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# On bimeasurings☆

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**Abstract**

We introduce and study bimeasurings from pairs of bialgebras to algebras. It is shown that the universal bimeasuring bialgebra construction, which arises from Sweedler's universal measuring coalgebra construction and generalizes the finite dual, gives rise to a contravariant functor on the category of bialgebras adjoint to itself. An interpretation of bimeasurings as algebras in the category of Hopf modules is considered.


*MSC:* 16W30; 16S10

## 0. Introduction

Measurings have first been introduced and studied by M.E. Sweedler [7]. They correspond to homomorphisms of algebras over a coalgebra which are cofree as comodules [1]. There is a universal measuring coalgebra $M(B, A)$ and measuring $\theta: M(B, A) \otimes B \to A$ for every pair of algebras $A$ and $B$ such that $C$-measurings from $B$ to $A$ correspond bijectively to coalgebra maps from $C$ to $M(B, A)$. If $B$ is a bialgebra and $A$ is commutative then $M(B, A)$ carries a natural bialgebra structure [3]. If in addition, $C$ is a bialgebra then one may consider maps $\psi: C \otimes B \to A$ which measure in both variables $C$ and $B$. In the cocommutative case

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these bimeasurings account for the "mixed" term in the second Sweedler cohomology group

$$H^2(C \otimes B, A) \simeq H^2(C, A) \oplus H^2(B, A) \oplus P(B, C, A)$$

as shown in [3]. If $A$ is commutative, then universal bimeasuring bialgebras (and universal bimeasuring) $B(C, A)$ and $B(B, A)$ exist so that bimeasurings $\theta: C \otimes B \to A$ bijectively correspond to bialgebra maps from $C$ to $B(B, A)$ as well as bialgebra maps from $B$ to $B(C, A)$. In fact

$$\mathrm{Bialg}(C, B(B, A)) \simeq \mathrm{Bimeas}(C \otimes B, A) \simeq \mathrm{Bialg}(B, B(C, A))$$

and hence the functor $B(\_, A)$ on the category of bialgebras is adjoint to itself. In the special case $A = k$, this gives a new proof that the finite dual construction $\_^\circ = B(\_, k)$ is adjoint to itself [8] (see [5] for a proof). Moreover, there is a natural injective map

$$B(C, A) \otimes B(B, A) \to B(C \otimes B, A),$$

which is always an isomorphism in the cocommutative case, and restricts to the well-known isomorphism $C^\circ \otimes B^\circ \simeq (C \otimes B)^\circ$ when $A = k$.

There is a natural notion of bimeasuring from an abelian matched pair of Hopf algebras $H = C \bowtie B$ to a commutative algebra $A$ extending that of ordinary cocommutative bimeasurings. These skew-bimeasurings form an abelian group under convolution isomorphic to the first matched pair cohomology group $\mathscr{H}^1(C, B, A)$ with coefficients in $A$ described in [2]. This group also corresponds to a subgroup of the group of $A$-linear automorphisms of the trivial $H$-comodule $H \otimes A$ and thus to a group of Hopf algebra structures on $H \otimes A$, each making $H \otimes A$ an algebra in the category of Hopf modules.

## 1. Preliminaries

### 1.1. Notation

All vector spaces (algebras, coalgebras, bialgebras) will be over a ground field $k$. If $A$ is an algebra and $C$ a coalgebra, then $\mathrm{Hom}(C, A)$ denotes the convolution algebra of all linear maps from $C$ to $A$. The unit and the multiplication on $A$ are denoted by $\eta: k \to A$ and $\mathrm{m}: A \otimes A \to A$; the counit and the comultiplication on $C$ are denoted by $\varepsilon: C \to k$ and $\Delta: C \to C \otimes C$. We use Sweedler's sigma notation for comultiplication: $\Delta(c) = c_1 \otimes c_2$, $(1 \otimes \Delta)\Delta(c) = c_1 \otimes c_2 \otimes c_3$, etc. If $f: U \otimes V \to W$ is a linear map, then we often write $f(u, v)$ instead of $f(u \otimes v)$.

### 1.2. Abelianization

Let $H$ be an algebra and $I \subseteq H$ the algebra ideal generated by all commutators, i.e. all elements of the form $[x, y] = xy - yx$. If $H$ is a Hopf algebra (bialgebra), then $I$ is a Hopf ideal (biideal). This is easily observed by the following identities:

$$S[x, y] = [S(y), S(x)],$$

$$\Delta[x, y] = x_1 y_1 \otimes x_2 y_2 - y_1 x_1 \otimes y_2 x_2$$
$$= [x_1, y_1] \otimes x_2 y_2 + y_1 x_1 \otimes [x_2, y_2].$$

We call the quotient algebra (Hopf algebra, bialgebra) $H_{\mathrm{ab}} = H/I$ the abelianization of $H$. It is the largest commutative quotient of $H$ in the sense that if $K$ is a commutative algebra (bialgebra) and $f: H \to K$ is an algebra (bialgebra) map, then there exists a unique algebra (bialgebra) map $\overline{f}: H_{\mathrm{ab}} \to K$ such that $f = \overline{f}\pi$, where $\pi: H \to H_{\mathrm{ab}}$ is the canonical projection.

