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Fuzzification of Rational and Recognizable Sets*

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Abstract. In this paper we present a different framework for the study of fuzzy finite machines and their fuzzy languages. Unlike the previous work on fuzzy languages, characterized by fuzzification at the level of their acceptors/generators, here we follow a top-down approach by starting our fuzzification with more abstract entities: monoids and particular families in monoids. Moreover, we replace the unit interval (in fact, a finite subset of the unit interval) as support for fuzzy values with the more general structure of a lattice. We have found that completely distributive complete lattices allow the fuzzification at the highest level, that of recognizable and rational sets. Quite surprisingly, the fuzzification process has not followed thoroughly the classical (crisp) theory. Unlike the case of rational sets, the fuzzification of recognizable sets has revealed a few remarkable exceptions from the crisp theory: for example the difficulty of proving closure properties with respect to complement, meet and inverse morphisms. Nevertheless, we succeeded to prove the McKnight and Kleene theorems for fuzzy sets by making the link between fuzzy rational/recognizable sets on the one hand and fuzzy regular languages and FT-NFA languages (languages defined by NFA with fuzzy transitions) on the other. Finally, we have drawn the attention to fuzzy rational transductions, which have not been studied extensively and which bring in a strong note of applicability.

Keywords: Fuzzy Transducer, Rational Set, Recognizable Set, Lattice

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1. Introduction

In the mid 60's, L. A. Zadeh ([39]) established the basis of fuzzy set theory, a field which has not ceased to intrigue mathematicians and computer scientists ever since. Soon after its foundation, the theory was extended to formal languages; first indirectly, by the work of J. A. Goguen ([10]) who introduced a general "Principle of Fuzzification", and later explicitly, by Zadeh and E. T. Lee, in [14]. Since then, there have been many attempts to define devices which either accept or generate elements/words of fuzzy languages. Some of them were "machine oriented", in the sense that they were aiming at simply adding fuzzy values to either transitions or states of classical automata (see for example FT-NFA as defined in [17] or Moore fuzzy automata as in [34]). Some others used heavier algebraic concepts, such as sigma algebras, steady or complete semirings, semimodules, etc. (see, for example, [23]). These two trends follow two major approaches in automata theory: a machine oriented and an algebraic approach. However, there exist other formalisms worth considering. Arguably the most remarkable one is the formalism proposed in mid 60's by S. Eilenberg ([8]), aiming at the study of two particular families of subsets in arbitrary monoids: sets defined by rational operations and sets defined by monoid actions. To our knowledge, there has been no attempt to study the fuzzification of rational and recognizable sets in arbitrary monoids. To some extent, our approach follows a line similar to the development of the general theory of formal power series on noncommutative variables with coefficients in a semiring. We believe that this method leads, in a natural way, to the concept of fuzzy machine and, in particular, to fuzzy finite automata and transducers. This paper addresses the need for a "fuzzy" theory which follows the classical theory of rational and recognizable sets in arbitrary monoids, using fuzzy set theoretic concepts. In this paper we are proving that this endeavor is possible, and that it leads to a formalism which does not always follow the "crisp" counterpart (i.e., it is not a trivial rewriting of what has already been done). Moreover, we prove (or ensure) that restricting our new formalism to special cases leads to existing results on fuzzy machines.

Why fuzzy machines? It is well known that both stochastic and fuzzy automata can be viewed as particular types of weighted automata, i.e., automata with transitions taking values in a semiring. From this point of view, all results and algorithms concerning weighted automata can readily be employed when dealing with stochastic and fuzzy automata. However, it is in our perception that the general theory of automata over semirings has a very broad spectrum and it sometimes misses particularities which would make it extremely applicable. For example, in the context of semirings, the notion of partial order is a second class citizen. In contrast, a lattice is built upon this very notion, and consequently, fuzzy automata are versatile tools for classification of words according to their associated value. In addition, arbitrary semirings can seldom be used for weighted automata. Indeed, one must always ensure that the semiring allows infinite sums (hence the use of complete semirings), or that the associated automata trigger sums involving locally finite sets (see, for example, [9, p. 127]). In contrast, since a complete lattice is a complete semiring, this difficulty is avoided for fuzzy automata in a natural manner. When comparing fuzzy and stochastic (probabilistic) automata, one observes that they have different interpretations and serve different purposes. Moreover, it has been noticed that the probabilities computed by stochastic automata decrease predictably, as a function of the length of the input (this consequence is more pronounced in the deterministic case, where there is at most one successful computation for any give input). If the input is long enough (inevitable for infinite languages), the computed probabilities become too small for any practical purpose (they go beyond the precision of any particular machine). In contrast, fuzzy automata do not have this drawback. It is our belief that particularizing the general theory of automata over semirings to automata over lattices (i.e., fuzzy automata) brings in specific and applicable results which make fuzzy machines rather interesting, from an algebraic, as well as a practical, point of view.

The need of defining fuzzy automata at a more abstract level and in a more general framework has been felt before, and a prolific research activity in this direction has preceded our work. Recent studies worth mentioning are: [21] concerned with the connection between the category of fuzzy automata and that of chains of nondeterministic automata, [22] which introduces generalized fuzzy automata over complete residuated lattices, [12] in which are studied fuzzy recognizers and recognizable sets (notions having little in common with the concept of recognizability defined in this paper), [1] where types of fuzzy languages and deterministic fuzzy automata are defined, [15] dealing with fuzzy machines over lattice-ordered monoids (in contrast with the present paper where the lattice and monoid structures are separated), or [16, 11, 13] where operations with fuzzy machines (such as coverings, cascades and wreath products) were studied, to mention just a few sources. These efforts have been fuelled and inspired by early developments of the late 60's and the 70's when fuzzy systems, fuzzy automata, and their applications reached a peak of their popularity. Examples of pioneering work in the field are: [37] and later [38, 28] where was first discussed the notion of fuzzy automaton with a nonfuzzy initial state and nonfuzzy inputs, in rigorous mathematical terms, [19] where a fuzzy automaton with a fuzzy initial state was introduced for the first time in a formal languages context, [32, 28, 29] where various types of valued sequential machines, including fuzzy automata over the unit interval, were defined and analyzed, [30] where one can find initial attempts to reduce fuzzy automata by means of a max-min algebra of real numbers, and where various criteria of reducibility and minimality are provided (see also the newer [24], in a broader context), etc.. In the past, there have been many attempts to find an ideal algebraic structure as a support for fuzzy automata, such as: ordered semigroups ([36], [31]), or ordered (or not) semirings ([35], and the newer [23]), boolean lattices ([20]), normalized convex fuzzy sets in the unit interval ([18]), and the list goes on. The legacy is indeed impressive, coming from many research fields and following different approaches; and consequently, a certain degree of decentralization has occurred. For example, one can find in the present literature several ways of defining a same type of fuzzy automaton, definitions which are not equivalent despite their identical terminology. More alarmingly, there exist results in different sources which are valid within their own context, however they become contradictory when placed side-by-side in an effort to consolidate the theory. The present paper does not claim to have solved this unification problems, which may very well be insurmountable; however it is a step toward its resolution by exposing a rigorous framework built incrementally from very basic concepts and gradually covering aspects most relevant to the topic, from a formal language point of view. It aims at a compromise formula balancing the complexity of the framework and its formalism power. There are several fundamental differences between our work and the previously-mentioned endeavors, many originating in our novel approach of combining basic concepts of fuzzy set theory with the abstract notion of computable set in arbitrary monoids as reflected in the well-known duality "rational set – recognizable set". In doing this, we took a special care to preserve the nature of both worlds, hoping to combine (by fuzzification) the notions without diluting them into each other. Consequently, we hope to have accomplished a fuzzification as easily accessible as possible to a formal language theorist (for a more set-theoretic treatment of the topic, which goes beyond the formal language context consult [7]).

It has become apparent by now that one of the goals of this paper is to revive the topic of fuzzy machines, and bring to attention a fresh point of view. Some of the results presented here can arguably be viewed as derived from the theory of formal power series over arbitrary semirings with noncommuting variables, in as much as fuzzy automata can be viewed as weighted automata. Some other results, with a fuzzy set theoretic specific, or those concerning arbitrary monoids, have no match in the context of formal power series (formal power series were first studied in [33, 9] and further developed in [27] and [4], to mention just a few). Whether a fuzzy subset of a monoid can be considered a formal power series is a matter of debate, since formal power series require noncommuting variables most of the time and in most cases the variables are elements of finitely generated free monoids. Their differences, subtle or not, lead to the state of having two fully developed and independent theories: a fuzzy set theory and a formal power series theory. Moreover, unlike most of the work on formal power series, focussed mainly on either free monoids or direct products of free monoids, in this paper we focus on arbitrary monoids. We also rebuild some classical definitions; for example, our notion of fuzzy recognizable set is not in line with the notion of recognizable K-subset (as in [9]) or recognizable series (as in [27]). Furthermore, using completely distributive complete lattices as proper algebraic structure for monoid fuzzification, we bring forward properties of fuzzy sets specifically derived from these structures. Finally, our formalism appears to be more friendly than that used in formal power series, the only algebraic structures used here being that of a lattice and a monoid.

The paper is structured as follows. In Section 2 we introduce basic concepts of lattice theory, with a focus on completely distributive complete lattices. We have found that these lattices are the proper algebraic objects for monoid fuzzification. It is important to note that indeed, this is a generalization of the previously used unit interval, since in fact, only a finite chain of the unit interval was used in past work. In Section 3 we proceed with the fuzzification of monoids, as support for fuzzy rational and recognizable sets. It is in this section where we define operations with fuzzy sets and study their properties. We continue in Section 4 and 5 with our main goal, that of fuzzifying the families of rational and recognizable sets. In these sections we address their closure properties, we define their corresponding abstract machines and we give links to their "crisp" counterpart. In Section 6 we make the connection between fuzzy rational and recognizable sets, using particular types of monoids and lattices. As proof of soundness, in Section 7 we apply the results developed in the previous sections to fuzzy relations in general and transductions in particular, with potential applications. A simple example of a fuzzy finite transducer can also be found here. Finally, in Section 8 we conclude our work and outline further directions.

