Name: Key	A#:	Section:

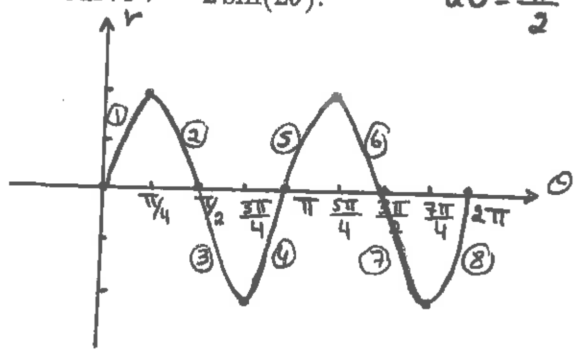
1. Let C be the polar curve $r = 2\sin(2\theta)$.

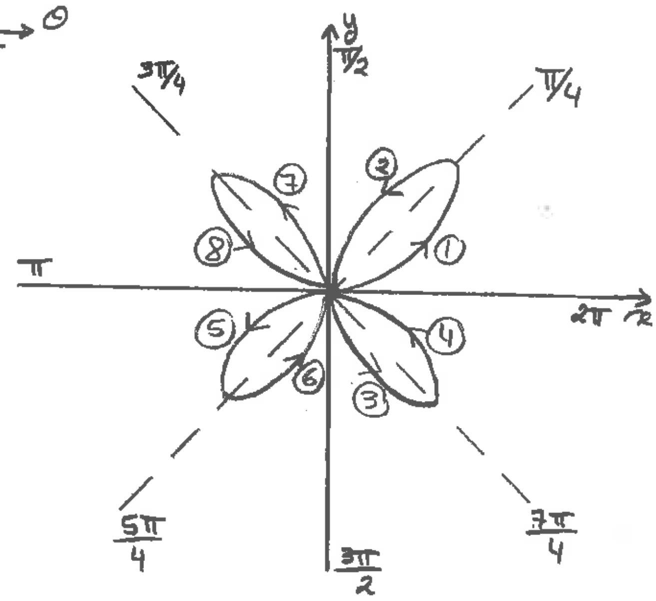
$2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$, Period of $\sin x$ is 2π

The Amplitude is 2

(a) Sketch C.

θ	$r = 2\sin 2\theta$
0	0
$\pi/4$	2
$\pi/2$	0
$3\pi/4$	-2
π	0
$5\pi/4$	2
$3\pi/2$	0
$7\pi/4$	-2
2π	0





(b) Find a Cartesian equation for C.

$r = 2\sin 2\theta$

$r = 4\sin\theta\cos\theta$

$r^2 = 16\sin^2\theta\cos^2\theta$

$\therefore r^2 = 16\sin^2\theta\cos^2\theta$

$= 16\left(\frac{y^2}{r^2}\right)\left(\frac{x^2}{r^2}\right)$

$= \frac{16y^2x^2}{r^4}$

Hence $r^6 = 16y^2x^2$ and $(x^2+y^2)^3 = 16x^2y^2$

$x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r}$

$y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r}$

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$

(c) Find the area enclosed by C.

$r = 2\sin 2\theta$, $0 \le \theta \le 2\pi$

$$A = \frac{1}{2}\int_0^{2\pi} (2\sin 2\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4\sin^2 2\theta\, d\theta$$

$$= 2\int_0^{2\pi} \sin^2 2\theta\, d\theta$$

$$= 2\int_0^{2\pi} \frac{1}{2}(1-\cos 4\theta)\, d\theta$$

$$= \int_0^{2\pi}(1-\cos 4\theta)\, d\theta = \left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = 2\pi$$

or $A = 4\int_0^{\pi/2} \frac{1}{2}(2\sin 2\theta)^2 d\theta$

(d) Find the equation of the tangent line to C at the point $(r, \theta) = (\sqrt{3}, \frac{\pi}{6})$.

