ON BIMEASURINGS II

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Abstract. We answer some questions posed in [GM] concerning the universal bimeasuring bialgebra via a construction of suitable subcoalgebras of the universal measuring coalgebra.

1. Introduction

If A and B are algebras and C is a coalgebra, then a linear map $\psi\colon B \otimes C \to A$ is called a measuring if it induces an algebra map from B to the convolution algebra $\mathrm{Hom}_k(C, A)$, or equivalently, if $\psi(bb', c) = \psi(b, c_1)\psi(b', c_2)$ and $\psi(1_B, c) = \varepsilon(c)1_A$. Measurings have first been introduced and studied by M.E. Sweedler [Sw]. They correspond to homomorphisms of algebras over a coalgebra which are cofree as comodules [GP]. There is a universal measuring coalgebra $M(B, A)$ and measuring $\theta\colon M(B, A) \otimes B \to A$ for every pair of algebras A and B such that C-measurings from B to A correspond bijectively to coalgebra maps from C to $M(B, A)$.

If B and C are bialgebras, then we call a map $\psi\colon B\otimes C \to A$ a bimeasuring, if it measures in both variables. If A is commutative then $M(B, A)$ carries a natural bialgebra structure [GM] making it a universal bimeasuring bialgebra (to avoid confusion, we denote it by $B(B, A)$). More precisely, bimeasurings from $B \otimes C$ to A are in a bijective correspondence with bialgebra maps from C to $B(B, A)$. In fact, there are natural isomorphisms

$$\mathrm{Bialg}(B, B(C, A)) \cong \mathrm{Bimeas}(B \otimes C, A) \cong \mathrm{Bialg}(C, B(B, A))$$

so that $B(-, A)$ defines a contravariant functor on the category of bialgebras that is adjoint to itself.

In [GM] we also construct a homomorphism

$$\alpha\colon B(S, A) \otimes B(T, A) \to B(S \otimes T, A).$$

While we have shown that α is injective, we have conjectured that it is, in general, not surjective. In this paper we confirm this conjecture. This is done in two steps. First, in Section 2 we show that, if $A = C^*$, where $C \neq k$ is a coalgebra, and B is an augmented algebra which has a finite dimensional augmented quotient with nonzero square of the augmentation ideal, then the universal

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measuring coalgebra $M(B,A)$ is not cocommutative. This is accomplished by constructing non-cocommutative subcoalgebras L and R of $T(B\otimes C)^\circ$ both measuring B to C^*. Since $M(B,C^*)$ is the maximal subcoalgebra of $T(B\otimes C)^\circ$ that measures B to C^* it then follows that L and R are subcoalgebras of $M(B,C^*)$.

The construction also shows, that in general, the canonical linear map $p\colon M(B,A)\to \mathrm{Hom}_k(B,A)$ is not injective, and hence that $M(B,A)$ can not be identified with a subspace of $\mathrm{Hom}_k(B,A)$. Section 3 contains some examples and applications of these concepts and constructions.

In Section 4 we then proceed by showing that any natural transformation $\beta\colon B(-\otimes -,A)\to B(-,A)\otimes B(-,A)$ satisfying $(\beta\otimes 1)\beta=(1\otimes\beta)\beta$ would induce alternative coalgebra structures on $B(H,A)$ for all commutative bialgebras H. But then, due to the "dual" Eckman-Hilton Theorem, $B(H,A)$ would have to be cocommutative. Hence no such transformation can exist and in particular, if $A=C^*$, where $C\neq k$ is a cocommutative coalgebra, then α is not invertible (or equivalently $\alpha\colon B(S,A)\otimes B(T,A)\to B(S\otimes T,A)$ is not surjective for general S and T).

1.1. **Notation.** All vector spaces (algebras, coalgebras, bialgebras) will be over a ground field k. The categories of vector spaces, algebras, coalgebras and bialgebras are denoted by $\mathrm{Vect}, \mathrm{Alg}, \mathrm{Coalg}$ and Bialg, respectively. If A is an algebra and C a coalgebra, then $\mathrm{Hom}_k(C,A)$ denotes the convolution algebra of all linear maps from C to A. The unit and the multiplication on A are denoted by $\eta\colon k\to A$ and $\mathrm{m}\colon A\otimes A\to A$; the counit and the comultiplication on C are denoted by $\varepsilon\colon C\to k$ and $\Delta\colon C\to C\otimes C$. We use Sweedler's notation for comultiplication: $\Delta(c)=c_1\otimes c_2$, $(1\otimes\Delta)\Delta(c)=c_1\otimes c_2\otimes c_3$ etc. If $f\colon U\otimes V\to W$ is a linear map than we often write $f(u,v)$ instead of $f(u\otimes v)$. If B is an augmented algebra, then we denote the augmentation ideal by $B^+=\ker\varepsilon$ and its square by $(B^+)^2$.

2. Construction of non-cocommutative subcoalgebras of the universal measuring coalgebra

Here we show that if $A=C^*$, where $C\neq k$ is a coalgebra and B is a finite dimensional augmented algebra such that $(B^+)^2\neq 0$, then the universal measuring coalgebra $M(B,A)$ is not cocommutative and the canonical map $p\colon M(B,A)\to\mathrm{Hom}_k(B,A)$ is not injective. This is done by constructing two distinct, non-commutative subcoalgebras L and R of $M(B,A)$, for which $p(L)=p(R)$. The constructions involve the "lifting" of linear functionals $B^+\otimes C\to k$ to elements of $M(B,A)\subseteq (T(B\otimes C))^\circ$.