2. Notions of Lattice Theory

In this section we present a few notions of lattice theory, with a focus on their transfinite properties. Our main purpose is to reach the concept of completely distributive complete lattice, used throughout this paper.

Definition 2.1. By a `partial ordering` of a set L we understand a relation (viewed as a subset of a Cartesian product) " $\leq$ " $\subseteq L \times L$ which is `reflexive` ($\forall x : \; x \leq x$), `antisymmetric` ($\forall x, y : x \leq y, y \leq x \Rightarrow x = y$), and `transitive` ($\forall x, y, z : \; x \leq y, y \leq z \Rightarrow x \leq z$). The order is called "partial" since there may exist incomparable elements in L.

Definition 2.2. Let L be a set and $\leq$ a partial ordering of L.

1. L is a `lattice` if all nonempty, finite subsets of L have a `least upper bound` and a `greatest lower bound` with respect to $\leq$.

2. L is a `complete lattice` if the previous property holds for arbitrary nonempty subsets of L.

We denote by $\vee$ (join) and $\wedge$ (meet) the operators which give the least upper bound and the greatest lower bound of a set (since $\leq$ is a partial order, the upper and lower bounds of a subset of L may not belong to the subset). If L is a complete lattice, we denote $0 = \bigwedge L$ and $1 = \bigvee L$; and is useful to define (by convention) "empty meets and joins", as $\bigvee \emptyset = \bigwedge \emptyset = \bigwedge L$. Observe that any finite lattice is complete.

For a lattice L we use the notation $(L, \leq, \vee, \wedge, 0, 1)$, where some of the operators may be omitted if they are given by the context or they are undefined. The following is an important property of complete lattices:

Birkhoff's law. ([5, p. 53]) In a complete lattice $(L, \leq, \vee, \wedge)$ the following self-dual law holds:

$$\bigvee_{\psi \in \Psi} \Big(\bigvee_{\varphi \in \phi_\psi} a_\varphi \Big) = \bigvee_{\varphi \in \Phi} a_\varphi \ ,$$

where Ψ is an index set, $\{\phi_\psi\}_{\psi \in \Psi}$ is a family of index sets (indexed by Ψ), $\Phi = \bigcup_{\psi \in \Psi} \phi_\psi$ and a_φ are elements of L.

Consequently, all complete lattices obey to the following generalized (transfinite) laws:

1. (`generalized commutativity`) Any nonempty subset S of L has a meet $\vee S$ and a join $\wedge S$ depending only on S.

2. (`generalized associativity`) If $\{S_\phi\}_{\phi \in \Phi}$ is a family of nonempty subsets of L indexed by Φ and if we denote $S = \bigcup_{\phi \in \Phi} S_\phi$ then

$$\bigvee_{\phi \in \Phi} (\bigvee S_\phi) = \bigvee S, \ \ \bigwedge_{\phi \in \Phi} (\bigwedge S_\phi) = \bigwedge S \ .$$

We say that a lattice L is `distributive` if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all $a, b, c \in L$. The property is self-dual, in the sense that if we interchange $\vee$ and $\wedge$, the equality remains true. It is easy to check that any totally ordered lattice (or `chain`) is distributive. In this paper we are interested in a stronger form of distributivity, namely the `generalized (transfinite) distributivity`: arbitrary meets distribute over arbitrary joins and vice versa. The property is formalized as following.

Let $(L, \leq)$ be a complete lattice and $\{S_\phi\}_{\phi \in \Phi}$ be a family of nonempty subsets of L indexed by Φ. Denote $\mathcal{F} = \{f : \Phi \to \bigcup_{\phi \in \Phi} S_\phi / f(\phi) \in S_\phi\}$ the set of all `choice functions` which map each index ϕ into an element of S_ϕ.

Definition 2.3. ([25]) $(L, \leq)$ is `completely distributive` if

$$\bigwedge_{\phi \in \Phi} (\bigvee S_\phi) = \bigvee_{f \in \mathcal{F}} (\bigwedge f(\Phi))$$

Remark 2.1. The property of complete distributivity is self-dual.

Definition 2.4. We say that a lattice L has no 0-divisors if and only if

$$\forall l_1, l_2 \in L : \ \left(\ l_1 \wedge l_2 = 0 \ \right) \Rightarrow \left(\ l_1 = 0 \text{ or } l_2 = 0 \ \right) \ .$$

Remark 2.2. Notice that if the order in L is total, then L has no 0-divisors.

In what will follow we consider only completely distributive complete lattices, abbreviated "c.d.c. lattices". Notice that a finite lattice is a c.d.c. lattice if and only if it is distributive (in the finite sense).

3. Monoid Fuzzification

Let M be an arbitrary set (for now) and $(L, \leq, \vee, \wedge, 0, 1)$ be a lattice.

Definition 3.1. An L-fuzzy set (or fuzzy set, when L is understood) on M is a function $\mu : M \to L$ (see, for example, the definition in [10]).

If we denote by L^M the family of all L-fuzzy sets on M then $(L^M, \leq, \vee, \wedge, \mu_\emptyset, \mu_M)$ has a lattice structure, where:

- $\forall \mu, \nu \in L^M : \ \mu \leq \nu \ \Leftrightarrow \ \forall m \in M$: $\mu(m) \leq \nu(m)$ in L;
- $\forall \mu, \nu \in L^M, \forall m \in M : \ \mu \vee \nu \in L^M, \ (\mu \vee \nu)(m) = \vee\{\mu(m), \nu(m)\}$;
- $\forall \mu, \nu \in L^M, \forall m \in M : \ \mu \wedge \nu \in L^M, \ (\mu \wedge \nu)(m) = \wedge\{\mu(m), \nu(m)\}$;
- $\mu_\emptyset = \{(m, 0)/m \in M\}$ and $\mu_M = \{(m, 1)/m \in M\}$.
  (in this definition we used the extensional representation of a function)

The meets and joins of fuzzy sets can be extended over arbitrary families. Following the convention, in L^M we have $\bigvee \emptyset = \bigwedge \emptyset = \mu_\emptyset$. Furthermore, the laws of L are ported to L^M, thus L^M becomes a c.d.c. lattice if L is c.d.c., a fact which will be assumed from now on.

If $\nu \in L^M$, we denote $supp(\nu) = \{m \in M/\nu(m) \neq 0\}$ to be the support of ν. A singleton in L^M is an element whose support has cardinality 1 (it has exactly one element), and we use the notation

$$\mu_m^l = \{(m, l)\} \cup \{(n, 0)/n \in M, n \neq m\}$$

for the singletons of L^M. We adopt the following nomenclature: $\mu_\emptyset$ is the null fuzzy set and μ_M is the uniform fuzzy set, i.e., the fuzzy set which associates value 1 to all the elements of M.

Let us define the following two important subfamilies of L^M:

1. the family of singletons in L^M, denoted by

$$S(L^M) = \{\mu \in L^M/\mu \text{ is a singleton } \} \ \text{, and}$$

2. the family of L-fuzzy sets with finite support, denoted by:

$$FS(L^M) = \{\nu \in L^M / \mid supp(\nu) \mid < \aleph_0\} \ .$$

Notice that $supp(\mu_\emptyset) = \emptyset$, hence $\mu_\emptyset \in FS(L^M)$. However, $\mu_\emptyset \notin S(L^M)$.

Remark 3.1. $FS(L^M)$ is closed under meets (possibly transfinite) and finite joins. Any element of $FS(L^M)$ is the result of finite (possibly empty) joins of elements of $S(L^M)$.

If we consider M to have the algebraic structure of a monoid, we can enrich the structure of L^M with an operation derived from the monoid operation, as follows.

Let $(M, \cdot, 1_M)$ be a monoid, $(L, \leq, \vee, \wedge, 0, 1)$ be a c.d.c lattice and consider L^M, the lattice of L-fuzzy sets on M. We call $\mu^1_{1_M}$ the `unit fuzzy set`. Let us define an operation "$\cdot$"(multiplication) over L^M as following:

$$\forall \mu, \nu \in L^M : \;\; (\mu \cdot \nu)(m) = \bigvee_{m=u\cdot v} \Big\{ \mu(u) \wedge \nu(v) \Big\} \;. \tag{1}$$

Notice that this operation does not involve empty joins and meets, since any element $m \in M$ accepts at least two decompositions: $m = m \cdot 1_M = 1_M \cdot m$. Furthermore, since L is complete, we have that $\vee$ is defined over arbitrary subsets of L, hence the operation "$\cdot$" is well-defined (despite the fact that m can have an infinite number of decompositions: for example, taking M to be the monoid $(\mathbb{R}, \cdot, 1)$, a real number $m \neq 1$ has an infinite number of non-trivial decompositions, $m = (m^{\frac{1}{k}})^k$, $k \in \mathbb{N}$). Operation (1) is a reflection of what is known in fuzzy set theory as the `extension principle`, introduced in [39] and elaborated upon in [7, p. 36].

Notation. Where there is no source of confusion, the multiplication of either elements of M or fuzzy sets will be represented by juxtaposition (by omitting the dot).

Lemma 3.1. Multiplication is associative and distributes over $\vee$ in L^M.

