Math 1211: Worksheet #9 Winter 2016

Name: Key	A#:	Section:

1. Determine whether the following series converge or diverge:

5 (a) $\sum_{n=1}^{\infty} \frac{1}{n-\sqrt{n}}$

For $n \geqslant 1$, $n - \sqrt{n} \leqslant n$

$$\frac{1}{n-\sqrt{n}} \geqslant \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n} \rightsquigarrow$ Divergent Harmonic Series.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n-\sqrt{n}}$ is also divergent by the Comparison Test.

5 (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^5+1}}{2n^3+2n+1}$

$a_n = \frac{\sqrt{n^5+1}}{2n^3+2n+1}$, $b_n = \frac{\sqrt{n^5}}{n^3} = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^5+1}}{2n^3+2n+1} \cdot \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^6+n}}{2n^3+2n+1} = \lim_{n\to\infty} \frac{\sqrt{1+1/n^5}}{2+2/n^2+1/n^3} = \frac{1}{2} > 0$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \rightsquigarrow$ p-series with $p = \frac{1}{2} < 1$, $\therefore$ divergent

Hence, $\sum_{n=1}^{\infty} \frac{\sqrt{n^5+1}}{2n^3+2n+1}$ is also divergent by the Limit Comparison Test.

(c) $\sum_{n=1}^{\infty} \frac{2^n + n^3}{3^n - n^2}$ $\quad a_n = \frac{2^n + n^3}{3^n - n^2}, \quad b_n = \frac{2^n}{3^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n + n^3}{3^n - n^2} \cdot \frac{3^n}{2^n}$$

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$$= \lim_{n\to\infty} \frac{6^n + 3^n n^3}{6^n - 2^n n^2}$$

$$= \lim_{n\to\infty} \frac{1 + n^3/2^n}{1 - n^2/3^n} = 1 > 0$$

$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \rightsquigarrow$ geometric series with $r = \frac{2}{3} < 1$, $\therefore$ convergent.

Hence, $\sum_{n=1}^{\infty} \frac{2^n + n^3}{3^n - n^2}$ is also convergent by the Limit Comparison Test.

5

(d) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ $\quad$ Or $\rightsquigarrow$

$\ln n < \sqrt{n}$

$\frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \rightsquigarrow$ p-series with $p = \frac{3}{2} > 1 \therefore$ convergent.

Hence, $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is also convergent by the Comparison Test.

Let $f(x) = \frac{\ln x}{x^2}$

$$f'(x) = \frac{x^2(1/x) - \ln x(2x)}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$$

$f'(x) < 0$ when $\ln x > \frac{1}{2}$ (i.e when $x > \sqrt{e}$), and hence $f(x)$ is decreasing for $x \geq 2$.

Furthermore, $f(x)$ is continuous and positive. $\therefore$ the Integral test applies.

let $u = \ln x \quad dv = x^{-2}$

$du = \frac{1}{x}dx \quad v = -x^{-1}$

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty} \int_2^R \frac{\ln x}{x^2}\,dx$$

$$= \lim_{R\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^R$$

$$= \lim_{R\to\infty} \left[-\frac{\ln R}{R} - \frac{1}{R} + \frac{\ln 2}{2} + \frac{1}{2}\right]$$

$$= -\lim_{R\to\infty} \frac{\ln R}{R} + \frac{\ln 2 + 1}{2}$$

$$\overset{H}{=} -\lim_{R\to\infty} \frac{1/R}{1} + \frac{\ln 2 + 1}{2} = \frac{\ln 2 + 1}{2} \quad \therefore \text{Convergent.}$$

Hence $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ is also convergent by the Integral test, and hence $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent as well.

v9.1

Name: Key	A#:	Section:

1. Determine whether the following series converge or diverge:

5pt (a) $\sum_{n=1}^{\infty} \frac{n^2}{3n^3-1}$

$a_n = \frac{n^2}{3n^3-1}$, $b_n = \frac{n^2}{n^3} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{3n^3-1} \cdot \frac{n}{1}$$
$$= \lim_{n\to\infty} \frac{n^3}{3n^3-1}$$
$$= \lim_{n\to\infty} \frac{1}{3 - 1/n^3} = \frac{1}{3} > 0$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ ⇝ Divergent Harmonic Series

∴ $\sum_{n=1}^{\infty} \frac{n^2}{3n^3-1}$ is also divergent by the Limit Comparison test.