If $H$ is a Hopf algebra, then $I$ is also the Hopf ideal generated by $\langle [x, y]_H - \varepsilon(xy)\rangle$, where $[x, y]_H = S(x_1)S(y_1)x_2 y_2$.

### 1.3. Cocommutative part

For a coalgebra $H$ we define $H_{\mathrm{c}}$, the cocommutative part of H, to be the largest cocommutative subcoalgebra of $H$ (it is obtained as a sum of all cocommutative subcoalgebras of $H$, hence it always exists). If $H$ is a bialgebra, then $H_{\mathrm{c}}$ is a bialgebra as well (the algebra generated by $H_{\mathrm{c}}$ is also a cocommutative subcoalgebra of $H$ and must therefore be equal to $H_{\mathrm{c}}$). Finally, if $H$ is a Hopf algebra, then so is $H_{\mathrm{c}}$. This is seen by noting that $S(H_{\mathrm{c}})$ is also a cocommutative subcoalgebra of $H$. If $f: K \to H$ is a coalgebra (bialgebra) map and $K$ is cocommutative, then clearly $f(K) \subseteq H_{\mathrm{c}}$; in other words, there exists a unique coalgebra (bialgebra) map $\overline{f}: K \to H_{\mathrm{c}}$ such that $f = \iota\overline{f}$ (here $\iota: H_{\mathrm{c}} \to H$ is the obvious map).

### 1.4. Measuring

Let $A$, $B$, be algebras, $C$ a coalgebra.

**Proposition 1.1** (*Sweedler [7], 7.0.1*). *A map $\psi: C \otimes B \to A$ corresponds to an algebra map $\rho: B \to \mathrm{Hom}(C, A)$, $\rho(b)(c) = \psi(c, b)$ if and only if*

(1) $\psi(c, bb') = \psi(c_1, b)\psi(c_2, b')$,
(2) $\psi(c, 1) = \varepsilon(c)$.

If the equivalent conditions from the proposition above are satisfied, we say that $\psi$ is a *measuring*, or that $C$ *measures* $B$ to $A$.

**Theorem 1.2** (*Sweedler [7], 7.0.4*). *If $A$ and $B$ are algebras, then there exists a unique measuring $\theta: M \otimes B \to A$ so that for any measuring $f: C \otimes B \to A$, there exists a unique coalgebra map $\overline{f}: C \to M$, s.t. $f = \theta(\overline{f} \otimes 1)$.*

The measuring $\theta: M \otimes B \to A$ from the theorem above is called the *universal measuring* and the coalgebra $M = M(B, A)$ the *universal measuring coalgebra*. The functor $M(\_, A): \mathrm{Alg}^{\mathrm{op}} \to \mathrm{Coalg}$ is right adjoint to $\mathrm{Hom}(\_, A): \mathrm{Coalg} \to \mathrm{Alg}^{\mathrm{op}}$. In particular, if $A=k$ then $M(B, A)=M(B, k)=B^{\circ}$ (the finite dual) and if $B=k$ then $M(B, A)=M(k, A)=k$.

In the construction of the universal bimeasurings, we shall use the following technical lemma.

**Lemma 1.3.** *Let A and B be algebras and* $\theta: M \otimes B \to A$ *the universal measuring. If C is a coalgebra and f and g coalgebra maps from C to M such that* $\theta(f \otimes 1_B) = \theta(g \otimes 1_B)$, *then* $f = g$.

**Proof.** Observe that $\theta(f \otimes 1) = \theta(g \otimes 1): C \otimes B \to A$ is a measuring and hence by the universal property, we have $f = g$. □

If we restrict ourselves to the category of cocommutative coalgebras, then we talk about *universal cocommutative measurings* and *universal cocommutative measuring coalgebras*. These were considered in [1]. In this case, if $C$ is cocommutative, then $C$-measurings $\psi: C \otimes B \to A$ are in bijective correspondence with $C$-algebra maps $\chi: C \otimes B \to C \otimes A$, given by $\chi = (1 \otimes \psi)(\Delta \otimes 1)$ and $\psi = (\varepsilon \otimes 1)\chi$.

**Proposition 1.4.** *If A and B are algebras, then the universal cocommutative measuring coalgebra* $M_c(B, A)$ *is isomorphic to the cocommutative part* $M(B, A)_c$ *of the universal measuring coalgebra* $M(B, A)$.

**Proof.** Note that $M(B, A)_c$ has the required universal property. □

## 2. Bimeasuring

**Definition 2.1.** If $N$ and $T$ are bialgebras and $A$ an algebra, then a map $\psi: N \otimes T \to A$ is a *bimeasuring* if $N$ measures $T$ to $A$ and $T$ measures $N$ to $A$, i.e.

$$\psi(nm, t) = \psi(n, t_1)\psi(m, t_2), \quad \psi(1_N, t) = \varepsilon(t),$$
$$\psi(n, ts) = \psi(n_1, t)\psi(n_2, s), \quad \psi(n, 1_T) = \varepsilon(n)$$

for $n, m \in N$ and $t, s \in T$.