Proof:
Let $\mu, \nu, \xi \in L^M$. For associativity, we verify that for every $m \in M$ we have

$$\Big((\mu \cdot \nu) \cdot \xi \Big)(m) = \Big(\mu \cdot (\nu \cdot \xi) \Big)(m) = \bigvee_{m=u\cdot v\cdot w} \Big\{ \mu(u) \wedge \nu(v) \wedge \xi(w) \Big\} \;.$$

Indeed, we have

$$(\mu(\nu\xi))(m) = \bigvee\nolimits_{m=uw'} \Big\{ \mu(u) \wedge (\nu\xi)(w') \Big\} =$$

$$= \bigvee\nolimits_{m=uw'} \Big\{ \bigwedge \{ \mu(u), \bigvee\nolimits_{w'=vw} \{ \nu(v) \wedge \xi(w) \} \} \Big\} =$$

(here we invoke the generalized distributivity in L)

$$= \bigvee\nolimits_{m=uw'} \Big\{ \bigvee\nolimits_{w'=vw} \{ \mu(u) \wedge \nu(v) \wedge \xi(w) \} \Big\} =$$

(we invoke generalized associativity)

$$= \bigvee\nolimits_{m=uvw} \Big\{ \mu(u) \wedge \nu(v) \wedge \xi(w) \Big\} \;.$$

On the other hand, we have

$((\mu\nu)\xi)(m) = \bigvee_{m=u'w} \Big\{ (\mu\nu)(u') \wedge \xi(w) \Big\} =$

$= \bigvee_{m=u'v} \Big\{ \Big(\bigvee_{u'=uv} \{\mu(u) \wedge \nu(v)\} \Big) \wedge \xi(w) \Big\} =$

(we invoke yet again the generalized distributivity)

$= \bigvee_{m=u'v} \Big\{ \bigvee_{u'=uv} \{\mu(u) \wedge \nu(v) \wedge \xi(w)\} \Big\} =$

(we invoke generalized associativity)

$= \bigvee_{m=uvw} \Big\{ \mu(u) \wedge \nu(v) \wedge \xi(w) \Big\}$.

We have used the fact that $(m = uvw) \Leftrightarrow (m = u'w$ and $u' = uv)$, given by the associativity in M, and we also used the fact that meets distribute over arbitrary(transfinite) joins in L. The last equality is given by general associativity - as mentioned in-line.

We now prove that multiplication distributes over joins:

$\big(\mu \cdot (\nu \vee \xi)\big)(m) = \bigvee_{m=u\cdot v} \Big\{ \mu(u) \wedge (\nu \vee \xi)(v) \Big\} =$ (by distributivity)

$= \bigvee_{m=u\cdot v} \Big\{ \Big(\mu(u) \wedge \nu(v)\Big) \vee \Big(\mu(u) \wedge \xi(v)\Big) \Big\} =$ (by gen. commutativity)

$= \Big(\bigvee_{m=u\cdot v} \{\mu(u) \wedge \nu(v)\} \Big) \vee \Big(\bigvee_{m=u\cdot v} \{\mu(u) \wedge \xi(v)\} \Big) = \Big((\mu\nu) \vee (\mu\xi) \Big)(m)$.

We have used the law of generalized commutativity in L. Distributivity "to the right" is proven in a similar way. □

By the previous lemma (by distributivity over joins, in particular), L^M becomes a multiplicative lattice. Distributivity over meets does not hold in general. However, we are able to prove the following property.

Lemma 3.2. For any $\mu, \nu, \xi \in L^M$ the following inequality holds:

$$\mu(\nu \wedge \xi) \leq (\mu\nu) \wedge (\mu\xi) \ .$$

Proof:

$\big(\mu \cdot (\nu \wedge \xi)\big)(m) = \bigvee_{m=u\cdot v} \Big\{ \mu(u) \wedge (\nu \wedge \xi)(v) \Big\} =$ (by definition)

$= \bigvee_{m=u\cdot v} \Big\{ \mu(u) \wedge \nu(v) \wedge \xi(v) \Big\} =$ (by idempotence)

$= \bigvee_{m=u\cdot v} \Big\{ \mu(u) \wedge \mu(u) \wedge \nu(v) \wedge \xi(v) \Big\} =$ (by commutativity)

$= \bigvee_{m=u\cdot v} \Big\{ \Big(\mu(u) \wedge \nu(v)\Big) \wedge \Big(\mu(u) \wedge \xi(v)\Big) \Big\} \leq$ (from transfinite distributivity)

$\leq \Big(\bigvee_{m=u\cdot v} \{\mu(u) \wedge \nu(v)\} \Big) \wedge \Big(\bigvee_{m=u\cdot v} \{\mu(u) \wedge \xi(v)\} \Big) = \Big((\mu\nu) \wedge (\mu\xi) \Big)(m)$. □

The inequality derived from transfinite distributivity is similar to the following relations: $(A_1 \wedge B_1) \vee (A_2 \wedge B_2) = (A_1 \vee A_2) \wedge (A_1 \vee B_2) \wedge (B_1 \vee A_2) \wedge (B_1 \vee B_2) \leq (A_1 \vee A_2) \wedge (B_1 \vee B_2)$. Notice that the inequality of the lemma holds also when we multiply "to the right": $(\nu \wedge \xi)\mu \leq (\nu\mu) \wedge (\xi\mu)$.

Corollary 3.1. $\big(S(L^M) \cup \{\mu_\emptyset\}, \cdot, \mu^1_{1_M}\big)$ is a submonoid of $\big(FS(L^M), \cdot, \mu^1_{1_M}\big)$ which is a submonoid of $(L^M, \cdot, \mu^1_{1_M})$.

Proof:
Here we prove only that $FS(L^M)$ is closed under "·", the rest being straightforward. If $\nu, \xi \in FS(L^M)$, then

$$\nu = \mu^{l_1}_{m_1} \vee \cdots \vee \mu^{l_k}_{m_k} \ ,$$
$$\xi = \mu^{t_1}_{n_1} \vee \cdots \vee \mu^{t_r}_{n_r} \ ,$$

for some singletons $\mu^{l_1}_{m_1}, \ldots, \mu^{l_k}_{m_k}, \mu^{t_1}_{n_1}, \ldots, \mu^{t_r}_{n_r}$, and is easy to see that

$$(\nu \cdot \xi)(m) = \bigvee_{m=m_i n_j,} \Big\{\nu(m_i) \wedge \xi(n_j)\Big\} \ ,$$

where the join is for all $1 \leq i \leq k$ and all $1 \leq j \leq r$.

Then, there will be at most $k \times r$ arguments which are mapped by $\nu \cdot \xi$ into non-null values (all possible combination $\{(m_i, n_j)\}_{i,j}$). This proves that $\nu \cdot \xi$ belongs to $FS(L^M)$, hence that $FS(L^M)$ is closed under "·". Notice that in the above proof we can also express $\nu \cdot \xi$ as

$$(\nu \cdot \xi)(m) = \bigvee_{m=m_i n_j} (l_i \wedge t_j) \ .$$

□

Remark 3.2. If L has no 0-divisors then $(S(L^M), \cdot, \mu^1_{1_M})$ becomes a monoid, more precisely a submonoid of $(S(L^M) \cup \{\mu_\emptyset\}, \cdot, \mu^1_{1_M})$.

Corollary 3.2. In L^M, "·" satisfies the following transfinite distributivity laws:

$$\mu \cdot \bigvee_{\phi \in \Phi} \nu_\phi = \bigvee_{\phi \in \Phi} (\mu \cdot \nu_\phi) \ , \quad \Big(\bigvee_{\phi \in \Phi} \nu_\phi\Big) \cdot \mu = \bigvee_{\phi \in \Phi} (\nu_\phi \cdot \mu)$$

for an arbitrary index set Φ, and these laws are self-dual. Furthermore,

- $\mu^1_{1_M} \cdot \nu = \nu \cdot \mu^1_{1_M} = \nu, \ \forall \nu \in L^M$;
- $\mu_\emptyset \wedge \nu = \mu_\emptyset \cdot \nu = \nu \cdot \mu_\emptyset = \mu_\emptyset, \ \forall \nu \in L^M$.

Proof:
We have, for example,

$\Big[\mu \cdot \big(\bigvee_{\phi \in \Phi} \nu_\phi\big)\Big](m) = \bigvee_{m=uv} \Big[\mu(u) \wedge \big(\bigvee_{\phi \in \Phi} \nu_\phi(v)\big)\Big] =$

(here we use transfinite distributivity in L)

$= \bigvee_{m=uv} \bigvee_{\phi \in \Phi} \Big[\mu(u) \wedge \nu_\phi(v)\Big] =$ (generalized commutativity)

$= \bigvee_{\phi \in \Phi} \bigvee_{m=uv} \Big[\mu(u) \wedge \nu_\phi(v)\Big] = \Big[\bigvee_{\phi \in \Phi} (\mu \cdot \nu_\phi)\Big](m) \ .$

For the other claims, we can check, for example, that

$$(\nu \cdot \mu^1_{1_M})(m) = \bigvee_{m=uv} \left\{ \nu(u) \wedge \mu^1_{1_M}(v) \right\} = \nu(m) \wedge \mu^1_{1_M}(1_M) = \nu(m) \ .$$

The other claims are straightforward. □

Remark 3.3. With these properties, L^M becomes a `complete lattice ordered semigroup`, abbreviated `closg` - notion defined for example in [10, p. 155]. By the fact that L^M is also a distributive lattice, we affirm that it is a `distributive closg`.