Recall that

Theorem 2.1. [Sw] *If B and A are algebras and*

$$\mathcal{C}=\mathcal{C}(\mathrm{Hom}_k(B,A))\xrightarrow{p}\mathrm{Hom}_k(B,A)$$

is the cofree coalgebra over $\mathrm{Hom}_k(B, A)$*, then the universal measuring coalgebra* $M = M(B, A)$ *is the maximal subcoalgebra of* $\mathcal{C}$*, for which*

$$\theta\colon M \otimes B \to \mathcal{C} \otimes B \overset{p\otimes 1}{\to} \mathrm{Hom}_k(B, A) \otimes B \overset{\mathrm{ev}}{\to} A$$

measures.

Now note that

Proposition 2.2. [Sw] *If* V *is a vector space, then* $\mathcal{C}(V^*) \cong (\mathrm{T}\, V)^\circ$ *and* $p\colon (\mathrm{T}\, V)^\circ \to V^*$ *is given by* $p(\alpha)(v) = \alpha(v)$*.*

Proof. The string of natural isomorphisms

$$\begin{aligned}\mathrm{Coalg}(C, \mathcal{C}(V^*)) &\cong \mathrm{Vect}(C, V^*) \cong (C \otimes V)^* \cong \mathrm{Vect}(V, C^*)\\ &\cong \mathrm{Alg}(\mathrm{T}(V), C^*) \cong \mathrm{Coalg}(C, \mathrm{T}(V)^\circ)\end{aligned}$$

implies that $\mathcal{C}(V^*) = (\mathrm{T}\, V)^\circ$ as coalgebras. Explicitly, if $f\colon C \to V^*$ a linear map, then $\overline{f}\colon C \to (\mathrm{T}\, V)^\circ$, given by

$$\overline{f}(c)(v^1 \otimes v^2 \otimes \ldots \otimes v^n) = f(c_1)(v^1) \ldots f(c_n)(v^n)$$

is the unique coalgebra map for which $f = p\overline{f}$. $\square$

Corollary 2.3. *If* B *and* C *are vector spaces, then*

$$\mathcal{C}((B \otimes C)^*) \cong \mathcal{C}(\mathrm{Hom}_k(B, C^*)) \cong (\mathrm{T}(B \otimes C))^\circ$$

and

$$p\colon (\mathrm{T}(B \otimes C))^\circ \to \mathrm{Hom}_k(B, C^*)$$

is given by

$$p(f)(b)(c) = f(b \otimes c).$$

Proof. This is an immediate consequence of the previous proposition, since $\mathrm{Hom}_k(B, C^*) \cong (B \otimes C)^*$. Moreover, the natural map $\chi\colon B^* \otimes C^* \to (B \otimes C)^*$, given by $\chi(f \otimes g)(b \otimes c) = f(b)g(c)$, is an isomorphism if either B or C is finite dimensional. $\square$

From now on assume that B is an augmented algebra with augmentation ideal $B^+ = \ker\{\varepsilon\colon B \to k\}$ and that C is a coalgebra. By Theorem 2.1 and Corollary 2.3 the universal measuring coalgebra $M = M(B, C^*)$ is the maximal subcoalgebra of $(\mathrm{T}(B \otimes C))^\circ$ for which $\theta = \mathrm{ev}(p \otimes 1)\colon M \otimes B \to C^*$, $\theta(f, b)(c) = f(b \otimes c)$, measures.

Define

$$\psi\colon B \otimes C \to (B^+ \otimes C) \oplus k$$

by

$$\psi(b \otimes c) = (\bar{b} \otimes c) \oplus \varepsilon(b)\varepsilon(c),$$

where $\bar{b} = b - \varepsilon(b)$. Thus ψ is the composite

$$\psi\colon B \otimes C \cong (B^+ \oplus k) \otimes C \cong (B^+ \otimes C) \oplus C \overset{1\oplus\varepsilon}{\to} (B^+ \otimes C) \oplus k$$

and is clearly surjective.

Consider the binary operations

$$m_l\colon (B^+\otimes C)\otimes(B^+\otimes C)\to(B^+\otimes C)$$

defined by

$$m_l((a\otimes c)\otimes(b\otimes d))=ab\otimes c\varepsilon(d)$$

and

$$m_r\colon (B^+\otimes C)\otimes(B^+\otimes C)\to(B^+\otimes C)$$

defined by

$$m_r((a\otimes c)\otimes(b\otimes d))=ab\otimes\varepsilon(c)d.$$

Then m_l and m_r are associative binary operations. By adjoining an identity, they induce two distinct algebra structures on $(B^+\otimes C)\oplus k$ with unit $u=(0,1)$, where the multiplications are given by

$$M_l((a\otimes c,x)\otimes(b\otimes d,y))=(ab\otimes c\varepsilon(d)+xb\otimes d+a\otimes cy,xy)$$

and

$$M_r((a\otimes c,x)\otimes(b\otimes d,y))=(ab\otimes\varepsilon(c)d+xb\otimes d+a\otimes cy,xy),$$

respectively. We will denote these algebras by B_l and B_r, respectively.