$r = \sqrt{3}$ $\qquad$ $r = 2\sin 2\theta$

$\theta = \frac{\pi}{6}$ $\qquad$ $\frac{dr}{d\theta} = 2\cos 2\theta (2) = 4\cos 2\theta$

$\pi/6 \to \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\frac{\pi}{3} \to \left(1/2, \frac{\sqrt{3}}{2}\right)$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr/d\theta \sin\theta + r\cos\theta}{dr/d\theta \cos\theta - r\sin\theta}$$

$$= \frac{4\cos 2\theta \sin\theta + 2\sin 2\theta \cos\theta}{4\cos 2\theta \cos\theta - 2\sin 2\theta \sin\theta}$$

At $\frac{\pi}{6}$, $$\frac{dy}{dx} = \frac{4\cos 2\pi/6 \sin \pi/6 + 2\sin 2\pi/6 \cos \pi/6}{4\cos \frac{2\pi}{6} \cos\frac{\pi}{6} - 2\sin\frac{2\pi}{6} \sin \pi/6}$$

$$= \frac{4\cos \pi/3 \sin \pi/6 + 2\sin \pi/3 \cos \pi/6}{4\cos \pi/3 \cos \pi/6 - 2\sin \pi/3 \sin \pi/6} = \frac{4(1/2)(1/2) + 2(\sqrt{3}/2)(\sqrt{3}/2)}{4(1/2)(\sqrt{3}/2) - 2(\sqrt{3}/2)(1/2)}$$

$$= \frac{1 + 3/2}{\sqrt{3} - \sqrt{3}/2} = \frac{5}{\sqrt{3}}$$

$x = r\cos\theta$ $\qquad \therefore x = \sqrt{3}\cos \pi/6 = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$

$y = r\sin\theta$ $\qquad \therefore y = \sqrt{3}\sin \pi/6 = \sqrt{3}\left(1/2\right) = \frac{\sqrt{3}}{2}$

$$\therefore y - \frac{\sqrt{3}}{2} = \frac{5}{\sqrt{3}}\left(x - \frac{3}{2}\right)$$

(e) Find all points of intersection of C with the circle $r = 1$.

$r = 2\sin 2\theta$ and $r = 1$

$2\sin 2\theta = 1$

$\Rightarrow \sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

and $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$ and $\frac{17\pi}{12}$

By symmetry $\theta = \frac{23\pi}{12}, \frac{19\pi}{12}, \frac{11\pi}{12}$, and $\frac{7\pi}{12}$

(f) Give an expression, in terms of definite integrals, for the area enclosed by both C and the circle $r = 1$. **Do not evaluate.**

$$A = 4\left[2 \cdot \frac{1}{2}\int_0^{\pi/12} (2\sin 2\theta)^2 \, d\theta + 2 \cdot \frac{1}{2}\int_{\pi/12}^{\pi/4} 1^2 \, d\theta\right]$$

Math 1211: Worksheet #7 **Winter 2016**

Name: Key	A#:	Section:

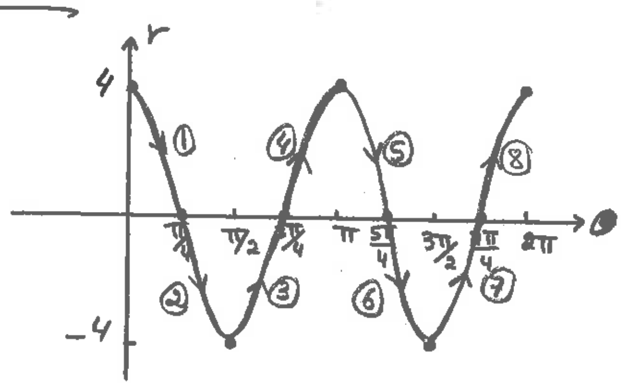
1. Let C be the polar curve $r = 4\cos 2\theta$.

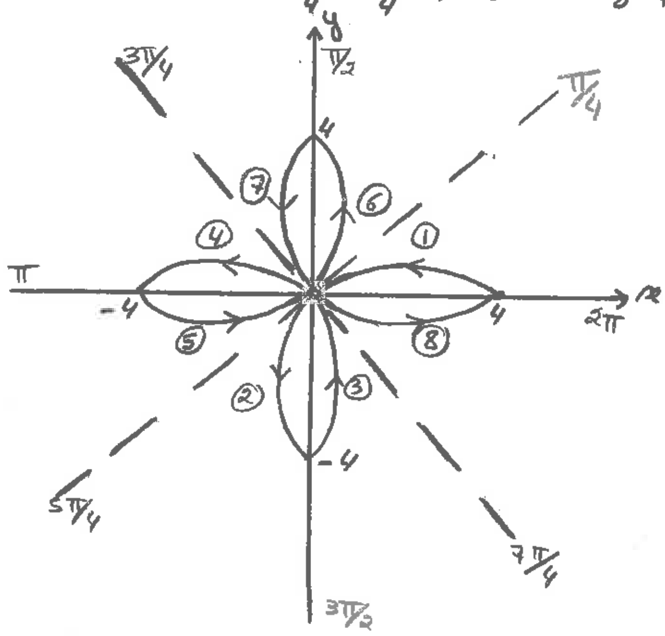
$2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$, Period $= \frac{2\pi}{2} = \pi$, Amplitude is 4

$\frac{p}{4} = \frac{\pi}{4} \rightsquigarrow$ increment by $\pi/4$

(a) Sketch C.