Since multiplication is associative in L^M we can define the unary operator "*"(or "star", or iteration) as following:

$$\forall \nu \in L^M : \ \ \nu^* = \bigvee_{i=0}^{\infty} \nu^i \ \ ,$$

with $\nu^0 = \mu^1_{1_M}$ (by convention) and $\nu^i = \nu^{i-1} \cdot \nu$, for all $i \geq 1$. As usual, we denote

$$\nu^+ = \bigvee_{j=1}^{\infty} \nu^j \ \ .$$

It is worth noticing the following facts:

- if $\mu^l_m, \mu^t_n \in S(L^M)$, we have $\mu^l_m \cdot \mu^t_n = \mu^{l \wedge t}_{m \cdot n}$, hence $(\mu^l_m)^i = \mu^l_{m^i}, \ \ \forall i \geq 1$;
- $(\mu_\emptyset)^* = (\mu_\emptyset)^0 = \mu^1_{1_M}$;
- $(\mu_\emptyset)^+ = \mu_\emptyset$.

If $\nu \in FS(L^M)$, we can write ν as a finite (eventually empty) meet of singletons: $\nu = \mu^{l_1}_{m_1} \vee \cdots \vee \mu^{l_k}_{m_k}$, for some $k \geq 0$, and if $\nu \neq \mu_\emptyset$ one can express any positive power of ν as

$$\forall p \geq 1 : \ \ \nu^p = \bigvee_{1 \leq i_1, \ldots, i_p \leq k} \mu^{l_{i_1} \wedge \cdots \wedge l_{i_p}}_{m_{i_1} \cdots m_{i_p}} \ \ .$$

Observation. Notice that $\nu \in FS(L^M)$ does not necessarily mean that $\nu^* \in FS(L^M)$. Indeed, the finite support of ν may generate an infinite submonoid of M which in turn may become exactly the support of ν^*, as the following example shows.

Example 3.1. Let $M = \{a, b\}^*$, the monoid of words over the alphabet $\{a, b\}$, and $L = \{0, 1\}$, with $0 < 1$. Take the following fuzzy set:

$$\nu = \{(a, 1), (b, 1)\} \cup \left\{ (w, 0) / w \in \{a, b\}^* \setminus \{a, b\} \right\} \ .$$

We have that $\nu \in FS(L^M)$, since only a and b have associated non-null values, and $\nu^* \notin FS(L^M)$ since one can easily check that $\nu^* = \{(w, 1) / w \in \{a, b\}^*\}$.

In general, if $\nu \in L^M$ and $m \neq 1_M$, then

$$\nu^*(m) = \bigvee_{m=m_1 \dots m_p} \bigwedge_{j=1}^{p} \nu(m_j) \ , \tag{2}$$

and $\nu^*(1_M) = 1$ (since $\nu^0 = \nu^1_{1_M}$). It is worth putting in words that `the iteration of any fuzzy set has always value` 1 `assigned to the unity`.

Remark 3.4. According to relation (2), if $m \in M$ has a factorization $m = m_1 \cdot m_2 \cdot \dots \cdot m_k$ such that $\mu(m_i) = 1$ for all $i \in \{1, ..., k\}$, then $\mu^*(m) = 1$.

Corollary 3.3. Let $\nu, \xi \in L^M$ be fuzzy sets such that $\mid \nu(M) \mid + \mid \xi(M) \mid < \aleph_0$. Then $\mid (\nu \cdot \xi)(M) \mid < \aleph_0$ and $\mid \nu^*(M) \mid < \aleph_0$.

Proof:
Let $S = \nu(M) \cup \xi(M)$, with S being a finite subset of L. Consider the following two subsets of L, derived from S:

$$S^\wedge = \{l \in L / \exists T \subseteq S : \ \ l = \wedge T\} \text{ , and}$$

$$(S^\wedge)^\vee = \{l \in L / \exists T \subseteq S^\wedge : \ \ l = \vee T\} \ .$$

It is clear that $(S^\wedge)^\vee$ is finite and includes S (by commutativity and idempotence in L). Since $\nu(M) \subseteq S$ and by equation (2), we infer that $\nu^*(M) \subseteq (S^\wedge)^\vee$. Indeed, although (2) may involve transfinite joins, by idempotence in L the value of $\nu^*(m)$ must be included in $(S^\wedge)^\vee$ for any $m \in M$. Similarly, since both $\nu(M) \subseteq S$ and $\xi(M) \subseteq S$ and by the equation (1), we infer that $(\nu \cdot \xi)(M) \subseteq (S^\wedge)^\vee$. This proves that $\nu^*(M)$ and $(\nu \cdot \xi)(M)$ are finite. □

Notation wise, let us denote by μ^l_M the `constant fuzzy set` which associates to all elements of M the value l.

Corollary 3.4. Any fuzzy set $\nu \in L^M$ verifies the following inequality:

$$\nu^* \geq \nu \wedge \mu_M^{\nu(1_M)} \ .$$

Proof:
Let ν be an arbitrary fuzzy set in L^M and p be an integer greater than 1. Any element $m \in M$ can be factorized as $m = m \cdot (1_M)^{p-1}$ and then

$$\nu^p(m) = \bigvee_{m=m_1 \dots m_p} \left\{ \bigwedge_{i=1}^{p} \nu(m_i) \right\} \geq \nu(m) \wedge \nu(1_M) \ ,$$

which leads to the inequality $\nu^*(m) \geq \nu(m) \wedge \nu(1_M)$. From here the conclusion follows shortly. □

An element m is prime in M if and only if it can not be written as a product of non-unit elements of M. In other words, $m \in M$ is prime if and only if $m = m_1 \cdot m_2 \Rightarrow m_1 = 1_M$ or $m_2 = 1_M$.

Corollary 3.5. If m is prime in M and ν is a fuzzy set in L^M such that $\nu(1_M) = 0$ then $\nu^*(m) = 0$.

Proof:
It suffices to observe that if m is prime, then any fuzzy set $\xi \in L^M$ verifies the equality

$$\xi^*(m) = \xi(m) \wedge \xi(1_M) \ .$$

□

Definition 3.2. Let M_1, M_2 be monoids and L_1, L_2 be c.d.c. lattices. An application $\tilde{h} : L_1^{M_1} \to L_2^{M_2}$ is a `morphism of fuzzy sets` (or fuzzy morphism) if it preserves arbitrary meets, and joins.

Remark 3.5. It is important to notice that any fuzzy set can be written as a meet of singletons, and any product of fuzzy sets can be written as a combination of meets and joins of singletons (this equally applies to the star and plus of a fuzzy set).

Proof:
If $\nu \in L^M$ is an arbitrary fuzzy set, then

$$\nu = \bigvee_{m \in M} \mu_m^{\nu(m)} \ ,$$

and if $\nu, \xi \in L^M$, we can express their product as

$$\nu \cdot \xi = \bigvee_{m \in M} \Big(\bigvee_{m = m_1 m_2} \mu_m^{\nu(m_1) \wedge \xi(m_2)} \Big)$$

□

Corollary 3.6. A fuzzy morphism preserves finite products, star and plus.

Consequently, the definition of fuzzy morphisms is in line with the definition of `closg homomorphisms` as found in [10, p. 155] (recall that $L_1^{M_1}$ and $L_2^{M_2}$ can be viewed as closg - complete lattice ordered semigroups).

4. Fuzzification of Rational Sets

Since we have assigned to L^M a monoid structure, it makes sense to talk about the family of rational sets of fuzzy sets: $Rat(L^M)$, and the family of recognizable sets of fuzzy sets: $Rec(L^M)$. However, the study of $Rat(L^M)$ and $Rec(L^M)$ is beyond the scope of the present paper. Here we define the so called "`fuzzy rational sets`" and "`fuzzy recognizable sets`", and we should point out that there is a difference between

"rational sets of fuzzy sets" and "fuzzy rational sets",

and between

"recognizable sets of fuzzy sets" and "fuzzy recognizable sets".

For example, a rational set of fuzzy sets is an element of $Rat(L^M)$, whereas a fuzzy rational set is an element of L^M obtained from singletons by rational operations, as will be describe in the following (Definition 4.1).

Note 1. During this study, there has been some debate on whether to adopt the nomenclature "*fuzzy rational sets*" or "*rational fuzzy sets*". Arguably, in this case, the most accurate is the latter construct, which is also in line with the existing terminology (for example "*regular fuzzy languages*", or "*finite fuzzy automata*", as in [17]). However, in other circumstances, this construct can not be applied properly. For example, the terminology "*fuzzy rational expressions*" seems more appropriate than "*rational fuzzy expressions*". To alleviate this dilemma, in this paper we adopt the following naming convention:

> *A concept "$\mathcal{C}$" in a classical (crisp) theory has the analogous "*`fuzzy` $\mathcal{C}$*" in the corresponding fuzzy theory.*

According to this, in the following we will talk about "fuzzy rational sets", "fuzzy recognizable sets", "fuzzy finite transducers", etc. .

Consider yet again the following two structures: a monoid $(M, \cdot, 1_M)$ and a c.d.c. lattice $(L, \leq, \wedge, \vee, 0, 1)$. Recall that we defined $S(L^M)$ as the set of all singletons in L^M and $FS(L^M)$ as the set of all finite unions of singletons in L^M. Observe that $S(L^M)$ does not contain $\mu_\emptyset$ whereas $FS(L^M)$ does. We define L-fuzzy rational sets in M by "fuzzifying" the classical definition (as found in [2]) of rational sets, as follows:

Definition 4.1. The family of L-`fuzzy rational sets` on M is the least family $\tilde{R}at(M)$ satisfying the following conditions:

(i) $FS(L^M) \subseteq \tilde{R}at(M)$;

(ii) $\forall \nu, \nu' \in \tilde{R}at(M) \; : \; \nu \vee \nu' \in \tilde{R}at(M)$ and $\nu \cdot \nu' \in \tilde{R}at(M)$;

(iii) $\forall \nu \in \tilde{R}at(M) \; : \; \nu^+ \in \tilde{R}at(M)$.