By the universal property of the tensor algebra functor $T\colon$ Vect $\to$ Alg there exist unique algebra maps $\Psi_l\colon T(B\otimes C)\to B_l$ and $\Psi_r\colon T(B\otimes C)\to B_r$ such that

$$\begin{tikzcd}
T(B\otimes C) \arrow[d, "\Psi_l"'] & B\otimes C \arrow[l, "\mathrm{incl}"'] \arrow[r, "\mathrm{incl}"] \arrow[d, "\psi"'] & T(B\otimes C) \arrow[d, "\Psi_r"'] \\
B_l \arrow[r, equal] & (B^+\otimes C)\oplus k \arrow[r, equal] & B_r
\end{tikzcd}$$

commutes, where incl is the canonical inclusion. It follows by induction on n that

$$\begin{aligned}\Psi_l(\mathbf{b}) &= \varepsilon(\mathbf{b})\\ &+ \sum_{i=1}^{n}\varepsilon(b^1)\dots\varepsilon(b^{i-1})\bar{b^i}b^{i+1}\dots b^n\otimes\varepsilon(c^1)\dots\varepsilon(c^{i-1})c^i\varepsilon(c^{i+1})\dots\varepsilon(c^n)\end{aligned}$$

and similarly

$$\begin{aligned}\Psi_r(\mathbf{b}) &= \varepsilon(\mathbf{b})\\ &+ \sum_{i=1}^{n}b^1\dots b^{i-1}\bar{b^i}\varepsilon(b^{i+1})\dots\varepsilon(b^n)\otimes\varepsilon(c^1)\dots\varepsilon(c^{i-1})c^i\varepsilon(c^{i+1})\dots\varepsilon(c^n)\end{aligned}$$

for $\mathbf{b}=(b^1\otimes c^1)\otimes\dots\otimes(b^n\otimes c^n)\in(B\otimes C)^n\subseteq T(B\otimes C)$, where $\varepsilon(\mathbf{b})=\varepsilon(b^1)\dots\varepsilon(b^n)\varepsilon(c^1)\dots\varepsilon(c^n)$.

Take the "Hopf duals" to obtain coalgebra maps $l = \Psi_l^\circ \colon B_l^\circ \to T(B\otimes C)^\circ$ and $r = \Psi_r^\circ \colon B_r^\circ \to T(B\otimes C)^\circ$ such that the diagram

$$\begin{tikzcd}
\mathcal{C}(\mathrm{Hom}_k(B,C^*)) \arrow[r, "p"] \arrow[d, "\cong"'] & \mathrm{Hom}_k(B,C^*) \arrow[d, "\cong"'] & \mathcal{C}(\mathrm{Hom}_k(B,C^*)) \arrow[l, "p"'] \arrow[d, "\cong"'] \\
T(B\otimes C)^\circ \arrow[r, "p"] & (B\otimes C)^* & T(B\otimes C)^\circ \arrow[l, "p"'] \\
B_l^\circ \arrow[u, "l"] \arrow[r, "\mathrm{incl}"] & ((B^+\otimes C)\oplus k)^* \arrow[u, "\psi^*"] & B_r^\circ \arrow[u, "r"] \arrow[l, "\mathrm{incl}"']
\end{tikzcd}$$

commutes, and more explicitly

$$l(f)(\mathbf{b}) = \varepsilon(\mathbf{b}) + \sum_{i=1}^{n} f(\bar{b}^i b^{i+1}\dots b^n \otimes c^i)\prod_{j=1}^{i-1}\varepsilon(b^j)\prod_{j\neq i}\varepsilon(c^j)$$

and

$$r(f)(\mathbf{b}) = \varepsilon(\mathbf{b}) + \sum_{i=1}^{n} f(b^1\dots b^{i-1}\bar{b}^i \otimes c^i)\prod_{j=i+1}^{n}\varepsilon(b^j)\prod_{j\neq i}\varepsilon(c^j).$$

Also note that the tensors $(b^1\otimes c^1)\otimes\dots\otimes(b^n\otimes c^n)$ together with $1_{\mathrm{T}(B\otimes C)}$ span $\mathrm{T}(B\otimes C)$.

By $\varepsilon\colon T(B\otimes C)\to k$ we mean the composite $T(B\otimes C)\overset{\oplus\varepsilon}{\to}\bigoplus k \overset{\sum}{\to} k$, i.e. $\varepsilon((b^1\otimes c^1)\otimes\dots\otimes(b^n\otimes c^n)) = \varepsilon(b^1)\varepsilon(c^1)\dots\varepsilon(b^n)\varepsilon(c^n)$.

Lemma 2.4. *If B is finite dimensional and $f\in(B^+\otimes C)^*$, then $l(f), r(f)\in T(B\otimes C)^\circ$ and*

$$\Delta l(f) = (l\otimes l)m_l^*(f) + l(f)\otimes\varepsilon + \varepsilon\otimes l(f),$$

$$\Delta r(f) = (r\otimes r)m_r^*(f) + r(f)\otimes\varepsilon + \varepsilon\otimes r(f).$$

Proof. First observe that $\Delta l(f)(\mathbf{a}\otimes\mathbf{b}) = l(f)(\mathbf{a}\cdot_{T(B\otimes C)}\mathbf{b}) = l(f)(\mathbf{a}\otimes\mathbf{b})$, where $\mathbf{a} = (a^1\otimes c^1)\otimes\dots\otimes(a^n\otimes c^n)$ and $\mathbf{b} = (b^1\otimes d^1)\otimes\dots\otimes(b^m\otimes d^m)$. A direct computation shows that

$$l(f)(\mathbf{a}\otimes\mathbf{b}) = \sum_{i=1}^{n} f(\bar{a}^i a^{i+1}\dots a^n b^1\dots b^m\otimes c^i)\prod_{j=1}^{i-1}\varepsilon(a^j)\prod_{j\neq i}\varepsilon(c^j)\prod_{k=1}^{m}\varepsilon(d^k)$$

