_{-2}^{1} \left((8 - (y^2+2))^2 - (8 - (4-y))^2 \right) dy$$

$$\left[= \pi \int_{-2}^{1} \left((6-y^2)^2 - (4+y)^2 \right) dy \right.$$

iv. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $y = 1$.

$$2\pi \int_{-2}^{1} (1-y)\left((4-y) - (y^2+2) \right) dy \quad \leftarrow \text{"SHELLS"}$$

<u>OR</u> $$\pi \int_2^3 \left((1+\sqrt{x-2})^2 - (1-\sqrt{x-2})^2 \right) + \pi \int_3^6 \left((1+\sqrt{x-2})^2 - (1+x-4)^2 \right) dx$$

$\uparrow$ "WASHERS"

v4.1

Name: Key	A#:	Section:

8

1. Find the area between $y = \cos x$ and $y = \cos 2x$ over the interval $0 \le x \le \pi$.

Points of intersection:

$\cos x = \cos 2x$

$\Rightarrow \cos x - \cos 2x = 0$

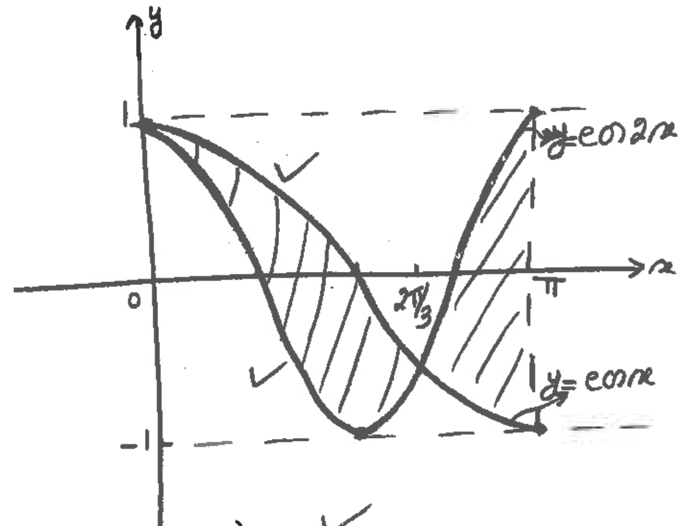
$\Rightarrow \cos x - (2\cos^2 x - 1) = 0$

$\Rightarrow (2\cos x + 1)(-\cos x + 1) = 0$

$\Leftrightarrow 2\cos x + 1 = 0$ or $-\cos x + 1 = 0$

$\Leftrightarrow \cos x = \frac{-1}{2}$ or $\cos x = 1$

$\Leftrightarrow x = \frac{2\pi}{3}$ or $x = 0$ (in $[0,\pi]$) ✓



$$\text{Area} = \int_0^{2\pi/3} (\cos x - \cos 2x)\,dx + \int_{2\pi/3}^{\pi} (\cos 2x - \cos x)\,dx$$

$$= \left[\sin x - \tfrac{1}{2}\sin 2x\right]_0^{2\pi/3} + \left[\tfrac{1}{2}\sin 2x - \sin x\right]_{2\pi/3}^{\pi}$$

$$= \left(\sin\tfrac{2\pi}{3} - \tfrac{1}{2}\sin\tfrac{4\pi}{3}\right) - \left(\sin 0 - \tfrac{1}{2}\sin 0\right) + \left(\tfrac{1}{2}\sin 2\pi - \sin\pi\right) - \left(\tfrac{1}{2}\sin\tfrac{4\pi}{3} - \sin\tfrac{2\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) - 0 + 0 - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \boxed{\frac{3\sqrt{3}}{2}}$$

← Don't deduct marks for error in final calculation, just note the mistake.

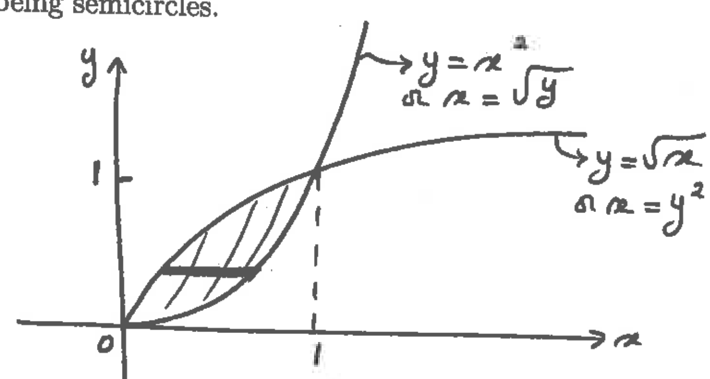
5

2. Find the volume of the solid whose base is the region bounded by $y = x^2$ and $y = \sqrt{x}$, with cross sections perpendicular to the y-axis being semicircles.