It follows immediately that if $\nu \in \tilde{R}at(M)$ then $\nu^* \in \tilde{R}at(M)$, since $\mu^1_{1_M} \in \tilde{R}at(M)$ by (i). Notice that condition (i) can be replaced by (i'): $S(L^M) \cup \{\mu_\emptyset\} \subseteq \tilde{R}at(M)$. Thus, both $FS(L^M)$ and $S(L^M) \cup \{\mu_\emptyset\}$ can be used as a base of our recursive definition.

Denote by $\tilde{R}atE(M)$ the set of all L-`fuzzy rational expressions`, i.e., of all parenthesized infix formulae obtained from the elements of $S(L^M)$ (viewed as atomic formulae), the nullary operator $\mu_\emptyset$, the binary operators $\vee$ and $\cdot$, and the unary operator *. The fuzzy set defined by a fuzzy regular expression $\tilde{E}$ is the join of all fuzzy sets (singletons or null fuzzy set) which can be "expressed" by $\tilde{E}$. We illustrate what we mean by "expressed" in the following example (a more formal description can be given using the semigroup homomorphism defined in the proof of Lemma 4.2).

Example 4.1. Let M and L be as usual, and denote by μ^l_m a singleton fuzzy set which assigns the fuzzy value l to the element m. Consider the following fuzzy rational expression:

$$\tilde{E} = \Big(\big((\mu^{l_1}_{m_1})^* \vee \mu^{l_2}_{m_2} \big) \cdot \mu^{l_3}_{m_3} \Big) \vee \mu_\emptyset \; .$$

The following are the fuzzy sets which are `expressed` by $\tilde{E}$:

$$\mu_\emptyset, \quad \mu^{l_2}_{m_2}\mu^{l_3}_{m_3}, \quad \mu^{l_3}_{m_3}, \quad \mu^{l_1\wedge l_3}_{m_1 m_3}, \quad \mu^{l_1\wedge l_3}_{(m_1)^2 m_3}, \quad \mu^{l_1\wedge l_3}_{(m_1)^3 m_3}, \quad \mu^{l_1\wedge l_3}_{(m_1)^4 m_3}, \ \ldots$$

The fuzzy set `defined` by $\tilde{E}$ is the join of all these fuzzy sets. Here we used that $(\mu^{l_1}_{m_1})^0 = \mu^1_{1_M}$ therefore $(\mu^{l_1}_{m_1})^0\mu^{l_3}_{m_3} = \mu^{l_3}_{m_3}$, and that $l_3 \wedge l_3 \wedge \ldots \wedge l_3 = l_3$.

It is clear that a fuzzy rational expression defines exactly one fuzzy set and that the family of sets defined by $\tilde{R}atE(M)$ is exactly $\tilde{R}at(M)$. Notation wise, if $\tilde{E}$ is a fuzzy rational expression then $\mid \tilde{E} \mid$ will denote the fuzzy set defined by $\tilde{E}$.

A few results concerning rational sets in arbitrary monoids can be proven to hold for fuzzy rational sets as well. In the following we mention some of them together with some properties particular to fuzzy rational sets.

Proposition 4.1. $\tilde{R}at(M)$ is closed under join, product, plus, star and fuzzy morphisms that map singletons into rational fuzzy sets.

Proof:
The closure under join, product, star and plus follows directly from the definition of $\tilde{R}at(M)$. We prove a stronger version of closure under fuzzy morphisms, namely that rationality is preserved regardless of the supporting lattice.

Assume $\tilde{h} : L_1^{M_1} \to L_2^{M_2}$ to be a fuzzy morphism which maps singletons into rational fuzzy sets. If $\nu \in L_1^{M_1}$ is represented by the expression $\tilde{E}_\nu \in \tilde{R}atE(M_1)$ then $\tilde{h}(\nu)$ will be expressed by a rational expression $\tilde{h}(\tilde{E}_\nu)$ obtained as following.

Let $\{\mu^{l_1}_{m_1}, \ldots, \mu^{l_k}_{m_k}\}$ be the set of all singletons which occur in $\tilde{E}_\nu$ and denote by $\tilde{e}_i$ the rational expression corresponding to $\tilde{h}(\mu^{l_i}_{m_i})$, $1 \le i \le k$ (recall that $\tilde{h}$ maps singletons into rational sets). Then, denote by $\tilde{h}(\tilde{E}_\nu)$ the rational expression obtained from $\tilde{E}_\nu$ by replacing each occurrence of $\mu^{l_i}_{m_i}$ with $\tilde{e}_i$, $1 \le i \le k$. One can observe that $\mid \tilde{h}(\tilde{E}_\nu) \mid = \tilde{h}(\nu)$ and then $\tilde{h}(\nu)$ is a fuzzy rational set in ${L_2}^{M_2}$. □

Notice that $\tilde{R}at(M)$ may not be closed under fuzzy morphisms in general. For example a fuzzy morphism may map a singleton into a fuzzy set with support that may not be obtained through rational operations (which would contradict Corollary 4.1).

We now wish to make a connection between the rationality of a fuzzy set and the rationality of its support; and in order to do so, the following properties are helpful.

Lemma 4.1. If $\nu, \xi \in L^M$ then

(i) $supp(\nu \vee \xi) = supp(\nu) \cup supp(\xi)$;

(ii) $supp(\nu \wedge \xi) \subseteq supp(\nu) \cap supp(\xi)$;

(iii) $supp(\nu \cdot \xi) \subseteq supp(\nu) \cdot supp(\xi)$;

(iv) $supp(\nu^*) \subseteq supp(\nu)^*$.

If in addition L has no 0-divisors then (ii′) $supp(\nu \wedge \xi) = supp(\nu) \cap supp(\xi);$

(iii′) $supp(\nu \cdot \xi) = supp(\nu) \cdot supp(\xi);$

(iv′) $supp(\nu^*) = supp(\nu)^*.$

Proof:
Recall that in a lattice L with no 0-divisors we have a finite meet equal to zero if and only if at least one of its terms is equal to zero. Moreover, in an arbitrary complete lattice, a join (finite or not) is not zero if and only if one of its term is not zero.

We prove (iii′) and (iv′), the rest being straightforward. If $m \in supp(\nu \cdot \xi)$ then $(\nu \cdot \xi)(m) \neq 0$, hence by the definition of multiplication there exists a decomposition $m = m_1 \cdot m_2$ such that $\nu(m_1) \wedge \xi(m_2) \neq 0$. This implies that $\nu(m_1) \neq 0$ and $\xi(m_2) \neq 0$, hence that $m_1 \in supp(\nu)$ and $m_2 \in supp(\xi)$. Then $m = m_1 \cdot m_2 \in supp(\nu) \cdot supp(\xi)$. Conversely, if $m \in supp(\nu) \cdot supp(\xi)$, then $m = m_1 \cdot m_2$ for some $m_1 \in supp(\nu)$ and $m_2 \in supp(\xi)$. Then $\nu(m_1) \wedge \xi(m_2) \neq 0$ (since L has no 0-divisors), hence $(\nu \cdot \xi)(m) \neq 0$, i.e., $m \in supp(\nu \cdot \xi)$.

If $m \in supp(\nu)^*$ then there exists a decomposition $m = m_1 \cdot ... \cdot m_k$ such that $m_i \in supp(\nu),\ \ \forall i \in \{1, ..., k\}$. Then, $\nu^k(m) \neq 0$ (since L has no 0-divisors), hence $\nu^*(m) \neq 0$. Conversely, if $m \in supp(\nu^*)$ then $m \in supp(\nu^i)$ for at least one $i \geq 0$. Then, there exists a decomposition $m = m_1 \cdot ... \cdot m_i$ such that $m_j \in supp(\nu)$, for all $j \in \{1, ..., i\}$. Then, $m \in supp(\nu)^i \subseteq supp(\nu)^*$. □

Corollary 4.1. If L has no 0-divisors, then

$$\nu \in \tilde{R}at(M) \Rightarrow supp(\nu) \in Rat(M) \ .$$

Proof:
The proof is by structural induction on $\nu \in \tilde{R}at(M)$, using the properties of Lemma 4.1. □

Let $\tilde{E} \in \tilde{R}atE(M)$ be a fuzzy rational expression and recall that we denote by $\mid \tilde{E} \mid$ the fuzzy rational set defined by $\tilde{E}$.

Definition 4.2. By `defuzzification of` $\tilde{E}$ we denote the rational expression $\vartheta\tilde{E} \in RatE(M)$ obtained from $\tilde{E}$ by replacing

- all terms $\mu^l_m \in S(L^M)$ of E with the corresponding $m \in M$,
- the term $\mu_\emptyset$ by $\emptyset$ – if present, and
- the operator $\vee$ by $+$,

and leaving the rest unchanged.