$$+\sum_{i=1}^{m} f(\bar{b}^i b^{i+1}\dots b^m\otimes d^i)\prod_{j=1}^{n}\varepsilon(a^j)\prod_{j=1}^{i-1}\varepsilon(b^j)\prod_{j=1}^{n}\varepsilon(c^j)\prod_{j\neq i}\varepsilon(d^j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{a}^i a^{i+1}\dots a^n \bar{b}^j b^{j+1}\dots b^m\otimes c^i)\prod_{s=1}^{i-1}\varepsilon(a^s)\prod_{t\neq i}\varepsilon(c^t)\prod_{s=1}^{j-1}\varepsilon(b^s)\prod_{k=1}^{m}\varepsilon(d^k)$$

$$+\sum_{i=1}^{n} f(\bar{a^i}a^{i+1}\dots a^n\otimes c^i)\prod_{s=1}^{i-1}\varepsilon(a^s)\prod_{t\neq i}\varepsilon(c^t)\prod_{s=1}^{n}\varepsilon(b^s)\prod_{k=1}^{m}\varepsilon(d^k)$$
$$+\sum_{j=1}^{m} f(\bar{b^j}b^{j+1}\dots b^m\otimes d^j)\prod_{s=1}^{n}\varepsilon(a^s)\prod_{t=1}^{n}\varepsilon(c^t)\prod_{s=1}^{j-1}\varepsilon(b^s)\prod_{k\neq j}\varepsilon(d^k)$$
$$= (l\otimes l)m_l^*(f)(\mathbf{a}\otimes\mathbf{b}) + (l(f)\otimes\varepsilon)(\mathbf{a}\otimes\mathbf{b}) + (\varepsilon\otimes l(f))(\mathbf{a}\otimes\mathbf{b}),$$

where the second equality is a consequence of the decomposition

$$b^1\dots b^m = \varepsilon(b^1)\dots\varepsilon(b^m) + \sum_{j=1}^{m}\varepsilon(b^1)\dots\varepsilon(b^{j-1})\bar{b^j}b^{j+1}\dots b^m.$$

Hence

$$l(f)(\mathbf{a}\cdot_{T(B\otimes C)}\mathbf{b}) = ((l\otimes l)m_l^*(f) + l(f)\otimes\varepsilon + \varepsilon\otimes l(f))\,(\mathbf{a}\otimes\mathbf{b}).$$

Similar arguments work for r. □

Corollary 2.5. *$L = l(B_l^\circ)$ and $R = r(B_r^\circ)$ are subcoalgebras of $(\mathrm{T}(B\otimes C))^\circ$.*

Remark 2.6. *Note that, in order to construct m_l and m_r, we do not need C to be a coalgebra. More precisely, if V is a vector space, $f\colon V\to k$ a nonzero linear functional and B an augmented algebra, then $m_l, m_r\colon (B^+\otimes V)\otimes(B^+\otimes V)\to(B^+\otimes V)$, given by $m_l((a\otimes v)\otimes(b\otimes w)) = ab\otimes vf(w)$ and $m_r((a\otimes v)\otimes(b\otimes w)) = ab\otimes f(v)w$, induce algebra structures B_l and B_r on $k\oplus(B^+\otimes V)$ with unit $(0,1)$, which are quotients of $T(B\otimes V)$. This gives rise to subcoalgebras L and R of $T(B\otimes V)^\circ$ as above.*

Observe that if $(B^+)^2\neq 0$ and $C\neq k$, then $B_l\neq B_r$ and neither B_l nor B_r is commutative. We claim that the composite linear maps

$$\theta_l\colon B_l^\circ\otimes B\overset{l\otimes 1}{\to} T(B\otimes C)^\circ\otimes B\overset{p\otimes 1}{\to}\mathrm{Hom}_k(B,C^*)\otimes B\overset{\mathrm{ev}}{\to} C^*$$

and

$$\theta_r\colon B_r^\circ\otimes B\overset{r\otimes 1}{\to} T(B\otimes C)^\circ\otimes B\overset{p\otimes 1}{\to}\mathrm{Hom}_k(B,C^*)\otimes B\overset{\mathrm{ev}}{\to} C^*$$

are measurings.

Proposition 2.7. *The maps $\theta_l\colon B_l^\circ\otimes B\to C^*$ and $\theta_r\colon B_r^\circ\otimes B\to C^*$, given by composing $\mathrm{ev}(p\otimes 1)\colon (\mathrm{T}(B\otimes C))^\circ\otimes B\to C^*$ with $l=\Psi_L^\circ$ and $r=\Psi_r^\circ$, respectively, are measurings. In particular, $L=l(B_l^\circ)$ and $R=r(B_r^\circ)$ are subcoalgebras of the universal measuring coalgebra $M(B,C^*)$*

Proof. To prove that θ_l is a measuring it suffices to show that $\theta_l(f\otimes ab) = \theta_l(f_1\otimes a)\theta_l(f_2\otimes b)$. Using the explicit form of $\theta_l\colon B_l^\circ\otimes B\to C^*$ given by

$$\theta_l(f\otimes b)(c) = \mathrm{ev}(p\Psi_l^*\otimes 1)(f\otimes b)(c) = p\Psi_l^*(f)(b)(c)$$
$$= \Psi_l^*(f)(b\otimes c) = f\Psi_l(b\otimes c) = f(\bar{b}\otimes c, \varepsilon(b)\varepsilon(c))$$

as well as the algebra structure of B_l and the coalgebra structure of B_l° one obtains