The cross section ⊥ to the y-axis has radius $r = \frac{1}{2}(\sqrt{y} - y^2)$ ✓

Area of semicircle is

$$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{1}{2}(\sqrt{y} - y^2)\right)^2$$



Diameter $= \sqrt{y} - y^2$

$$V = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \frac{1}{2}\pi\left(\frac{1}{2}(\sqrt{y} - y^2)\right)^2 dy$$

$$= \frac{\pi}{8}\int_0^1 (\sqrt{y} - y^2)^2\,dy$$

$$= \frac{\pi}{8}\int_0^1 (y + y^4 - 2y^2\sqrt{y})\,dy$$

$$= \frac{\pi}{8}\int_0^1 (y + y^4 - 2y^{5/2})\,dy = \frac{\pi}{8}\left[\frac{y^2}{2} + \frac{y^5}{5} - \frac{2y^{7/2}}{7/2}\right]_0^1$$

$$= \frac{\pi}{8}\left(\frac{1}{2} + \frac{1}{5} - \frac{4}{7}\right)$$

$$= \boxed{\frac{9\pi}{560}}$$

3. Let $\mathcal{R}$ be the region bounded between the curves $y^2 = 3 - x$ and $y = x - 1$.

4 (a) Sketch $\mathcal{R}$, noting all points of intersection of its bounding curves.

Points of Intersection:

$$\begin{cases} x = 3 - y^2 \\ x = y + 1 \end{cases}$$

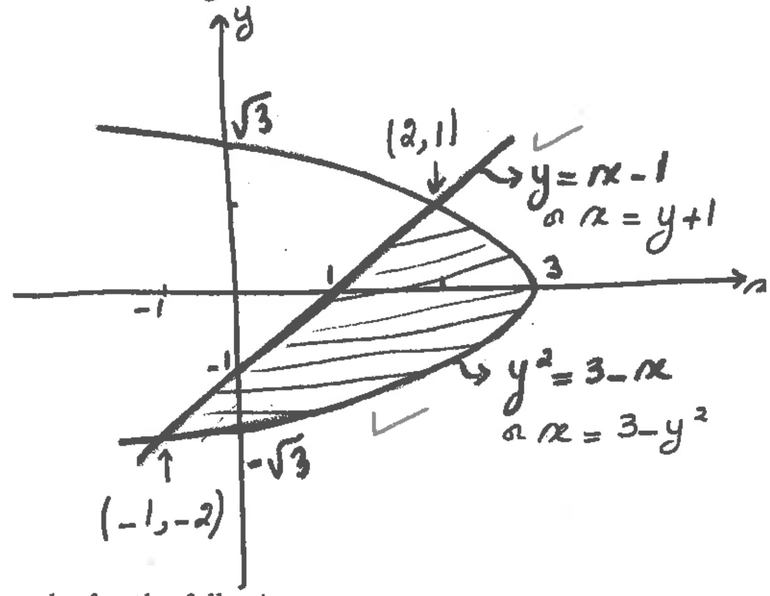
$3 - y^2 = y + 1$

$\Rightarrow y + y^2 - 2 = 0$

$\Leftrightarrow (y+2)(y-1) = 0$

$\Leftrightarrow y = -2$ or $y = 1$

$\therefore (-1, -2)$ and $(2, 1)$



2 (b) Give expressions, in terms of definite integrals, for the following.
Do not evaluate your integrals!

i. The area of $\mathcal{R}$.

$$\text{Area} = \int_{-2}^{1} (x_R - x_L)\,dy = \int_{-2}^{1} (3 - y^2) - (y+1)\,dy = \int_{-2}^{1} (2 - y^2 - y)\,dy$$

Or

$$\text{Area} = \int_{-1}^{2} [(x-1) - (-\sqrt{3-x})]\,dx + \int_{2}^{3} [\sqrt{3-x} - (-\sqrt{3-x})]\,dx$$

$$= \int_{-1}^{2} ((x-1) + \sqrt{3-x})\,dx + \int_{2}^{3} 2\sqrt{3-x}\,dx$$

2 ii. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = -1$.