**Definition 2.2.** Let $T$ be a bialgebra and $A$ an algebra. If a bimeasuring $\theta: B \otimes T \to A$ is such that for every bimeasuring $f: N \otimes T \to A$, there exists a unique bialgebra map $\overline{f}: N \to B$ with the property $f = (\overline{f} \otimes 1)\theta$, then $\theta$ is called the (left) *universal bimeasuring* and $B = B(T, A)$ is called the (left) *universal bimeasuring bialgebra*.

If we limit ourselves to cocommutative $B$'s and $N$'s, we talk about the *universal cocommutative bimeasurings* and we denote the *universal cocommutative bimeasuring bialgebra* (if it exists) by $B_c(T, A)$.

### 2.1. Bimeasurings over commutative algebras

The following proposition shows that universal bimeasurings exist whenever the algebra $A$ is commutative.

**Proposition 2.3.** *If T is a bialgebra, A a commutative algebra, and* $\theta: M \otimes T \to A$ *the universal measuring, then there exists a unique algebra structure on M so that T measures M to A, i.e.* $\theta(fg, t) = \theta(f, t_1)\theta(g, t_2)$ *and* $\theta(1_M, t) = \varepsilon(t)$.

*Furthermore, with this algebra structure M becomes a bialgebra and $\theta$ the universal bimeasuring. If T is a Hopf algebra, then so is $M_c$.*

**Proof.** Observe that $\omega: M \otimes M \otimes T \to A$, given by $\omega(m \otimes m', t) = \theta(m, t_1)\theta(m', t_2)$ is a measuring and defines the multiplication $\mathrm{m}: M \otimes M \to M$, to be the unique coalgebra map so that $\theta(\mathrm{m} \otimes 1) = \omega$.

Similarly the unit $\eta: k \to M$ is the unique coalgebra map so that $\theta(\eta \otimes 1) = \eta_A \varepsilon_N$.

The associativity and the unit conditions follow from Lemma 1.3 by noting that $\theta(\mathrm{m}(\mathrm{m} \otimes 1_M) \otimes 1_T) = \theta(\mathrm{m}(1_M \otimes \mathrm{m}) \otimes 1_T)$, $\theta(\mathrm{m}(\eta \otimes 1_M) \otimes 1_T) = \theta(\tau_l \otimes 1_T)$ and $\theta(\mathrm{m}(1_M \otimes \eta) \otimes 1_T) = \theta(\tau_r \otimes 1_T)$ (here $\tau_l$ and $\tau_r$ denote the canonical isomorphisms from $k \otimes M$ and $M \otimes k$, respectively to $M$).

Since the multiplication and the unit are coalgebra maps, $M$ must be a bialgebra and $\theta: M \otimes T \to A$ a bimeasuring. We claim $\theta$ is the universal bimeasuring. Let $f: N \otimes T \to A$ be a bimeasuring. Since $f$ is a measuring, there exists a unique coalgebra map $\overline{f}: N \to M$ so that $\theta(\overline{f} \otimes 1) = f$. It remains to show that $f$ is also an algebra map. This follows from Lemma 1.3, since we have $\theta(\mathrm{m}(\overline{f} \otimes \overline{f}) \otimes 1) = \theta(\overline{f}\mathrm{m} \otimes 1)$ and $\theta(\overline{f}\eta_N \otimes 1) = \theta(\eta_M \otimes 1)$.

Observe that, if $T$ is a Hopf algebra, then there is a unique coalgebra map $S: M^{\mathrm{cop}} \to M$ such that the diagram

$$\begin{array}{ccc} M^{\mathrm{cop}} \otimes T & \xrightarrow{S\otimes 1} & M \otimes T \\ {\scriptstyle 1\otimes S_T}\downarrow & & \downarrow{\scriptstyle \theta_T} \\ M^{\mathrm{cop}} \otimes T^{\mathrm{op}} & \xrightarrow{\bar{\theta}} & A \end{array}$$

commutes. Here $\bar{\theta}: M^{\mathrm{cop}} \otimes T^{\mathrm{op}} \to A$, defined by $\bar{\theta}(x \otimes t) = \theta(x \otimes t)$, satisfies the measuring conditions $\bar{\theta}(x \otimes t * t') = \theta(x \otimes t't) = \theta(x_1 \otimes t')\theta(x_2 \otimes t) = \theta(x_2 \otimes t)\theta(x_1 \otimes t') = \bar{\theta}(x_2 \otimes t)\bar{\theta}(x_1 \otimes t')$. This coalgebra map $S: M^{\mathrm{cop}} \to M$ is the obvious candidate for an antipode on $M$. Even though $\theta(1 * S \otimes 1) = \theta(\eta\varepsilon \otimes 1) = \theta(S * 1 \otimes 1)$, we cannot invoke Lemma 1.3 since $1 * S = m(1 \otimes S)\Delta: M \to M$ and $S * 1 = m(S \otimes 1)\Delta: M \to M$ may not be coalgebra maps, and hence $S$ may not be convolution inverse to the identity. So $M$ may not be a Hopf algebra. However, $M_c$ is invariant under $S$, and the restrictions $1 * S: M_c \to M_c$ and $S * 1: M_c \to M_c$ are coalgebra maps, the cocommutative version of Lemma 1.3 implies that $M_c$ is a cocommutative Hopf algebra with antipode $S$. □