$$\begin{aligned}
&[\theta_l(f_1 \otimes a)\theta_l(f_2 \otimes b)](c) \\
&\quad = m_{C^*}(\theta_l(f_1 \otimes a) \otimes \theta_l(f_2 \otimes b))(c) = \theta_l(f_1 \otimes a)(c_1)\theta_l(f_2 \otimes b)(c_2) \\
&\quad = f_1(\bar{a} \otimes c_1, \varepsilon(a)\varepsilon(c_1))f_2(\bar{b} \otimes c_2, \varepsilon(b)\varepsilon(c_2)) \\
&\quad = f((\bar{a} \otimes c_1, \varepsilon(a)\varepsilon(c_1))(\bar{b} \otimes c_2, \varepsilon(b)\varepsilon(c_2)) \\
&\quad = f(\bar{a}\bar{b} \otimes c + \bar{a}\varepsilon(b) \otimes c + \varepsilon(a)\bar{b} \otimes c, \varepsilon(ab)\varepsilon(c)) \\
&\quad = f(\bar{ab} \otimes c, \varepsilon(ab)\varepsilon(c)) = \theta_l(f \otimes ab)(c)
\end{aligned}$$

as required, since $\bar{ab} = ab - \varepsilon(ab) = \bar{a}\bar{b} + \bar{a}\varepsilon(b) + \varepsilon(a)\bar{b}$. A similar computation shows that $\theta_r \colon B_r^\circ \otimes B \to C^*$ is a measuring as well. It follows that $L = l(B_l^\circ)$ and $R = r(B_r^\circ)$ are subcoalgebras of $M(B, C^*)$, since $M(B, C^*)$ is the maximal subcoalgebra of $T(B \otimes C)^\circ$ measuring B to C^*. $\square$

Theorem 2.8. *If the augmented algebra B has a finite dimensional quotient B' such that $(B'^+)^2 \neq 0$ and C has dimension at least 2, then the universal measuring coalgebra $M(B, C^*)$ is not cocommutative and the map*

$$p \colon M(B, C^*) \to \mathrm{Hom}_k(B, C^*),$$

is not injective.

Proof. By the construction of B_l and B_r and by the nature of the finite dual coalgebra of an algebra as the direct limit of the dual coalgebras of the finite dimensional quotient algebras we may assume that B and C are finite dimensional. If $(B^+)^2 \neq 0$ and C is at least 2-dimensional, then neither of the algebras B_l nor B_r is commutative, hence neither the subcoalgebra $L = l(B_l^\circ)$ nor $R = r(B_r^\circ)$ of $M(B, C^*)$ is cocommutative.

Moreover, from the explicit expressions for $l(f)$ and $r(f)$ in $T(B \otimes C)^*$ it follows that $l(f) \neq r(f)$ for some $f \in ((B^+ \otimes C) \oplus k)^*$. However, since $l(f)$ and $r(f)$ are in $M(B, C^*)$, and $pl(f) = \psi^*(f) = pr(f)$ for all $f \in ((B^+ \otimes C) \oplus k)^*$, we see that $p \colon M(B, C^*) \to \mathrm{Hom}_k(B, C^*)$ is not injective. $\square$

Remark 2.9. *Observe that if B is a bialgebra over a field k of characteristic not 2 then $(B^+)^2 = 0$ if and only if $B^+ = 0$, i.e. if and only if $B = k$. This is because whenever $b \in B^+$ then $\Delta(b) = b \otimes 1 + 1 \otimes b + \sum b_i \otimes b_i'$ with $b_i, b_i' \in B^+$, and thus if $(B^+)^2 = 0$ then $0 = \Delta(b^2) = 2b \otimes b$. Also note that in characteristic 2, $B = kC_2$ and $B = k[x]](x^2)$ are the only non-trivial examples of Hopf algebras with $(B^+)^2 = 0$. Here is why. If $\dim B^+ > 1$ let x and y be inearly independent elements of B^+. Then $0 = \Delta(xy) = x \otimes y + y \otimes x$, since $(B^+)^2 = 0$, in contradiction to the linear independence of $x \otimes y$ and $y \otimes x$ in $B \otimes B$. Thus $\dim B^+ \leq 1$. If $\dim B^+ = 1$ and $x \neq 0$ is an element of B^+ then $\Delta(x) = x \otimes 1 + 1 \otimes x + ax \otimes x$ for some scalar $a \in k$. If $a = 0$ then*

$B = k[x]/(x^2)$. If $a \neq 0$ then $B = kC_2$, where $t = 1 + ax$ is the generator of C_2.

3. Examples and Applications

Aside from the finite dimensional case, the conditions of Theorem 2.8 are also satisfied for proper augmented algebras with the square of the augmentation ideal not zero. We say that the algebra B is proper if it satisfies the following equivalent conditions:
- B is separated in its cofinite ideal topology;
- The intersection of all cofinite ideals of B is zero;
- The canonical map $B \to \varprojlim B/J$, where J runs through the cofinite ideals, is injective;
- For every non-zero $b \in B$ there is a cofinite ideal J in B which excludes b.

Lemma 3.1. *If B is a proper augmented algebra such that $(B^+)^2 \neq 0$ then B has a finite dimensional augmented quotient with the same property.*

The following are examples of proper augmented algebra:
- $B = C^*$, for every coaugmented coalgebra;
- $B = H^*$ and $B = H^\circ$ for every bialgebra H;
- Every commutative Noetherian augmented algebra ([Sw], Section 6.1);
- The universal envelope of a proper, in particular of a finite dimensional, Lie algebra in characteristic zero, by a result of Harish-Chandra [H-C] (for a Hopf algebraic proof see [M]).