Lemma 4.2. If L has no 0-divisors and $\tilde{E} \in \tilde{R}atE(M)$, then

$$\mid \vartheta\tilde{E} \mid \ = supp(\mid \tilde{E} \mid) \ .$$

Proof:
If we replace any occurrence in $\tilde{E}$ of an element in $S(L^M) \cup \{\mu_\emptyset\}$ by a new symbol and if we denote by Σ the alphabet of these new symbols, we obtain a regular expression E over Σ. By doing so, we implicitly define a semigroup homomorphism

$$i : \Sigma^* \to S(L^M) \cup \{\mu_\emptyset\} \ ,$$

which maps each symbol of Σ back to the singleton (or $\mu_\emptyset$) which it replaces, in addition $i(\varepsilon) = \mu_\emptyset$. It is important to notice that the obtained regular expression E is `unambiguous`, in the sense described in [6, p. 150]. By the fact that L has no 0-divisors we infer that a word $u \in \mid E \mid$ is mapped to $\mu_\emptyset$ if and only if either u is the empty word or u contains a symbol mapped to $\mu_\emptyset$. By the same observation we have that

$$\mid \tilde{E} \mid \ = \bigvee_{u \in |E|} i(u), \text{ and } \quad \mid \vartheta\tilde{E} \mid \ = \left\{ m \in M \middle/ u \in \mid E \mid \text{ and } i(u) = \mu^l_m \neq \mu_\emptyset \right\} \ .$$

Then, $m \in supp(\mid \tilde{E} \mid)$ if and only if there exists $u \in \mid E \mid$ such that $i(u)(m) = l \neq 0$, that is, $m \in \mid \vartheta\tilde{E} \mid$. □

Notice that if the condition that L does not have 0-divisors is not satisfied, then we can be certain only of the following relation:

$$\mid \vartheta\tilde{E} \mid \ \supseteq supp(\mid \tilde{E} \mid) \ .$$

Proposition 4.2. If $\nu \in \tilde{R}at(M)$ then $\mid \nu(M) \mid < \aleph_0$.

Proof:
We prove this property by structural induction on ν. It is clear that all singletons and $\mu_\emptyset$ have the property. If $\mu, \xi \in \tilde{R}at(M)$ such that $\mid \nu(M) \mid < \aleph_0$ and $\mid \xi(M) \mid < \aleph_0$, then clearly $\mid (\nu \vee \xi)(M) \mid < \aleph_0$. Moreover, by Corollary 3.3 we have that $\mid (\nu \cdot \xi)(M) \mid < \aleph_0$ and that $\mid \nu^*(M) \mid < \aleph_0$, fact which completes the induction. □

It is worth putting in words the fact that `any fuzzy rational set takes a finite number of values, when viewed as a function`.

Definition 4.3. An L-`fuzzy finite automaton` on M is a tuple $\tilde{A} = (Q, S(L^M), E, I, F)$, where

1. $E \subseteq Q \times S(L^M) \times Q$ is a `finite` set (of transitions); and
2. $I, F \subseteq Q$ are initial, and final sets (of states), respectively.

A `computation` (or path) in A is an element $c \in E^+$ of the following form:

$$c = (p_1, \mu_1, p_2)(p_2, \mu_2, p_3) \dots (p_{k-1}, \mu_{k-1}, p_k)(p_k, \mu_k, p_{k+1}).$$

The computation c is `successful` if $p_1 \in I$ and $p_{k+1} \in F$. We denote by $\mid c \mid$ the fuzzy set $\mu_1 \cdot \ldots \cdot \mu_{k+1}$, and we say that $\mid c \mid$ is the `label` of c. The fuzzy set defined by $\tilde{A}$ is denoted by $\mid \tilde{A} \mid$ and is given by

$$\mid \tilde{A} \mid = \bigvee \left\{ \ \mid c \mid \ / c \text{ is a successful computation in } \tilde{A} \right\}$$

The finiteness of E allows us to consider Q to be finite as well.

Notice carefully that we do not admit "empty" computations ($c \in E^+$). Consequently, the fuzzy finite automaton $(\{q\}, S(L^M), \emptyset, \{q\}, \{q\})$ defines the fuzzy set $\mu_\emptyset$ - in contrast with the crisp (i.e., classical) theory, where a similar situation leads to the set having the unity as its sole element.

Theorem 4.1. A fuzzy set is defined by a fuzzy finite automaton if and only if it is an element of $\tilde{R}at(M)$.

Proof:
A constructive proof of this result is to apply methods similar to the method of converting a finite automaton into a rational expression and vice versa. In order to do so, one may find helpful to use the semigroup morphism defined in the proof of Lemma 4.2. The details are straightforward. □

Remark 4.1. The definition of L-fuzzy finite automata can be changed to allow $E \subseteq Q \times FS(L^M) \times Q$, without changing the power of these automata.

The following subfamily of rational fuzzy sets will be used in Section 6.

Definition 4.4. By $\tilde{R}atR(M)$ we denote the family of `restricted rational fuzzy sets`, defined by fuzzy finite automata, called `restricted fuzzy finite automata`, which verify the following conditions:

1. they have only one initial state;
2. for each state q, there exists a transition $(q, \mu^1_{1_M}, q)$; and
3. there is no transition of the form $(p, \mu^l_{1_M}, q)$ with p, q different states and $l \in L \setminus \{0\}$.

In Section 6 we relate this family to the family of fuzzy recognizable sets in finitely generated free monoids.

Let $m \neq 1_M$ be an element of the monoid M. We say that m is a divisor of 1_M, or a `1-divisor`, if there exists $m' \in M$ such that $m \cdot m' = 1_M$ or $m' \cdot m = 1_M$. Notice that this notion is weaker than that of an invertible element in M.

Lemma 4.3.

1. If $\nu \in \tilde{R}at(M)$ and $\nu(1_M) \in \{0, 1\}$ then $\nu \in \tilde{R}atR(M)$.
2. If M has no 1-divisors and $\nu \in \tilde{R}at(M)$ then
$$\nu \in \tilde{R}atR(M) \Leftrightarrow \nu(1_M) \in \{0, 1\} \ .$$

Proof:
We prove the second affirmation. If $\nu \in \tilde{R}atR(M)$ is realized by a restricted fuzzy automaton $\tilde{A} = (Q, S(L^M), E, q_0, F)$, then we observe that the only situation ensuring $\nu(1_M) \neq 0$ is when $q_0 \in F$, in which case $\nu(1_M) = 1$ (we count the fact that M has no 1-divisors). Conversely, if $\nu(1_M) \in \{0, 1\}$ and ν is realized by a fuzzy finite automaton $\tilde{A} = (Q, S(L^M), E, I, F)$, then we can construct an equivalent restricted automaton $\tilde{A}'$, following an algorithm somehow similar to the elimination of ε-transitions in NFA:

1. We first perform a "$\mu_{1_M}^{\forall}$" closure in $\tilde{A}$: for every pair of transitions $(p, \mu_{1_M}^{l_1}, q)$, $(q, \mu_{1_M}^{l_2}, r)$ we add a transition $(p, \mu_{1_M}^{l_1 \wedge l_2}, r)$. Notice that the process is finite, due to the idempotence in L.

2. Then, for every pair of transitions $(p, \mu_{1_M}^{l}, q)$, $(q, \mu_{m}^{l'}, r)$ with $m \neq 1_M$, we add a transition $(p, \mu_{m}^{l \wedge l'}, r)$. In addition, for every pair of transitions (p, μ_{m}^{l}, q), $(q, \mu_{1_M}^{l'}, r)$ with $m \neq 1_M$, we add a transition $(p, \mu_{m}^{l \wedge l'}, r)$. Unlike the case of ε-removal in NFA, here we have this additional "interleave", fact which will make unnecessary the addition of extra final or initial states.

3. We eliminate all transitions of type $(p, \mu_{1_M}^{l}, q)$, for $l \in L \setminus \{0\}$, and we add the transitions $(p, \mu_{1_M}^{1}, p)$, for all $p \in Q$. The obtained fuzzy automaton is equivalent to $\tilde{A}$ modulus the value in 1_M, and is restricted, modulus the fact that we may have multiple initial states.

4. Finally we add a new states q_0 which becomes the only initial state, and a transition (q_0, μ_{m}^{l}, p) for each transition (q, μ_{m}^{l}, p) with q initial state in $\tilde{A}$. We also add the transition $(q_0, \mu_{1_M}^{1}, q_0)$. If $\nu(\mu_{1_M}^{1}) = 1$ then we add q_0 to the set of final states.

We obtain a fuzzy finite automaton $\tilde{A}'$ which is restricted and realizes ν, hence proving that $\nu \in \tilde{R}atR(M)$. □

Let $\nu \in L^M$ be an arbitrary fuzzy set. For any $l \in L$ we defined the following "step" fuzzy set: $\nu^{\geq l} \in L^M$, given by

$$\nu^{\geq l}(m) = \begin{cases} \nu(m), & if \;\; \nu(m) \geq l; \\ 0, & otherwise. \end{cases}$$

Proposition 4.3. If L is totally ordered, then

$$\nu \in \tilde{R}at(M) \Rightarrow \forall l \in L : \;\; \nu^{\geq l} \in \tilde{R}at(M) \;\; .$$

Proof:
Consider a fuzzy finite automaton $\tilde{A} = (Q, S(L^M), E, I, F)$ such that $\mid \tilde{A} \mid = \nu$ and choose an arbitrary $l \in L$. We construct the following finite automaton: $\tilde{B} = (Q, S(L^M), E', I, F)$, where

$$E' = \{(p, \mu_m^t, q)/(p, \mu_m^t, q) \in E \text{ and } t \geq l\} \;\; .$$

It remains to prove that $\mid \tilde{B} \mid = \nu^{\geq l}$. Since L is totally ordered, ν can only take values among the fuzzy values of the singletons labeling the transitions of $\tilde{A}$ (recall that the rationality of ν suffices to ensure that ν takes a finite number of values). More specifically, if $\nu(m) = t$ then there exists a successful computation

$$(p_0, \mu_{m_1}^{t_1}, p_1)...(p_{k-1}, \mu_{m_k}^{t_k}, p_k)$$

with $m = m_1 \cdot ... \cdot m_k$ and $t = t_j$ for some $j \in \{1, ..., k\}$. In addition, t is the smallest (with respect to the order in L) value among $\{t_1, ..., t_k\}$. Notice that this happens despite the fact that there may exist an infinity of successful paths corresponding to m.