**Theorem 2.4.** *If A is a commutative algebra, then the universal bimeasuring bialgebra construction gives rise to a contravariant functor $B(\_, A)$ on the category of bialgebras that is adjoint to itself.*

**Proof.** It is easy to see that the construction is functorial.

Let $T$ and $N$ be bialgebras. We shall display a canonical bijection

$$\psi_{T,N}: \mathrm{Bialg}(T, B(N, A)) \to \mathrm{Bialg}(N, B(T, A)).$$

It is observed from the diagram below:

$$\begin{tikzcd}
B(T,A)\otimes T \arrow[dr, "\theta_T"'] & N\otimes T \arrow[l, "1\otimes\overline{f}"'] \arrow[r, "1\otimes f"] \arrow[d] & N\otimes B(N,A) \arrow[dl, "\theta_N"] \\
 & A &
\end{tikzcd}$$

More precisely, if $\theta_T: B(T,A)\otimes T \to A$ and $\theta_N: N\otimes B(N,A) \to A$ are universal bimeasurings and $f: T \to B(N,A)$ is a bialgebra map, then $\theta_N(1\otimes f): N\otimes T \to A$ is a bimeasuring and we define $\psi_{T,N}(f) = \overline{f}: N \to B(T,A)$ to be the unique bialgebra map such that $\theta_T(\overline{f}\otimes 1) = \theta_N(1\otimes f): N\otimes T \to A$. If $g: S \to B(T,A)$ is a bialgebra map, then define $\xi_{N,T}(g) = \overline{g}: T \to B(N,A)$ to be the unique bialgebra map so that $\theta_T(1\otimes\overline{g}) = \theta_N(g\otimes 1)$ and note that $\xi_{N,T}$ is the inverse of $\psi_{T,N}$.

We shall conclude the proof by showing that $\psi_{R,N}(f\alpha) = B(\alpha,A)\psi_{T,N}(f)$, if $\alpha: R \to T$ is a bialgebra map. Indeed, if $\theta_R: B(R,A)\otimes R \to A$ is the universal bimeasuring, then $\theta_R(B(\alpha,A)\overline{f}\otimes 1) = \theta_R(B(\alpha,A)\otimes 1)(\overline{f}\otimes 1) = \theta_T(1\otimes\alpha)(\overline{f}\otimes 1) = \theta_T(\overline{f}\otimes 1)(1\otimes\alpha) = \theta_N(\overline{f}\otimes 1)(1\otimes\alpha) = \theta_N(1\otimes f\alpha) = \theta_R(\psi_{R,N}(f\alpha)\otimes 1)$. Hence we are done by Lemma 1.3. □

**Corollary 2.5** (*Takeuchi [8], Michaelis [5]*). *The finite dual construction $B \mapsto B^\circ$ defines a contravariant functor on the category of bialgebras that is adjoint to itself.*

**Remark.** If we fix a bialgebra $T$, then the universal bimeasuring bialgebra construction gives rise to a covariant functor $B(T,\_)$ from the category of commutative algebras to the category of bialgebras. It is easy to see that the functor preserves monomorphisms. In particular, there is a bialgebra monomorphism $T^\circ \to B(T,A)$ for any commutative algebra $A$ (arising from the unit $\eta: k \to A$). If the algebra $A$ is augmented, then the monomorphism is split.

### *2.2. Bimeasurings over noncommutative algebras*

It makes little sense to discuss bimeasurings when the algebra $A$ is not commutative. A point in case is the following:

**Proposition 2.6.** *Let $\psi: N\otimes T \to A$ be a bimeasuring. If either $N$ or $T$ is a Hopf algebra then $\psi(N\otimes T)$ generates a commutative subalgebra of $A$.*

**Proof.** Assume $N$ is a Hopf algebra and note that

$$\begin{aligned}\psi(n,t)\psi(m,s) &= \psi(S(m_1),t_1)\psi(m_2n_1,t_2s_1)\psi(S(n_2),s_2)\\ &= \psi(m,s)\psi(n,t).\end{aligned}$$

If $T$ is a Hopf algebra, then the argument is symmetric. □

Now suppose that $T$ is a Hopf algebra and that the algebra $A$ is not commutative. In view of the proposition above, it is clear that the universal bimeasuring $\theta: B(T, A) \otimes T \to A$ can only exist if every bimeasuring from $N \otimes T$ to $A$ maps into a fixed commutative subalgebra of $A'$ of $A$. The proposition below illustrates the fact that the universal bimeasurings exist in general only if $A$ is commutative.