Example 3.2. *Here is the explicit description of L and R for a simple choice of B and C^*. Let $B = kC_2$ and $C = (kG)^*$, where $C_2 = \langle t \rangle$ is the cyclic group of order two and G is a finite group. If $x = \frac{1-t}{2}$, then with the basis of $B^+ \otimes C$ consisting of $\tilde{p_g} := x \otimes p_g$ the multiplication in $B_l = k \oplus (B^+ \otimes C) = k\{1, \tilde{p_g} | g \in G\}$ is determined by $\tilde{p_g}\tilde{p_h} = p_h(e)\tilde{p_g}$ (here e denotes the unit in G and $p_g(h) = \delta_{g,h}$). Hence the structure of the coalgebra $L = B_l^* = k\{u, \tilde{g} | g \in G\}$ is given by*

$$\begin{aligned} &\varepsilon(u) = 1, \Delta(u) = u \otimes u \\ &\varepsilon(\tilde{g}) = 0, \Delta(\tilde{g}) = u \otimes \tilde{g} + \tilde{g} \otimes u + \tilde{g} \otimes \tilde{e}. \end{aligned}$$

The embedding $l \colon L \to B(B, C^*) \subseteq M(B \otimes C)^\circ$ is explicitely given as follows. If $\mathbf{a} = (b^1 \otimes c^1) \otimes \dots (b^n \otimes c^n) \in T(B \otimes C)$, where $b^i \in \{1, x\}$ and $c^j \in \{p_g | g \in G\}$, then

$$l(u)(\mathbf{a}) = \varepsilon(\mathbf{a}) = \begin{cases} 1; \mathbf{a} = (1 \otimes p_e) \otimes \dots \otimes (1 \otimes p_e) \\ 0; \textit{otherwise} \end{cases}$$

and

$$l(\tilde{g})(\mathbf{a}) = \begin{cases} 1; \mathbf{a} = (1 \otimes p_e) \otimes \dots \otimes (1 \otimes p_e) \otimes (x \otimes p_g) \otimes (b^i \otimes p_e) \otimes \dots \otimes (b^n \otimes p_e) \\ 0; \textit{otherwise} \end{cases}$$

Note that $B_r = B_l^{op}$ *and* $R = L^{co-op}$.

Remark 3.3. *If* B *is commutative then* $B_r = B_l^{op}$ *and* $R = L^{co-op}$. *If* $(B^+)^2 = 0$ *then* B_r *and* B_l *are equal and commutative, so that* R *and* L *are equal and cocommutative. By Remark 2.9 this last situation cannot occur for any bialgebra* $B \neq k$ *in characteristic different from 2.*

If B is an augmented algebra and A is a commutative algebra then

$$\mathrm{Der}_{\eta_A \varepsilon_B}(B, A) \cong \mathrm{Vect}(B^+/(B^+)^2, A)$$

as vector spaces. Although this vector space is of course in general not zero, even if B is finite dimensional, it is shown below that it is zero whenever B is a finite dimensional Hopf algebra over a field of characteristic zero.

Proposition 3.4. *Let* B *be a Hopf algebra and* A *a commutative algebra over a field of characteristic zero. If* $B^+ \neq (B^+)^2$ *then* $B(B, A)$ *is infinite dimensional.*

Proof. Recall that $\mathrm{Coalg}(k, B(B, A)) \cong \mathrm{Alg}(B, A)$ is a group under convolution with identity $\eta_A \varepsilon_B \colon B \to k \to A$, which also represents the identity of the Hopf algebra $B(B, A)$. Moreover, $\mathrm{Coalg}((k[x]/(x^2))^*, A) \cong \{(f, g) | f \in \mathrm{Alg}(B, A), g \in \mathrm{Der}_f(B, A)\}$, where $(k[x]/(x^2))^* \cong k \oplus ky$ is the infinitesimal coalgebra with $\Delta(y) = y \otimes 1 + 1 \otimes y$. In particular, if k has characteristic zero, then $\mathrm{Der}_{\eta_A \varepsilon_B}(B, A) \cong \mathrm{Vect}(B^+/(B^+)^2, A)$, is the Lie algebra of primitives of $B(B, A)$, on which the group of points $\mathrm{Alg}(B, A)$ acts by conjugation. The universal envelope $U\,\mathrm{Der}_{\eta_A \varepsilon_B}(B, A)$ is a Hopf subalgebra of $B(B, A)$, more precisely

$$U\,\mathrm{Der}_{\eta_A \varepsilon_B}(B, A) \rtimes k\,\mathrm{Alg}(B, A) \subseteq B(B, A).$$

It follows that $B(B, A)$ is infinite dimensional whenever $B^+/(B^+)^2 \neq 0$. If, in addition, k is algebraically closed then

$$U\,\mathrm{Der}_{\eta_A \varepsilon_B}(B, A) \rtimes k\,\mathrm{Alg}(B, A) \cong B_c(B, A),$$

the cocommutative part of $B(B, A)$, as noted in [GM]. □

Corollary 3.5. *If* B *is a finite dimensional Hopf algebra over a field of characteristic zero, then* $B^+ = (B^+)^2$ *and* $\mathrm{Der}_{\eta_A \varepsilon_B}(B, A) = 0$ *for every commutative algebra* A.