Assume now that $\nu(m) = t \geq l$ and consider a successful path in $\tilde{A}$ as above. This path will also exist in $\tilde{B}$ by its construction, since t is smaller than the value of every singleton which appears in the path. Moreover, since each successful path of $\tilde{B}$ is a successful path in $\tilde{A}$ we conclude that $\mid \tilde{B} \mid (m) = t$.

In the other case, when $\nu(m) = t < l$, there exist no successful path in $\tilde{A}$, as above, with all $t_1, ..., t_k$ greater than or equal to l. This directly implies that there is no successful path in $\tilde{B}$ corresponding to m, hence that $\mid \tilde{B} \mid (m) = 0$. □

This property may prove to be useful in answering the following question, which has not yet been addressed: if $\nu \in \tilde{R}at(M)$, is it true that $\nu^{-1}(l) \in Rat(M)$ for an arbitrary $l \in L$? If not, under what conditions it is true ? Proposition 4.3 is also used in the proof of Theorem 6.4.

5. Fuzzification of Recognizable Sets

Compared to the case of rational sets, the fuzzification of recognizable sets turns out to be a more complex matter. This can be explained in part by the fact that actions over arbitrary monoids are algebraically more complex than rational closures. One clear impact of this difference is the difficulty of stating the usual closure properties of recognizable sets in the "fuzzy" context. In the following we consider M to be an arbitrary monoid and L to be a c.d.c. lattice.

Definition 5.1. The element $\nu \in L^M$ is an L-`fuzzy recognizable set` on M if and only if there exists a finite monoid N, a monoid morphism $h : S(L^M) \cup \{\mu_\emptyset\} \to N$ and a set $P \subseteq N$ such that

$$\nu = \bigvee h^{-1}(P) \ .$$

Notice that ν does not necessarily belong to the monoid $S(L^M) \cup \{\mu_\emptyset\}$, or $FS(L^M)$ for that matter. Notice also that $h(\mu_\emptyset)$ acts as a "zero" in N, i.e., $h(\mu_\emptyset) \cdot n = n \cdot h(\mu_\emptyset) = h(\mu_\emptyset)$, for every n in the image of h.

In the following we particularize the notion of `action of a monoid over an arbitrary set` (as defined in [26, §II.2.1]) to the submonoid of singletons of L^M.

Definition 5.2. A (right) `fuzzy action` of $S(L^M)$ over (or, on) a set Q is a mapping

$$\delta : Q \times \left[S(L^M) \cup \{\mu_\emptyset\}\right] \to Q \ ,$$

such that

(i) $\forall q \in Q : \ \ \delta(q, \mu^1_{1_M}) = q$;

(ii) $\forall q \in Q, \ \ \mu, \mu' \in S(L^M) \cup \{\mu_\emptyset\} : \ \ \delta(\delta(q, \mu), \mu') = \delta(q, \mu \cdot \mu')$.

Notice that we allow null ($\mu_\emptyset$) "transitions" and, according to the above definition, if $\mu \cdot \mu' = \mu^1_{1_M}$, then $\delta(\delta(q, \mu), \mu') = q$, for all $q \in Q$.

Definition 5.3. A `fuzzy action automaton` over M by L is a tuple $\tilde{A} = (Q, S(L^M) \cup \{\mu_\emptyset\}, \delta, q_0, F)$ where

(i) Q is a finite set (of states);

(ii) $\delta : Q \times \left[S(L^M) \cup \{\mu_\emptyset\}\right] \to Q$ is a fuzzy action (the next state function);

(iii) $q_0 \in Q$ is an initial state;

(iv) $F \subseteq Q$ is a set of final states.

The fuzzy set defined by $\tilde{A}$ is given by

$$| \tilde{A} |= \bigvee \{\mu/\delta(q_0, \mu) \in F\}.$$

Remark 5.1. We have $| \tilde{A} | (1_M) = 1$ if and only if $q_0 \in F$. We have $| \tilde{A} | (1_M) = 0$ if and only if $\delta(q_0, \mu^l_{1_M}) \notin F$, for all $l \in L \setminus \{0\}$.

Notice that $\tilde{A}$ in Definition 5.3 does not necessarily have a finite representation, since although it has a finite number of states, it may have an infinite number of "transitions". Furthermore, since $| \tilde{A} |$ is a possibly infinite join of singletons, it may not be an element of $S(L^M)$, or $FS(L^M)$ for that matter. Finally, as in the "crisp" case, fuzzy action automata are close relatives of deterministic automata, due to the functionality of their transition table.

Remark 5.2. If L has no 0-divisors, then $S(L^M)$ becomes a submonoid of L^M and we can replace $S(L^M \cup \{\mu_\emptyset\})$ with $S(L^M)$ in all previous definitions (5.1, 5.2 and 5.3). Observe that $\mu_\emptyset$ is defined by a fuzzy action automaton with no final states (hence it is always recognizable, by the following theorem).

Theorem 5.1. An element of L^M is an L-fuzzy recognizable set if and only if it is defined by an L-fuzzy action automaton on M.

Proof:

I. Assume that $\nu \in \tilde{Rec}(M)$. Then there exists a finite monoid Q, a monoid morphism $h : S(L^M) \cup \{\mu_\emptyset\} \rightarrow Q$ and a set $F \subseteq Q$ such that $\nu = \bigvee h^{-1}(F)$. Consider the fuzzy action automaton $\tilde{A} = (Q, S(L^M) \cup \{\mu_\emptyset\}, \delta, 1_Q, F)$ where $\delta : Q \times [S(L^M) \cup \{\mu_\emptyset\}] \rightarrow Q$ is given by $\delta(q, \mu) = q \cdot h(\mu)$. It can be verified that $\tilde{A}$ is well defined and that $| \tilde{A} |= \nu$.

II. Consider an arbitrary L-fuzzy action automaton $\tilde{A} = (Q, S(L^M) \cup \{\mu_\emptyset\}, \delta, q_0, F)$ and the finite monoid of mappings $N = (Q^Q, \circ, id_Q)$ (where id_Q is the identity mapping on Q). Define the function $h : S(L^M) \cup \{\mu_\emptyset\} \rightarrow N$ as

$$\mu \rightarrow h(\mu) \quad : \quad h(\mu)(q) = \delta(q, \mu) \quad ,$$

and $P = \{f \in N/f(q_0) \in F\}$. Then one can check that h is a monoid morphism and that $| \tilde{A} |= \bigvee h^{-1}(P)$.

□

Notice that $\mu_\emptyset \in \tilde{Rec}(M)$, since we follow the convention that $\bigvee \emptyset = \mu_\emptyset$ and $\mu_\emptyset$ is "recognized" by the morphism of Definition 5.1, when $P = \emptyset$.

Remark 5.3. One may observe that $h(S(L^M) \cup \{\mu_\emptyset\})$ in part (II) of the proof of Theorem 5.1 is what one may call a `fuzzy transition monoid` associated to a fuzzy action automaton.

Notice also that unlike the case of fuzzy rational sets, it is not clear whether we can replace $S(L^M) \cup \{\mu_\emptyset\}$ with $FS(L^M)$ in either Definition 5.1 or Definition 5.3. It may very well lead to a different family of recognizable sets. This matter is left for further work.

As for the case of fuzzy rational sets, a few classical results about "crisp" recognizable sets can be obtained for fuzzy recognizable sets as well.

Proposition 5.1. $\tilde{Rec}(M)$ is closed under finite joins.

Proof:
Let $\nu_1, \nu_2 \in \tilde{Rec}(M)$ be "recognized" by the finite monoids N_1, N_2, the monoid morphisms h_1, h_2 and the subsets P_1, P_2, as in Definition 5.1. Then consider the product monoid $(N_1 \times N_2, \cdot, (1_{N_1}, 1_{N_2}))$ and the morphism $g : S(L^M) \cup \{\mu_\emptyset\} \to N_1 \times N_2$ given by $g(\mu) = (h_1(\mu), h_2(\mu))$. One can check that g is indeed a morphism, and that

$$(\nu_1 \vee \nu_2)(m) = \nu_1(m) \vee \nu_2(m) = \bigvee \left[h_1^{-1}(P_1)\right](m) \quad \vee \quad \bigvee \left[h_2^{-1}(P_2)\right](m) =$$
$$= \bigvee \{\mu(m)/h_1(\mu) \in P_1\} \quad \vee \quad \bigvee \{\mu(m)/h_2(\mu) \in P_2\} =$$

(here we use generalized join associativity and idempotence in L)

$$= \bigvee \{\mu(m)/(h_1(\mu), h_2(\mu)) \in (P_1 \times N_2) \cup (N_1 \times P_2)\} =$$
$$= \bigvee \left\{\mu(m)/\mu \in g^{-1}\big((P_1 \times N_2) \cup (N_1 \times P_2)\big)\right\} \ .$$

Then, we have proven that

$$\nu_1 \vee \nu_2 = \bigvee g^{-1}\big((P_1 \times N_2) \cup (N_1 \times P_2)\big) \ ,$$

which justifies that $\nu_1 \vee \nu_2$ is rational. □

So far, there exist no pertinent results about the closure of fuzzy recognizable sets under meet, inverse fuzzy morphisms, complement and difference. This matter will be the subject of further study. Moreover, we have not classified yet the support of a fuzzy recognizable set over an arbitrary monoid. However, the next result gives an answer for the case when the monoid does not have 1-divisors.