**Proposition 2.7.** *The universal bimeasuring bialgebra $B(k[x], A)$ exists if and only if the algebra A is commutative.*

**Proof.** It is sufficient to see that every element of $A$ is in the image of some bimeasuring $N \otimes k[x] \to A$. This is observed by noting that $\psi_\alpha: k[x] \otimes k[x] \to A$, given by $\psi(x^i, x^j) = \delta_{i,j} i! \alpha^i$ is a bimeasuring for all $\alpha \in A$ ($\delta_{i,j}$ denotes the Kronecker's delta function). □

## 3. Universal cocommutative bimeasuring bialgebras

**Proposition 3.1.** *Let T be a bialgebra and A an algebra* (*not necessarily commutative*). *If the universal bimeasuring bialgebra $B(T, A)$ exists*, *then the universal cocommutative bialgebra $B_c(T, A)$ exists as well and we have the equality $B_c(T, A) = (B(T, A))_c$.*

**Proof.** Clear. □

Hence if $A$ is a commutative algebra, then we always have $B_c(T, A) = B(T, A)_c$. The proposition below sheds some light on the structure of universal cocommutative bimeasurings.

**Proposition 3.2.** *Suppose the image of a bimeasuring $\psi: N \otimes T \to A$ generates a commutative subalgebra of A. If N is cocommutative*, *then $\psi$ factors through $T_{ab}$*, *i.e. there is a unique bimeasuring $\overline{\psi}: N \otimes T_{ab}$*, *such that $\psi = \overline{\psi}(1 \otimes \pi)$*, *where $\pi: T \to T_{ab}$ is the canonical projection.*

**Proof.** We compute

$$\psi(n, ts) = \psi(n_1, t)\psi(n_2, s) = \psi(n_2, s)\psi(n_1, t) = \psi(n_1, s)\psi(n_2, t) = \psi(n, st)$$

and conclude the proof by pointing out that if $\psi(n, t) = 0$ for some $t \in T$ and all $n \in N$, then $\psi(n, sts') = \psi(n_1, s)\psi(n_2, t)\psi(n_3, s') = 0$ for all $s, s' \in T$. □

**Corollary 3.3.** *Let N and T be cocommutative bialgebras. If $\psi: N \otimes T \to A$ is a bimeasuring with commutative image in A*, *then $\psi$ factors through $N_{ab} \otimes T_{ab}$*, *i.e. there is a unique bimeasuring $\overline{\psi}: N_{ab} \otimes T_{ab} \to A$ such that $\psi = \overline{\psi}(\pi \otimes \pi)$.*

**Proposition 3.4.** *If T is a perfect Hopf algebra* (*i.e. $T_{ab} = k$*), *then the universal bimeasuring bialgebra $B_c(T, A)$ exists for all algebras A and it is equal to the ground field k.*

**Proof.** Apply Lemma 2.6 and Proposition 3.2. □

**Proposition 3.5.** *If A is a commutative algebra and T a cocommutative bialgebra, then the universal bimeasuring bialgebra $B(T, A)$ is commutative.*

**Proof.** Apply Proposition 3.2. □

It is natural to ask the symmetric question: If $T$ is a commutative bialgebra, is the universal bimeasuring bialgebra $B(T, A)$ automatically cocommutative? We conjecture this is not the case in general; however, we can say the following.

**Proposition 3.6.** *If A is a commutative algebra and T a bialgebra, then $B_c(T, A) = B_c(T_{ab}, A)$.*

**Proof.** Apply Proposition 3.2. □

**Remark.** The proposition above is symmetric to Proposition 3.5 in the sense that the proposition in question is equivalent to saying that

$$B(T_c, A)_{ab} = B(T_c, A).$$

If $T$ is a Hopf algebra and $A$ is a commutative algebra, then the subsets $\mathrm{Alg}(T, A)$ and $\mathrm{Der}(T, A)$ of the convolution algebra $\mathrm{Hom}(T, A)$ form a group and a Lie algebra under convolution, respectively. Moreover, $\mathrm{Alg}(T, A)$ acts on $\mathrm{Der}(T, A)$ by "conjugation", giving rise to a canonical algebra map $\chi : U\mathrm{Der}(T, A) \rtimes k\mathrm{Alg}(T, A) \to \mathrm{Hom}(T, A)$. The map $\psi : \mathrm{Der}(T, A) \times \mathrm{Alg}(T, A) \times T \to A$, defined by $\psi(f, g, t) = f(t_1)g(t_2)$, extends to a bimeasuring

$$\psi : U\mathrm{Der}(T, A) \rtimes k\mathrm{Alg}(T, A) \otimes T \to A.$$

Thus, there is a unique bialgebra map $\bar{\psi} : U\mathrm{Der}(T, A) \rtimes k\mathrm{Alg}(T, A) \to B(T, A)$ such that $\theta(\bar{\psi} \otimes 1) = \psi$, whose image is in $B_c(T, A)$. The fact that $B_c(T, A)$ is a subcoalgebra of the cofree coalgebra $\mathscr{C}(\mathrm{Hom}(T, A))$ [7] now shows together with Proposition 1.4 that the group of points and the Lie algebra of primitives of $B_c(T, A)$ can be identified with $\mathrm{Alg}(T, A)$ and $\mathrm{Der}(T, A)$, respectively. If $k$ is an algebraically closed field of characteristic zero, it then follows by the structure theorem for cocommutative Hopf algebras that $U\mathrm{Der}(T, A) \rtimes k\mathrm{Alg}(T, A) \cong B_c(T, A)$.