Proof. If B is finite dimensional then $B^* = B(B, k)$ is finite dimensional as well. Then, by the Proposition above, $\mathrm{Vect}(B^+/(B^+)^2, k) \cong \mathrm{Der}_{\varepsilon_B}(B, k) = 0$, so that $B^+ = (B^+)^2$ and hence $\mathrm{Der}_{\eta_A \varepsilon_B}(B, A) \cong \mathrm{Vect}(B^+/(B^+)^2, A) = 0$ for every algebra A. □

Corollary 3.6. *Let* B *be a finite dimensional augmented algebra over a field of characteristic zero. If* $B^+ \neq (B^+)^2$ *then the augmented algebra structure on* B *can not be completed to a Hopf algebra structure.*

4. Comparison between $B(S,A)\otimes B(T,A)$ and $B(S\otimes T,A)$

If T and S are bialgebras and A is a commutative algebra then we define $\alpha = \alpha_{S,T,A}\colon B(S,A)\otimes B(T,A)\to B(S\otimes T,A)$ to be the unique bialgebra map for which $\theta_{S\otimes T}(\alpha,1\otimes 1) = \mathrm{m}_A(\theta_S\otimes\theta_T)\sigma_{2,3}$ [GM]. Here $\theta_X\colon B(X,A)\otimes X\to A$ denotes the universal bimeasuring. Note that α defines a natural transformation from $B(-,A)\otimes B(-,A)\colon \mathrm{Bialg}\times\mathrm{Bialg}\to\mathrm{Bialg}$ to $B(-\otimes -,A)\colon \mathrm{Bialg}\times\mathrm{Bialg}\to\mathrm{Bialg}$. Moreover α is also natural in the A-variable. More precisely, if $f\colon S'\to S$ and $g\colon T'\to T$ are morphisms of bialgebras and $h\colon A\to A'$ is a morphism of commutative algebras, then the diagram

$$\begin{array}{ccc} B(S,A)\otimes B(T,A) & \xrightarrow{\alpha_{S,T,A}} & B(S\otimes T,A) \\ {\scriptstyle B(f,h)\otimes B(g,h)}\Big\downarrow & & {\scriptstyle B(f\otimes g,h)}\Big\downarrow \\ B(S',A')\otimes B(T',A') & \xrightarrow{\alpha_{S',T',A'}} & B(S'\otimes T',A') \end{array}$$

commutes. Indeed

$$\begin{array}{rl} & \theta_{S'\otimes T',A'}(B(f\otimes g,h)\alpha_{S,T,A},1\otimes 1) = h\theta_{S\otimes T,A}(\alpha_{S\otimes T,A},f\otimes g) \\ = & h\theta_{S\otimes T,A}(\alpha_{S\otimes T,A},1\otimes 1)(1\otimes 1\otimes f\otimes g) \\ = & hm_A(\theta_{S,A}\otimes\theta_{T,A})\sigma_{2,3}(1\otimes 1\otimes f\otimes g) \\ = & m_{A'}(h\otimes h)(\theta_{S,A}\otimes\theta_{T,A})(1\otimes f\otimes 1\otimes g)\sigma_{2,3} \\ = & m_{A'}(\theta_{S',A'}(B(f,h),1)\otimes\theta_{T',A'}(B(g,h),1))\sigma_{2,3} \\ = & m_{A'}(\theta_{S',A'}\otimes\theta_{T',A'})\sigma_{2,3}(B(f,h)\otimes B(g,h)\otimes 1\otimes 1) = \\ = & \theta_{S'\otimes T',A'}(\alpha_{S',T',A'},1\otimes 1)(B(f,h)\otimes B(g,h)\otimes 1\otimes 1) \\ = & \theta_{S'\otimes T',A'}(\alpha_{S',T',A'}(B(f,h)\otimes B(g,h)),1\otimes 1). \end{array}$$

Proposition 4.1. *If $F = B(-,A)\colon \mathrm{Bialg}\to\mathrm{Bialg}$, then the natural transformation $\alpha\colon (F\otimes F)\to F(1\otimes 1)$ satisfies $\alpha(1\otimes\alpha) = \alpha(\alpha\otimes 1)$, i.e.*

$$\begin{array}{ccc} F(S)\otimes F(T)\otimes F(U) & \xrightarrow{\alpha_{S,T}\otimes 1} & F(S\otimes T)\otimes F(U) \\ {\scriptstyle 1\otimes\alpha_{T,U}}\Big\downarrow & & {\scriptstyle \alpha_{S\otimes T,U}}\Big\downarrow \\ F(S)\otimes F(T\otimes U) & \xrightarrow{\alpha_{S,T\otimes U}} & F(S\otimes T\otimes U). \end{array}$$

commutes for all bialgebras S, T and U.

Proof. A straightforward computation shows that $\theta(\alpha(\alpha\otimes 1),1)(f\otimes g\otimes h, s\otimes t\otimes u) = \theta(f,s)\theta(g,t)\theta(h,u) = \theta(\alpha(1\otimes\alpha),1)(f\otimes g\otimes h, s\otimes t\otimes u)$. □

We will now show that if $A = C^*$, where $C\neq k$ is a cocommutative coalgebra, then there is no natural transformation $\beta\colon F(1\otimes 1)\to F\otimes F$ satisfying $(\beta\otimes 1)\beta = (1\otimes\beta)\beta$. In particular, α_A can not be invertible (and hence for some bialgebras S and T, the homomorphism $\alpha_{S,T,A}$ is not surjective), since in that case α_A^{-1} would be such a transformation.