θ	$r = 4\cos 2\theta$
0	4
$\pi/4$	0
$\pi/2$	-4
$3\pi/4$	0
π	4
$5\pi/4$	0
$3\pi/2$	-4
$\frac{7\pi}{4}$	0
2π	4





(b) Find a Cartesian equation for C.

$$r = 4\cos 2\theta$$
$$r = 4(\cos^2\theta - \sin^2\theta)$$
$$r^2 = 16(\cos^2\theta - \sin^2\theta)^2$$
$$= 16(\cos^4\theta + \sin^4\theta - 2\cos^2\theta\sin^2\theta)$$
$$= 16\cos^4\theta + 16\sin^4\theta - 32\cos^2\theta\sin^2\theta$$
$$= 16\left(\frac{x}{r}\right)^4 + 16\left(\frac{y}{r}\right)^4 - 32\left(\frac{x}{r}\right)^2\left(\frac{y}{r}\right)^2$$
$$= \frac{16x^4}{r^4} + \frac{16y^4}{r^4} - \frac{32x^2y^2}{r^4}$$
$$\therefore r^6 = 16x^4 + 16y^4 - 32x^2y^2, \text{ Hence } \boxed{(x^2+y^2)^3 = 16x^4 + 16y^4 - 32x^2y^2}$$

$$\left\{ \begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned} \right.$$

$$\left\{ \begin{aligned} & x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r} \\ & y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r} \\ & r^2 = x^2 + y^2 \\ & r = \sqrt{x^2+y^2} \end{aligned} \right.$$

(c) Find the area enclosed by C.

$$A = \frac{1}{2}\int_0^{2\pi} (4\cos 2\theta)^2\, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} 16\cos^2 2\theta\, d\theta$$
$$= 8\int_0^{2\pi} \cos^2 2\theta\, d\theta$$
$$= 8\int_0^{2\pi} \frac{1}{2}(1 + \cos 4\theta)\, d\theta$$
$$= 4\int_0^{2\pi}(1 + \cos 4\theta)\, d\theta$$
$$= 4\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{2\pi}$$
$$= 4\left[2\pi + \frac{1}{4}\sin 8\pi - 0 - \frac{1}{4}\sin 0\right]$$
$$= 8\pi$$

$\stackrel{\text{or}}{=}$ $A = 8\left[\frac{1}{2}\int_0^{\pi/4} (4\cos 2\theta)^2\, d\theta\right]$

$\stackrel{\text{or}}{=}$ $A = 4\left[\frac{1}{2}\int_{\pi/4}^{3\pi/4} (4\cos 2\theta)^2\, d\theta\right]$

or $A = 4\left[\frac{1}{2}\int_{-\pi/4}^{\pi/4} (4\cos 2\theta)^2\, d\theta\right]$

v7.2

(d) Find the equation of the tangent line to C at the point $(r,\theta) = (2, \frac{\pi}{6})$.

$r = 2$, $\theta = \frac{\pi}{6}$

$r = 4\cos 2\theta$

$\frac{dr}{d\theta} = 4(-\sin 2\theta)2$
$= -8\sin 2\theta$

$x = r\cos\theta$

$y = r\sin\theta$

So at $\frac{\pi}{6}$, $x = 2\cos\frac{\pi}{6} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

and $y = 2\sin\frac{\pi}{6} = 2(\frac{1}{2}) = 1$

$\therefore\ y - 1 = \frac{\sqrt{3}}{7}\left(x - \frac{2\sqrt{3}}{2}\right)$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr/d\theta \sin\theta + r\cos\theta}{dr/d\theta\cos\theta - r\sin\theta}$$
$$= \frac{-8\sin 2\theta\sin\theta + 4\cos 2\theta\cos\theta}{-8\sin 2\theta\cos\theta - 4\cos 2\theta\sin\theta}$$