Or

For $n \geq 1$,
$3n^3 - 1 < 3n^3$
$$\frac{1}{3n^3-1} > \frac{1}{3n^3}$$
$$\frac{n^2}{3n^3-1} > \frac{n^2}{3n^3} = \frac{1}{3n}$$
$\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ ⇝ Divergent Harmonic series multiplied by $1/3$.
∴ Divergent.
Hence, $\sum_{n=1}^{\infty} \frac{n^2}{3n^3-1}$ is also divergent by the Comparison test.

5pt (b) $\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{2n^5+n+1}}$

$a_n = \frac{n+2}{\sqrt{2n^5+n+1}}$, $b_n = \frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+2}{\sqrt{2n^5+n+1}} \cdot \frac{n^{3/2}}{1}$$
$$= \lim_{n\to\infty} \frac{n^{5/2} + 2n^{3/2}}{\sqrt{2n^5+n+1}}$$
$$= \lim_{n\to\infty} \frac{1 + 2/n}{\sqrt{2 + 1/n^4 + 1/n^5}} = \frac{1}{\sqrt{2}} > 0$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ ⇝ p-series with $p = \frac{3}{2} > 1$, ∴ convergent.

Hence, $\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{2n^5+n+1}}$ is also convergent by the Limit Comparison Test.

5pb (c) $\sum_{n=1}^{\infty} \frac{2^{3n}+1}{3^{2n}+n^2}$

$$a_n = \frac{2^{3n}+1}{3^{2n}+n^2}, \quad b_n = \frac{2^{3n}}{3^{2n}} = \left(\frac{2^3}{3^2}\right)^n = \left(\frac{8}{9}\right)^n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^{3n}+1}{3^{2n}+n^2} \cdot \frac{3^{2n}}{2^{3n}}$$

$$= \lim_{n\to\infty} \frac{72^n + 3^{2n}}{72^n + 2^{3n}.n^2}$$

$$= \lim_{n\to\infty} \frac{1 + 1/2^{3n}}{1 + n^2/3^{2n}} = 1 > 0$$

$\sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^n \sim$ geometric series with $r = \frac{8}{9} < 1$ $\therefore$ convergent

Hence $\sum_{n=1}^{\infty} \frac{2^{3n}+1}{3^{2n}+n^2}$ is also convergent by the limit Comparison Test.

5pb (d) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Or $\sim\!\!>$

$\ln n < \sqrt{n}$

$\frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sim$ p-series with $p = \frac{3}{2} > 1$, $\therefore$ convergent.

Hence, $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is also convergent by the Comparison Test.

Let $f(x) = \frac{\ln x}{x^2}$

$$f'(x) = \frac{x^2(1/x) - \ln x(2x)}{x^4}$$

$$= \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$$

$f'(x) < 0$ when $\ln x > \frac{1}{2}$ (i.e. when $x > \sqrt{e}$)

and hence $f(x)$ is decreasing for $x \geqslant 2$.

Furthermore, $f(x)$ is continuous and positive,

$\therefore$ the integral test applies.