## 4. Tensor products and universal bimeasurings

Throughout this section, $T$ and $S$ will be bialgebras and $A$ a commutative algebra. We shall examine how the tensor product $B(T, A) \otimes B(S, A)$ of universal bimeasuring bialgebras $B(T, A)$ and $B(S, A)$ is related to the universal bimeasuring bialgebra $B(T \otimes S, A)$. Recall that if the algebra $A$ is the ground field $k$, then we have

$$B(T, k) \otimes B(S, k) = T^\circ \otimes S^\circ \simeq (T \otimes S)^\circ = B(T \otimes S, k).$$

We conjecture that, in general, the bialgebras $B(T, A) \otimes B(S, A)$ and $B(T \otimes S, A)$ are not isomorphic.

Since the algebra $A$ is commutative, the linear map

$$\psi: B(T, A) \otimes B(S, A) \otimes T \otimes S \to A,$$

given by

$$\psi(f \otimes g, t \otimes s) = \theta_T(f, t)\theta_S(g, s),$$

is a bimeasuring. Define

$$\alpha: B(T, A) \otimes B(S, A) \to B(T \otimes S, A)$$

to be the unique coalgebra map such that $\psi = \theta_{T\otimes S}(\alpha, 1)$. Furthermore, let $\overline{\iota_1}: B(T \otimes S, A) \to B(T, A)$, $\overline{\iota_2}: B(T \otimes S, A) \to B(S, A)$, $\overline{\pi_1}: B(T, A) \to B(T \otimes S, A)$ and $\overline{\pi_2}: B(S, A) \to B(T \otimes S, A)$ be bialgebra maps induced by $\iota_1 = 1 \otimes \eta: T \to T \otimes S$, $\iota_2 = \eta \otimes 1: S \to T \otimes S$, $\pi_1 = 1 \otimes \varepsilon: T \otimes S \to T$ and $\pi_2 = \varepsilon \otimes 1: T \otimes S \to S$, respectively. Using Lemma 1.3, it is easy to see that the following diagram commutes, and symmetrically for $B(S, A)$ with $\overline{\iota_2}$ and $\overline{\pi_2}$.

$$\begin{tikzcd}
B(T,A) \arrow[r, "1\otimes\eta"] \arrow[dr, "\overline{\pi_1}"'] & B(T,A)\otimes B(S,A) \arrow[r, "1\otimes\varepsilon"] \arrow[d, "\alpha"] & B(T,A) \\
& B(T\otimes S,A) \arrow[ur, "\overline{\iota_1}"'] &
\end{tikzcd}$$

Hence the composite map

$$B(T, A) \otimes B(S, A) \xrightarrow{\alpha} B(T \otimes S, A) \xrightarrow{\overline{\iota_1} * \overline{\iota_2}} B(T, A) \otimes B(S, A),$$

where $\overline{\iota_1} * \overline{\iota_2} = (\overline{\iota_1} \otimes \overline{\iota_2})\Delta$, is the identity and therefore, $\alpha$ must be an injective mapping. However, although $\theta\alpha(\overline{\iota_1} * \overline{\iota_2}) = \theta(1, 1)$, we cannot invoke Lemma 1.3 here, since $\alpha(\overline{\iota_1} * \overline{\iota_2})$ is not a coalgebra map. But the restriction $\alpha: B_c(T, A) \otimes B_c(S, A) \to B_c(T \otimes S, A)$ is easily seen to be an isomorphism by invoking the cocommutative version of Lemma 1.3.

## 5. Cocommutative bimeasurings and Hopf modules

Throughout this section, $T$ and $N$ denote cocommutative Hopf algebras and $A$ a commutative algebra. Furthermore let $\mu: N \otimes T \to N$ and $\nu: N \otimes T \to T$ be a pair of actions making $(N, T, \mu, \nu)$ into an abelian matched pair of Hopf algebras [4]. We can then talk about skew bimeasurings $\psi: N \otimes T \to A$, that is linear maps satisfying

$$\psi(nm, t) = \psi(n, m_1(t_1))\psi(m_2, t_2), \quad \psi(1, t) = \varepsilon(t),$$
$$\psi(n, ts) = \psi(n_1^{t_1}, s)\psi(n_2, t), \quad \psi(n, 1) = \varepsilon(n),$$

where we abbreviate $\mu(n,t) = n(t)$ and $\nu(n,t) = n^t$, or equivalently

$$\psi(ntm) = \psi(nt_1)\psi(t_2m), \quad \psi(t) = \varepsilon(t),$$
$$\psi(tns) = \psi(tn_1)\psi(n_2s), \quad \psi(n) = \varepsilon(n).$$

(Here we identify $nt \in T \bowtie N$ with $n \otimes t \in N \otimes T$). The set $P_{\mu,\nu}(N,T,A)$ of all such maps then becomes an abelian group under convolution. It is then easy to observe that the abelian group of skew bimeasurings is isomorphic to the first cohomology group of the matched pair (see [2], Sections 1.3 and 1.4, for definition and description of the cohomology groups $\mathcal{H}^*(N,T,A)$ of the abelian matched pair $(N,T) = (N,T,\mu,\nu)$ with coefficients in the algebra $A$).