At $\frac{\pi}{6}$, $\frac{dy}{dx} = \frac{-8\sin 2\pi/6 \sin\pi/6 + 4\cos 2\pi/6\cos\pi/6}{-8\sin 2\pi/6\cos\pi/6 - 4\cos 2\pi/6\sin\pi/6}$

$$= \frac{-8\sin\pi/3\sin\pi/6 + 4\cos\pi/3\cos\pi/6}{-8\sin\pi/3\cos\pi/6 - 4\cos\pi/3\sin\pi/6}$$
$$= \frac{-8(\sqrt{3}/2)(1/2) + 4(1/2)(\sqrt{3}/2)}{-8(\sqrt{3}/2)(\sqrt{3}/2) - 4(1/2)(1/2)}$$
$$= \frac{-2\sqrt{3} + \sqrt{3}}{-2(3) - 1} = \frac{\sqrt{3}}{7}$$

$\pi/6 \to (\sqrt{3}/2, 1/2)$

$\pi/3 \to (1/2, \sqrt{3}/2)$

$\rightarrow$ $\boxed{y - 1 = \frac{\sqrt{3}}{7}\left(x - \sqrt{3}\right)}$

(e) Find all points of intersection of C with the circle $r = 2$.

$r = 4\cos 2\theta$ and $r = 2$

$4\cos 2\theta = 2$

$\Rightarrow \cos 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

and $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

By symmetry $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$\theta = \frac{\pi}{6} + \pi n_1$ for $n_1 \in \mathbb{Z}$

or $\theta = \frac{5\pi}{6} + \pi n_2$ for $n_2 \in \mathbb{Z}$

$\{(2,\pi/6), (2, 5\pi/6), (2, 7\pi/6), (2, 11\pi/6), (2, \pi/3), (2, 2\pi/3), (2, 4\pi/3)$, and $(2, 5\pi/3)$ are all points of intersection of C with the circle $r = 2$.

(f) Give an expression, in terms of definite integrals, for the area enclosed by both C and the circle $r = 2$. Do not evaluate.

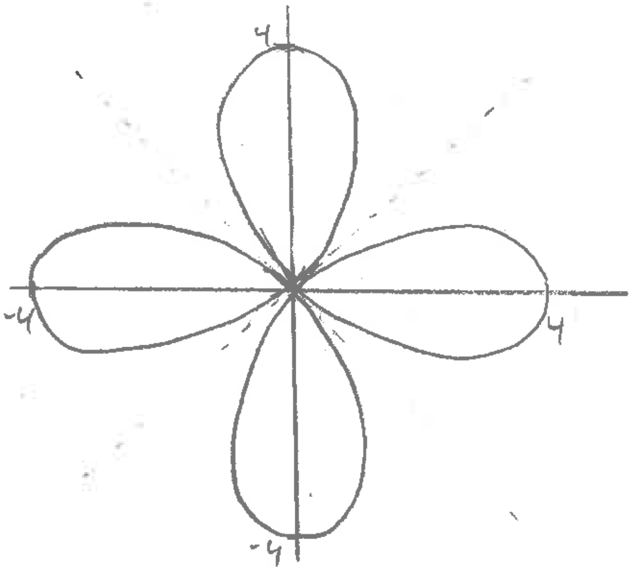
$$A = 8\int_0^{\pi/6} \frac{1}{2}(2)^2\,d\theta + 8\int_{\pi/6}^{\pi/4} \frac{1}{2}(4\cos 2\theta)^2\,d\theta$$

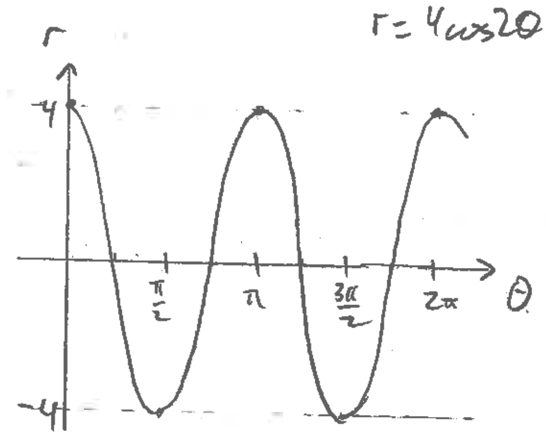
Math 1211: Worksheet #7 **Winter 2016**

Name: SOLUTIONS	A#:	Section:

1. Let C be the polar curve $r = 4\cos 2\theta$.

(a) Sketch C.