$$V = \pi \int_{-2}^{1} (R_o^2 - R_i^2)\,dy$$

$$= \pi \int_{-2}^{1} (1 + (3 - y^2))^2 - (1 + (y+1))^2\,dy$$

$$= \pi \int_{-2}^{1} ((4 - y^2)^2 - (2 + y)^2)\,dy$$

1 iii. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = 4$.

$$V = \pi \int_{-2}^{1} (4 - (y+1))^2 - (4 - (3 - y^2))^2\,dy$$

$$= \pi \int_{-2}^{1} ((3 - y)^2 - (1 + y^2)^2)\,dy$$

3 iv. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $y = 2$.

$$V = 2\pi \int_{-2}^{1} (2 - y)[(3 - y^2) - (y+1)]\,dy$$ $\rightsquigarrow$ Shells

or

$$V = \pi \int_{-1}^{2} ((2 + \sqrt{3-x})^2 - (3-x)^2)\,dx + \pi \int_{2}^{3} ((2 + \sqrt{3-x})^2 - (2 - \sqrt{3-x})^2)\,dx$$

$\hookrightarrow$ washer

v5.2

Math 1211: Worksheet #5 — Winter 2016

Name: SOLUTIONS	A#:	Section:

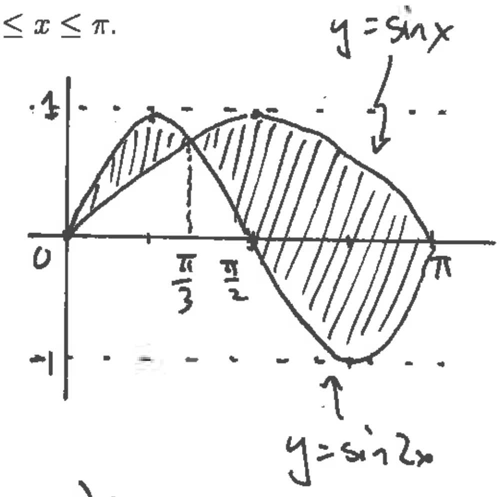
1. Find the area between $y = \sin x$ and $y = \sin 2x$ over the interval $0 \le x \le \pi$.

$\sin x = \sin 2x \iff \sin x = 2\sin x\cos x$

$\iff \sin x(1-2\cos x) = 0$

$\iff \sin x = 0$ OR $\cos x = \frac{1}{2}$

$\iff x = 0, \pi$ OR $x = \pi/3$



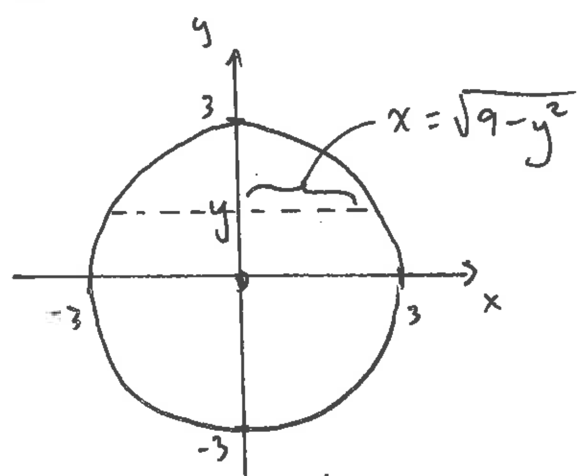
$$\text{Area} = \int_0^{\pi/3} (\sin 2x - \sin x)\,dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x)\,dx$$

$$= \left(-\tfrac{1}{2}\cos 2x + \cos x\right)\Big]_0^{\pi/3} + \left(-\cos x + \tfrac{1}{2}\cos 2x\right)\Big]_{\pi/3}^{\pi}$$

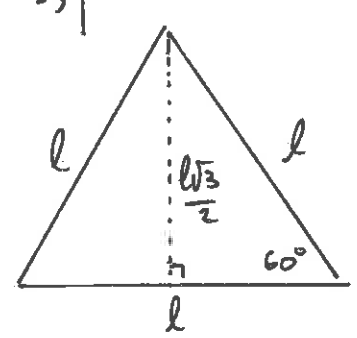
$$= \left(\tfrac{1}{4}+\tfrac{1}{2}\right) - \left(-\tfrac{1}{2}+1\right) + \left(1+\tfrac{1}{2}\right) - \left(-\tfrac{1}{2}-\tfrac{1}{4}\right)$$

$$= \boxed{\frac{5}{2}}$$

2. Find the volume of the solid whose base is the circle $x^2 + y^2 = 9$, with cross sections perpendicular to the y-axis being equilateral triangles.