**Proposition 5.1.** *If $(N,T,\mu,\nu)$ is an abelian matched pair of Hopf algebras, we have an isomorphism $P_{\mu,\nu}(N,T,A) \simeq \mathcal{H}^1(N,T,A)$. In particular, the abelian group $P(N,T,A)$ of all bimeasurings from $N \otimes T$ to $A$ is isomorphic to the cohomology group $\mathcal{H}^1(N,T,A)$ of the trivial matched pair $(N,T,1 \otimes \varepsilon, \varepsilon \otimes 1)$.*

There is a relation between bimeasurings, Hopf module isomorphisms and algebras in the category of Hopf modules, which we want to outline here. A Hopf module $(M,\delta,\mu)$ over a Hopf algebra $H$ is a $H$-comodule $\delta: M \to H \otimes M$ together with a compatible $H$-module structure $\mu: H \otimes M \to M$, so that the diagram

$$\begin{tikzcd}
H \otimes M \arrow[r, "\delta_{H\otimes M}"] \arrow[d, "\mu"'] & H \otimes H \otimes M \arrow[d, "1\otimes\mu"] \\
H \arrow[r, "\delta"] & H \otimes M
\end{tikzcd}$$

commutes, i.e. $\delta(hm) = h_1m_{-1} \otimes h_2m_0$, where $\delta_{H\otimes M} = (m_H \otimes 1 \otimes 1)\tau_{23}(\Delta \otimes \delta)$. A morphism of Hopf modules is just an $H$-linear and $H$-colinear map. The cotensor product $M \otimes^H N$ together with the diagonal action, which restricts from the diagonal action of $M \otimes N$, is a symmetric tensor in the category of Hopf modules $\mathrm{Vect}_H^H$. The vector space of coinvariants

$$A = M^{\mathrm{co}H} = \mathrm{equ}\left( M \underset{\iota\otimes 1}{\overset{\delta}{\rightrightarrows}} H \otimes M \right)$$

is precisely the image of $\rho = \mu(S \otimes 1)\delta: M \to M$, which then has the image factorization $\rho = \kappa\bar{\rho}: M \to A \to M$, where $\bar{\rho}: M \to A$ is the projection and $\kappa: A \to M$ the inclusion.

**Theorem 5.2** (*Sweedler [7], Montgomery [6]*). *$\theta = (1 \otimes \bar{\rho})\delta: M \to H \otimes A$ is an isomorphism of* Hopf *modules. The functor $(\ )^{\mathrm{co}H}: \mathrm{Vect}_H^H \to \mathrm{Vect}$ is a tensor preserving equivalence of categories with inverse $H \otimes \_: \mathrm{Vect} \to \mathrm{Vect}_H^H$.*

**Proof.** It is easy to check that $\theta$ is a homomorphism of Hopf modules and that $\theta\kappa(a) = 1 \otimes a$ for all $a \in A$. It then follows that $\mu(1 \otimes \kappa)\theta = \mathrm{id}_M$ and $\theta\mu(1 \otimes \kappa) = \mathrm{id}_{H\otimes A}$, so that $\theta$ is invertible and $\theta^{-1} = \mu(1 \otimes \kappa)$. $\square$

An algebra in $\mathrm{Vect}_H^H$ is a Hopf module $M$ together with Hopf module maps $v\colon H \to M$ and $\nabla\colon M \otimes^H M \to M$ satisfying the usual unitarity and associativity conditions. It follows that the equivalence described in the preceding theorem restricts to algebras $(\ )^{\mathrm{co}H}\colon \mathrm{Alg}_H^H \to \mathrm{Alg}$.

**Theorem 5.3.** *If $(M, \delta, \mu)$ is an algebra in $\mathrm{Alg}_H^H$ with algebra of coinvariants $A$, then the following groups are isomorphic*:

(1) $\mathrm{Reg}_+(H, A)$, *the group of convolution invertible normalized linear maps $\psi\colon H \to A$*,
(2) $\mathrm{Aut}_A^H(M)$, *the group of $A$-linear $H$-comodule automorphisms $\Phi\colon M \to M$*,
(3) *the group $\mathscr{A}$ of $A$-linear actions $\overline{\mu}\colon H \otimes M \to M$, such that $(M, \delta, \overline{\mu})$ is a* Hopf *module*.

**Proof.** By Theorem 5.2 it suffices to consider the $H$-comodule $H \otimes A$. Convolution invertible, normalized linear maps $\psi\colon H \to A$ are in bijective correspondence with $A$-linear $H$-comodule automorphisms $\phi\colon H \otimes A \to H \otimes A$, i.e. there is an isomorphism $\alpha\colon \mathrm{Reg}_+(H, A) \to \mathrm{Aut}_A^H(H \otimes A)$ given by $\alpha(\psi) = (1 \otimes m_A)(1 \otimes \psi \otimes 1)(\Delta_H \otimes 1)$ and $\alpha^{-1}(\phi) = (\varepsilon_H \otimes 1)\phi(1 \otimes \iota_A)$. In particular, if $\phi = \alpha(\psi)$, then $\phi(h \otimes a) = h_1 \otimes \psi(h_2)a$.