Theorem 4.2. *If $F\colon \mathrm{Alg} \to \mathrm{Vect}$ is a contravariant functor and $\beta\colon F(1\otimes 1) \to F\otimes F$ a natural transformation satisfying $(\beta\otimes 1)\beta = (1\otimes\beta)\beta$, more precisely*

$$\begin{tikzcd}
F(S\otimes T\otimes U) \arrow[r, "\beta_{S\otimes T,U}"] \arrow[d, "\beta_{S,T\otimes U}"'] & F(S\otimes T)\otimes F(U) \arrow[d, "\beta_{S,T}\otimes 1"'] \\
F(S)\otimes F(T\otimes U) \arrow[r, "1\otimes\beta_{T,U}"] & F(S)\otimes F(T)\otimes F(U)
\end{tikzcd}$$

commutes for all S, T, and U, then for every commutative algebra H, the data

$$\begin{aligned} \Delta = \beta F(m) \quad &: \quad F(H) \to F(H)\otimes F(H) \\ \varepsilon = F(\eta) \quad &: \quad F(H) \to F(k) = k \end{aligned}$$

defines a coalgebra structure on $F(H)$.

Proof. The diagrams

$$\begin{tikzcd}
F(H) \arrow[r, "F(m)"] \arrow[d, "F(m)"'] & F(H\otimes H) \arrow[r, "\beta"] \arrow[d, "F(m\otimes 1)"'] & F(H)\otimes F(H) \arrow[d, "F(m)\otimes 1"'] \\
F(H\otimes H) \arrow[r, "F(1\otimes m)"] \arrow[d, "\beta"'] & F(H\otimes H\otimes H) \arrow[r, "\beta"] \arrow[d, "\beta"'] & F(H\otimes H)\otimes F(H) \arrow[d, "\beta\otimes 1"'] \\
F(H)\otimes F(H) \arrow[r, "1\otimes F(m)"] & F(H)\otimes F(H\otimes H) \arrow[r, "1\otimes\beta"] & F(H)\otimes F(H)\otimes F(H)
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(H) \arrow[r, "F(m)"] \arrow[d, equal] & F(H\otimes H) \arrow[r, "\beta"] \arrow[d, "F(1\otimes\eta)"'] & F(H)\otimes F(H) \arrow[d, "1\otimes F(\eta)"'] \\
F(H) \arrow[r, "\simeq"] & F(H\otimes k) \arrow[r, "\beta"] & F(H)\otimes k
\end{tikzcd}$$

commute by the naturality of β and since $(\beta\otimes 1)\beta = (1\otimes\beta)\beta$, which means that $F(H)$ is a coalgebra with comultiplication $\Delta = \beta F(m)$ and counit $\varepsilon = F(\eta)$. □

Remark 4.3. *Note that, in the theorem above, it is sufficient to require the condition $(\beta_{S,T}\otimes 1)\beta_{S\otimes T,U} = (1\otimes\beta_{T,U})\beta_{S,T\otimes U}$ only for the case when $S = T = U$.*

Theorem 4.4. *If $F\colon \mathrm{Alg} \to \mathrm{Coalg}$ is a contravariant functor and $\beta\colon F(1\otimes 1) \to F\otimes F$ a natural transformation satisfying the condition $(\beta\otimes 1)\beta = (1\otimes\beta)\beta$, then the coalgebra $F(A)$ is cocommutative whenever A is commutative.*

Proof. If A is a commutative algebra, then by Theorem 4.2, the data

$$\begin{aligned} \Delta' = \beta F(m) \quad &: \quad F(A) \to F(A)\otimes F(A) \\ \varepsilon' = F(\eta) \quad &: \quad F(A) \to F(k) = k \end{aligned}$$

defines a coalgebra structure on $F(A)$. Also note that, by construction, Δ' and ε' are coalgebra maps. Now apply a coalgebra version of the Eckman-Hilton Theorem [EH], which says that a comonoid in the category of coalgebras is just a cocommutative coalgebra. More precisely, if (C, Δ, ε) and $(C, \Delta', \varepsilon')$ are coalgebra structures on C such that Δ' and ε' are coalgebra maps then $\varepsilon' = \varepsilon$, $\Delta' = \Delta$ and Δ is cocommutative. If Δ' and ε' are coalgebra maps then in particular $\varepsilon' = \varepsilon$ and $(\Delta' \otimes \Delta')\Delta = (1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)\Delta'$, so that

$$\Delta' = (1 \otimes \varepsilon \otimes \varepsilon \otimes 1)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)\Delta' = (1 \otimes \varepsilon \otimes \varepsilon \otimes 1)(\Delta' \otimes \Delta')\Delta = \Delta$$

and

$$\tau\Delta' = (\varepsilon \otimes 1 \otimes 1 \otimes \varepsilon)(1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)\Delta' = (\varepsilon \otimes 1 \otimes 1 \otimes \varepsilon)(\Delta' \otimes \Delta')\Delta = \Delta.$$

□

Corollary 4.5. *If $A = C^*$, where $C \neq k$ is a cocommutative coalgebra and $F = B(-, A)$, then there is no natural transformation $\beta\colon F(1 \otimes 1) \to F \otimes F$ satisfying $(\beta \otimes 1)\beta = (1 \otimes \beta)\beta$. In particular $\alpha_A\colon F \otimes F \to F(1 \otimes 1)$ is not invertible.*

Remark 4.6. *Careful reading of the proofs above shows that if $C \neq k$ is a cocommutative coalgebra and H is a finite dimensional commutative bialgebra such that $(H^+)^2 \neq 0$, then at least one of the maps $\alpha_{H,H}\colon B(H, C^*) \otimes B(H, C^*) \to B(H \otimes H, C^*)$ and $\alpha_{H,H\otimes H}\colon B(H \otimes H \otimes H, C^*) \to B(H \otimes H, C^*) \otimes B(H, C^*)$ is not surjective.*

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