The cross section through y is an equilateral triangle with sides $2\sqrt{9-y^2}$, which has area $\frac{(2\sqrt{9-y^2})^2\sqrt{3}}{4} = \sqrt{3}(9-y^2)$



$\text{Area} = \frac{1}{2} l \cdot \frac{l\sqrt{3}}{2} = \frac{l^2\sqrt{3}}{4}$

So Volume $= \int_{-3}^{3} \sqrt{3}(9-y^2)\,dy$

$= \sqrt{3}\left(9y - \frac{1}{3}y^3\right)\Big|_{-3}^{3}$

$= \boxed{36\sqrt{3}}$

3. Let $\mathcal{R}$ be the region bounded between the curves $\overbrace{y^2 = x - 1}^{x = y^2+1}$ and $\overbrace{y = x - 3}^{x = y+3}$.

(a) Sketch $\mathcal{R}$, noting all points of intersection of its bounding curves.

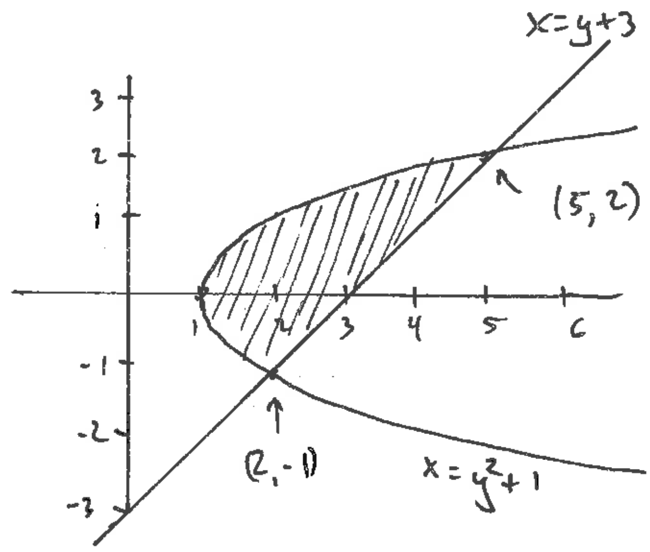
$y^2+1 = y+3 \iff y^2 - y - 2 = 0$

$\iff (y-2)(y+1) = 0$

$\iff y = 2$ or $y = -1$

So intersection pts are

$(x,y) = (5,2)$ and $(2,-1)$



(b) Give expressions, in terms of definite integrals, for the following.
Do not evaluate your integrals!

i. The area of $\mathcal{R}$.

$$\boxed{\int_{-1}^{2} \left((y+3) - (y^2+1)\right) dy} = \int_{-1}^{2} (2+y-y^2)\, dy$$

$$\left[\text{OR} \quad \int_1^2 2\sqrt{x-1}\, dx + \int_2^5 \left(\sqrt{x-1} - (x-3)\right) dx\right]$$

ii. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = -1$.

$$\boxed{\pi \int_{-1}^{2} \left((y+3+1)^2 - (y^2+1+1)^2 \right) dy} = \pi \int_{-1}^{2} \left((y+4)^2 - (y^2+2)^2 \right) dy$$

iii. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $x = 6$.

$$\boxed{\pi \int_{-1}^{2} \left((6 - (y^2+1))^2 - (6 - (y+3))^2 \right) dy}$$

$$= \pi \int_{-1}^{2} \left((5-y^2)^2 - (3-y)^2 \right) dy$$

iv. The volume of the solid obtained by revolving $\mathcal{R}$ about the line $y = 2$.

"SHELLS" $\rightarrow$ $$\boxed{2\pi \int_{-1}^{2} (2-y)\left((y+3) - (y^2+1)\right) dy} = 2\pi \int_{-1}^{2} (2-y)(2+y-y^2)\, dy$$

OR: $\pi \int_1^2 \left((2+\sqrt{x-1})^2 - (2-\sqrt{x-1})^2 \right) dx + \pi \int_1^2 \left((2-(x-3))^2 + (2-\sqrt{x-1})^2 \right) dx$

$\uparrow$ "WASHERS"

v5.3