The $A$-linear $H$-comodule automorphisms $\phi\colon H \otimes A \to H \otimes A$ correspond bijectively to $A$-linear actions $\tilde{\mu}\colon H \otimes H \otimes A \to H \otimes A$ such that $(H \otimes A, \Delta \otimes 1, \tilde{\mu})$ is a Hopf module over $H$ with coinvariants $A$. The bijection is given by the commutative diagram

$$\begin{array}{ccc} H \otimes H \otimes A & \xrightarrow{\mu} & H \otimes A \\ {\scriptstyle 1\otimes\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ H \otimes H \otimes A & \xrightarrow{\tilde{\mu}} & H \otimes A \end{array}$$

i.e. by the isomorphism $\beta\colon \mathrm{Aut}_A^H(H \otimes A) \to \mathscr{A}$ defined by $\beta(\phi) = \phi\mu(1 \otimes \phi^{-1})$ and $\beta^{-1}(\bar{\mu}) = \bar{\mu}(1 \otimes \iota_H \otimes 1)$. A tedious, but straightforward calculation shows that $(H \otimes A, \delta, \bar{\mu})$ is a Hopf module and, in fact, an algebra in the category of Hopf modules over $H$. On the other hand, $\phi = \beta^{-1}(\bar{\mu}) = \bar{\mu}(1 \otimes \iota_H \otimes 1)$ is an $A$-linear $H$-comodule map, since $\bar{\mu}$ is an $A$-linear action such that $(H \otimes A, \delta, \bar{\mu})$ is a Hopf module. By the arguments in the proof of Theorem 5.2, it follows that $\bar{\phi}\theta = \mathrm{id}_{H \otimes A} = \theta\bar{\phi}$. Moreover, $\beta^{-1}\beta(\phi) = \phi\mu(1 \otimes \phi^{-1}) = \phi$ and $\beta\beta^{-1}(\bar{\mu}) = \bar{\phi}\mu(1 \otimes \bar{\phi}^{-1}) = \bar{\phi}\mu(1 \otimes \theta) = \bar{\phi}\theta\bar{\mu} = \bar{\mu}$. $\square$

If $(N, T, \mu, v)$ is a matched pair of cocommutative Hopf algebras with bismash product $H = T \bowtie N$, then the relation between the action $\bar{\mu}\colon (T \bowtie N) \otimes N \otimes T \otimes A \to N \otimes T \otimes A$ and the skew bimeasuring $\psi\colon T \bowtie N \to A$ is given by

$$\begin{aligned} \bar{\mu}(nt \otimes m \otimes s \otimes a) &= \bar{\mu}(n \otimes \bar{\mu}(t \otimes m \otimes s \otimes a)) \\ &= \bar{\mu}(n \otimes t_1[m_1] \otimes t_2 s \otimes \psi(t_3 m_2)a) \\ &= n_1 \cdot t_1[m_1] \otimes n_2(t_1 s_1) \otimes \psi(n_3 t_2 s_2)\psi(t_3 m_2)a, \end{aligned}$$

where $t[n] = n_2^{S(n_1)(S(t))} = S(S(n)^{S(t)})$.

**Corollary 5.4.** *If* $(N, T, \mu, \nu)$ *is a matched pair of cocommutative Hopf algebras and A is a commutative algebra then the following groups are isomorphic*:

(1) $\mathrm{Bimeas}(N \otimes T, A)$, *the group of bimeasurings under convolution*,
(2) $\mathrm{Aut}^{T\bowtie N}_{N\otimes A}(N \otimes T \otimes A) \cap \mathrm{Aut}^{T\bowtie N}_{T\otimes A}(N \otimes T \otimes A)$, *the group of* $T\bowtie N$*-comodule automorphisms which are* $N \otimes A$*-linear as well as* $T \otimes A$*-linear*,
(3) $\mathscr{A}$, *the group of actions* $\bar{\mu}$: $(N\bowtie T) \otimes (N \otimes T \otimes A) \to N \otimes T \otimes A$ *diagonal in N as well as in T* (*i.e. the N-action is* $N \otimes A$*-linear and the T-action is* $T \otimes A$*-linear*).

**Proof.** The result follows directly from Theorem 5.3 by a lengthy, routine computation. We use the identities

$$t_1[n_1](t_2^{n_2}) = \varepsilon(n)t,$$
$$(t_1[n_1])^{t_2^{n_2}} = n\varepsilon(t)$$

connecting the distributive law $nt = n_1(t_1)n_2^{t_2}$ and its inverse $tn = t_1[n_1]t_2^{n_2}$, where $t[n] = S(S(n)^{S(t)})$ and $t^n = S(S(n)(S(t)))$ [2]. □

**Remark.** Motivated by the referee's report we have recently found a rather lengthy and technical proof of our conjecture in Section 4 that $\alpha : B(T, A) \otimes B(S, A) \to B(T \otimes S, A)$ is not an isomorphism, and moreover that no "natural" such isomorphism can exist. This question together with more details on the structure of $B(T, A)$ will be addressed in a subsequent paper.

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