Let $u = \ln x \quad dv = x^{-2}$

$du = \frac{1}{x}dx \quad v = -x^{-1}$

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty} \int_2^R \frac{\ln x}{x^2}\,dx$$

$$= \lim_{R\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^R$$

$$= \lim_{R\to\infty} \left[-\frac{\ln R}{R} - \frac{1}{R} + \frac{\ln 2}{2} + \frac{1}{2}\right]$$

$$= -\lim_{R\to\infty} \frac{\ln R}{R} + \frac{\ln 2 + 1}{2}$$

$$\overset{H}{=} -\lim_{R\to\infty} \frac{1/R}{1} + \frac{\ln 2+1}{2} = \frac{\ln 2+1}{2} \quad \therefore \text{Convergent.}$$

Hence, $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ is also convergent by the Integral test, and hence $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent as well.

v9.2

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Math 1211: Worksheet #9 **Winter 2016**

Name: SOLUTIONS	A#:	Section:

1. Determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{3n^3-\sqrt{n}}$

Note that $\dfrac{n^2+1}{3n^3-\sqrt{n}} \approx \dfrac{n^2}{3n^3} = \dfrac{1}{3n}$ for large n.

So compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

Get $\displaystyle\lim_{n\to\infty} \frac{\frac{n^2+1}{3n^3-\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3+n}{3n^3-\sqrt{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{3+\frac{1}{n^{5/2}}} = \frac{1}{3}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{3n^3-\sqrt{n}}$ also $\boxed{\text{diverges}}$

OR/ Note that $\dfrac{n^2+1}{3n^3-\sqrt{n}} > \dfrac{n^2}{3n^3-\sqrt{n}} > \dfrac{n^2}{3n^3} = \dfrac{1}{3n}$

Since $\sum \frac{1}{3n}$ diverges, so does $\sum \dfrac{n^2+1}{3n^3-\sqrt{n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{2n^5+n+1}}$

Note that $\dfrac{n+2}{\sqrt{2n^5+n+1}} \approx \dfrac{n}{\sqrt{2n^5}} = \dfrac{1}{\sqrt{2}\cdot n^{3/2}}$ for large n.

So we compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Get $\displaystyle\lim_{n\to\infty} \frac{\frac{n+2}{\sqrt{2n^5+n+1}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{5/2}+2n^{3/2}}{\sqrt{2n^5+n+1}} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\sqrt{2+\frac{1}{n^4}+\frac{1}{n^5}}} = \frac{1}{\sqrt{2}}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p series, $p=\frac{3}{2}>1$),

$\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{2n^5+n+1}}$ also $\boxed{\text{converges}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n3^n+1}{(5n+1)4^n}$

(Note that $\dfrac{n3^n+1}{(5n+1)4^n} \approx \dfrac{n3^n}{5n4^n} = \dfrac{1}{5}\left(\dfrac{3}{4}\right)^n$ for large n)

So we compare with $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$:

Get: $\displaystyle\lim_{n\to\infty} \frac{n3^n+1}{(5n+1)4^n} \Big/ \left(\frac{3}{4}\right)^n = \lim_{n\to\infty} \frac{n3^n4^n+4^n}{(5n+1)3^n4^n}$

DIVIDE top + bottom by $n3^n4^n$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n3^n}}{5+\frac{1}{n}}$$

$$= \frac{1}{5}$$

Since $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ converges (geometric with $r=\frac{3}{4}<1$),

$\displaystyle\sum_{n=1}^{\infty} \frac{n3^n+1}{(5n+1)4^n}$ also $\boxed{\text{converges}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Apply integral test, with $f(x) = \dfrac{\ln x}{x^2}$.

Note $f'(x) = \dfrac{\frac{1}{x}\cdot x^2 - 2x\ln x}{x^4} = \dfrac{1-2\ln x}{x^3} < 0$ for $\ln x > \frac{1}{2}$.

So $f(x)$ is decreasing eventually

And $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{R\to\infty}\left[-\frac{1}{x}\ln x\Big|_1^R + \int_1^R \frac{dx}{x^2}\right]$

PARTS, with $u=\ln x$, $dv=\frac{1}{x^2}dx$

$$= \lim_{R\to\infty}\left[-\frac{\ln R}{R} - \frac{1}{R} + 1\right]$$

$$= 0 - 0 + 1$$

$$= 1$$

Since $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ $\boxed{\text{converges}}$

OR Note that $\ln n < \sqrt{n}$ for large n. So $\dfrac{\ln n}{n^2} < \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{3/2}}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

v9.3