(b) Find a Cartesian equation for C.

$$r = 4\cos 2\theta$$
$$= 4(\cos^2\theta - \sin^2\theta)$$
$$= 4\left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2\right)$$

since $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow r^3 = 4(x^2 - y^2)$$
$$\Rightarrow (r^2)^3 = 16(x^2-y^2)^2 \Rightarrow \boxed{(x^2+y^2)^3 = 16(x^2-y^2)^2}$$

since $r^2 = x^2 + y^2$

(c) Find the area enclosed by C.

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (4\cos 2\theta)^2\, d\theta$$
$$= 8\int_0^{2\pi} \cos^2 2\theta\, d\theta$$
$$= 4\int_0^{2\pi} (1 + \cos 4\theta)\, d\theta$$
$$= 4\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{2\pi}$$
$$\boxed{= 8\pi}$$

(d) Find the equation of the tangent line to C at the point $(r, \theta) = (2, \frac{\pi}{6})$.

$$y = r\sin\theta = 4\cos 2\theta \sin\theta \Rightarrow \frac{dy}{d\theta} = -8\sin 2\theta \sin\theta + 4\cos 2\theta \cos\theta$$

$$x = r\cos\theta = 4\cos 2\theta \cos\theta \Rightarrow \frac{dx}{d\theta} = -8\sin 2\theta \cos\theta - 4\cos 2\theta \sin\theta$$

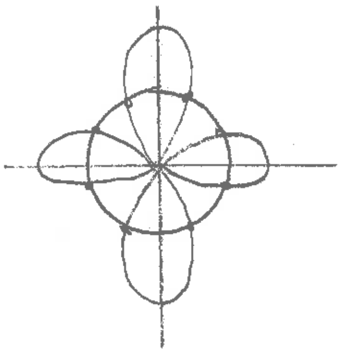
So $\left.\frac{dy}{d\theta}\right|_{\theta=\pi/6} = -8\sin\frac{\pi}{3}\sin\frac{\pi}{6} + 4\cos\frac{\pi}{3}\cos\frac{\pi}{6} = -\sqrt{3}$

$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = -8\sin\frac{\pi}{3}\cos\frac{\pi}{6} - 4\cos\frac{\pi}{3}\sin\frac{\pi}{6} = -7$

Then $\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-\sqrt{3}}{-7} = \frac{\sqrt{3}}{7}$. And $(r,\theta) = (2, \frac{\pi}{6}) \Rightarrow (x,y) = (\sqrt{3}, 1)$

Tangent line is $\boxed{y - 1 = \frac{\sqrt{3}}{7}(x - \sqrt{3})}$

(e) Find all points of intersection of C with the circle $r = 2$



Set $2 = 4\cos 2\theta$ to get $\cos 2\theta = \frac{1}{2}$.

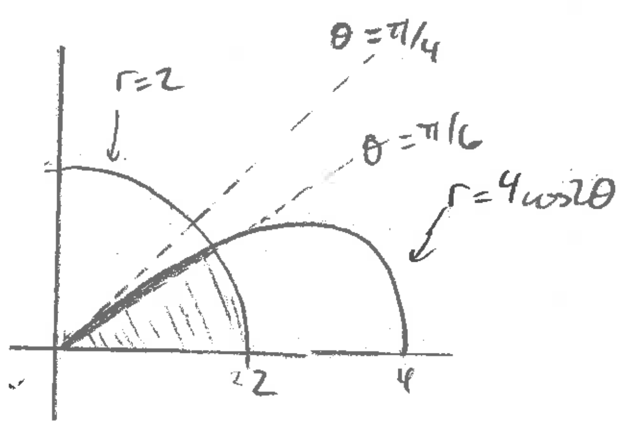
So $2\theta = \pi/3 \Rightarrow \theta = \pi/6$ is one solution.

By symmetry of figure, all intersection points are at

$$\theta = \pm\pi/6,\ \pm\frac{\pi}{2} \pm \frac{\pi}{6},\ \pi \pm \frac{\pi}{6}.$$

OR $\boxed{\theta = \pi/6, \pi/3, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6} \text{ all with } r = 2}$

(f) Give an expression, in terms of definite integrals, for the area enclosed by both C and the circle $r = 2$. **Do not evaluate.**



$$\text{Area} = 8\left[\underbrace{\frac{1}{2}\int_0^{\pi/6} 2^2\,d\theta}_{} + \frac{1}{2}\int_{\pi/6}^{\pi/4} (4\cos 2\theta)^2\,d\theta\right]$$

(or just $\pi/3$, since this is $\frac{1}{12}$ of a circle of radius 2)