Proposition 5.2. If L has no 0-divisors and M has no 1-divisors then

$$\nu \in \tilde{Rec}(M) \Rightarrow supp(\nu) \in Rec(M) \ .$$

Proof:
Let $\tilde{A} = (Q, S(L^M) \cup \{\mu_\emptyset\}, \delta, q_0, F)$ be such that $\nu = \mid \tilde{A} \mid$. We define the following action automaton over M:

$$B = \big(\mathcal{P}(Q), M, \delta', \{q_0\}, \{K \subseteq Q / K \cap F \neq \emptyset\}\big), \text{ with}$$

- $\mathcal{P}(Q)$ is the power-set(set of parts, or 2^Q) of Q; and
- $\delta' : \mathcal{P}(Q) \times M \to \mathcal{P}(Q)$, given by

$$\delta'(K, 1_M) = K \quad , and$$
$$\forall m \in M \setminus \{1_M\} : \quad \delta'(K, m) = \{\delta(q, \mu_m^l)/q \in K, \ \ \mu_m^l \in S(L^M)\} \ .$$

Let us prove that indeed δ' is an action (it is clearly a function). If $m_1, m_2 \in M \setminus \{1_M\}$ and $K \subseteq Q$, then we have

$$\delta'(\delta'(K, m_1), m_2) = \delta'(\{\delta(p, \mu^{l_1}_{m_1})/p \in K, \mu^{l_1}_{m_1} \in S(L^M)\}, m_2) =$$

$$= \{\delta(\delta(p, \mu^{l_1}_{m_1}), \mu^{l_2}_{m_2})/p \in K,\ \ \mu^{l_1}_{m_1}, \mu^{l_2}_{m_2} \in S(L^M)\} = \ \ \textit{(} \delta \textit{ is an action)}$$

$$= \{\delta(p, \mu^{l_1 \wedge l_2}_{m_1 m_2})/p \in K,\ \ l_1, l_2 \in L \setminus \{0\}\} = \ \ \textit{(} L \textit{ has no } 0\textit{-divisors)}$$

$$= \{\delta(p, \mu^{l}_{m_1 m_2})/p \in K,\ \ \mu^{l}_{m_1 m_2} \in S(L^M)\} = \ \ \textit{(} M \textit{ has no } 1\textit{-divisors)}$$

$$= \delta'(K, m_1 m_2)$$

The cases when either one or both of m_1 and m_2 are the unity (1_M) can be easily proven to obey action's laws as well. Consequently, $\mid B \mid$ is recognizable in M. If $m \in M \setminus \{1_M\}$, we have that

$$\mid \tilde{A} \mid (m) \neq 0 \Leftrightarrow \Big[\bigvee\{\mu/\delta(q_0, \mu) \in F\}\Big](m) \neq 0 \Leftrightarrow$$

$$\Leftrightarrow\ \ \exists \mu^l_m \in S(L^M):\ \ \delta(q_0, \mu^l_m) \in F \Leftrightarrow$$

$$\Leftrightarrow\ \ \{\delta(q_0, \mu^l_m)/\mu^l_m \in S(L^M)\} \cap F \neq \emptyset \Leftrightarrow$$

$$\Leftrightarrow\ \ \delta'(\{q_0\}, m) \cap F \neq \emptyset \Leftrightarrow m \in \mid B \mid\ .$$

If $\mid \tilde{A} \mid (1_M) = 0$, then it follows immediately that $supp(\nu) = \mid B \mid$, hence the conclusion. Otherwise, we distinguish two cases. If $q_0 \in F$, then $\mu^1_{1_M} \in \nu$ and $1_M \in \mid B \mid$, hence, yet again, $supp(\nu) = \mid B \mid$. Finally, if $q_0 \notin F$, then $1_M \notin \mid B \mid$; however, $supp(\nu) = \mid B \mid \cup \{1_M\}$, which is still recognizable (we invoke the closure under finite joins). It follows that $supp(\nu) \in Rec(M)$. □

Many questions concerning the properties of fuzzy recognizable sets are still open, some of which are outlined in Section 8.

6. Kleene and McKnight Theorems for Fuzzy Sets

In this section we are drawing various connection between the following families of sets: fuzzy rational (or restricted rational), fuzzy recognizable, fuzzy regular and the family of languages realized by FT-NFA ([17]). As usual, unless specified otherwise, we consider fuzzy sets over a c.d.c. lattice L. We start by relating fuzzy rational/recognizable sets and regular languages.

Corollary 6.1. Let Σ be an alphabet and $\nu \in \tilde{R}at(\Sigma^*)$ or $\nu \in \tilde{R}ec(\Sigma^*)$. If L has no 0-divisors, then $supp(\nu)$ is a regular language in Σ^*.

Proof:
One can apply Corollary 4.1 or Proposition 5.2 (since Σ^* has no 1-divisors) and the Kleene theorem (as found in [2, T. 2.1, p. 56]) for "crisp" languages. □

Remark 6.1. It is important to notice that if M is a free monoid, then M has no 1-divisors. Indeed, assume there exist $m_1, m_2 \in M \setminus \{1_M\}$, such that $m_1 \cdot m_2 = 1_M$. Consider the representation of m_1 and m_2 as a free product of generators, $m_1 = g_1 \cdot ... \cdot g_k$ and $m_2 = g_{k+1} \cdot ... \cdot g_t$. We have that $g_1 \cdot ... \cdot g_t = 1_M$, which contradicts the freeness of M.

Theorem 6.1. *(a McKnight theorem for fuzzy sets)* Let M be a monoid and L be a finite, distributive lattice. If M is finitely generated, then $\tilde{R}at(M) \supseteq \tilde{R}ec(M)$.

Proof:
Since L is finite, then L is a c.d.c. lattice as well. Denote $L = \{l_1, ..., l_m\}$ and assume that a set of generators for M is $\{g_1, ..., g_n\}$. Consider a fuzzy action automaton $\tilde{A} = (Q, S(L^M) \cup \{\mu_\emptyset\}, \delta, q_0, F)$. It suffices to prove that $\mid \tilde{A} \mid$ can be realized by a fuzzy finite automaton over M.

Construct the fuzzy finite automaton $\tilde{A}' = (Q, S(L^M), E, \{q_0\}, F)$ where for all $i \in \{1, ..., n\}, j \in \{1, ..., m\}$ we set:

$$(p, \mu_{g_i}^{l_j}, q) \in E \Leftrightarrow \delta(p, \mu_{g_i}^{l_j}) = q \ .$$

Clearly E is finite. We first prove that $\mid \tilde{A} \mid (m) = \mid \tilde{A}' \mid (m)$, for all $m \in M \setminus \{1_M\}$. Assume $\mid \tilde{A} \mid (m) = l \neq 0$ for some $m \in M \setminus \{1_M\}$. We have that

$$\mid \tilde{A} \mid (m) = \bigvee \{l' / \delta(q_0, \mu_m^{l'}) \in F\} \ ,$$

in other words, there exist $l'_1, ..., l'_k \in L$ such that $\delta(q_0, \mu_m^{l'_i}) \in F$, for all $i \in \{1, ..., k\}$; and $l = l'_1 \vee ... \vee l'_k$. Consider a representation of m as a product of generators, $m = g'_1 \cdot ... \cdot g'_t$. It can easily be checked that $\delta(q_0, \mu_{g'_1}^{l'_i} \cdot ... \cdot \mu_{g'_t}^{l'_i}) \in F$, for all $i \in \{1, ..., k\}$. If we consider an arbitrary $i \in \{1, ..., k\}$, let us denote $q_1 = \delta(q_0, \mu_{g'_1}^{l'_i})$, ..., $q_k = \delta(\delta(...\delta(q_0, \mu_{g'_1}^{l'_i})..., \mu_{g'_{k-1}}^{l'_i}), \mu_{g'_k}^{l'_i})$. Then $q_k \in F$ and there exists a successful computation c in $\mid \tilde{A}' \mid$, given by

$$c = (q_0, \mu_{g'_1}^{l'_i}, q_1)...(q_{k-1}, \mu_{g'_k}^{l'i}, q_k) \ .$$

In other words, for each $\mu_m^{l'_i}$, there exists a successful computation in $\mid \tilde{A}' \mid$ labeled $\mu_m^{l'_i}$. One can observe also that each successful computation in $\mid \tilde{A}' \mid$, labeled $\mu_m^{l'}$, implies $\delta(q_0, \mu_m^{l'}) \in F$, by the construction. Then we have proven that $\mid \tilde{A} \mid (m) = l \Leftrightarrow \mid \tilde{A}' \mid (m) = l$. Then $\tilde{A}'$ realizes ν, modulus the value in 1_M. One can observe that $\nu(1_M) = \mid A' \mid (1_M) \vee \mu_{1_M}^{\nu(1_M)}$, which implies that $\nu \in \tilde{R}at(M)$, as a join of fuzzy rational sets. □

Corollary 6.2. Let L be a finite, distributive lattice and M be a finitely generated monoid. If $\nu(1_M) \in \{0, 1\}$ and $\nu \in \tilde{R}ec(M)$ then $\nu \in \tilde{R}atR(M)$.

Proof:
We apply Theorem 6.1 and Lemma 4.3 (1). □

Theorem 6.2. If L is a finite, distributive lattice and M is free, then $\tilde{R}atR(M) \subseteq \tilde{R}ec(M)$.

Proof:
Consider a code (as in [3, p. 38]) $G \subseteq M$ such that $G^* = M$ (M is freely generated by G). If $\tilde{A} = (Q, S(L^M), E, q_0, F)$ is a restricted fuzzy finite automaton, we construct an equivalent fuzzy action (automaton) as following. First we take all $\mu_m^l \in S(L^M) \setminus \mu_{1_M}^1$ which are labels of some transitions in E (implying that m can not be 1_M, due to the restrictions of $\tilde{A}$). There is a finite number of such labels. Each label μ_m^l is expanded as a "maximal" join of singletons

$$\mu_m^l = \bigvee \{\mu_{g_1}^{l_1} \cdot ... \cdot \mu_{g_t}^{l_t} / l_1 \wedge ... \wedge l_t = l, \;\; g_1 \cdot ... \cdot g_t = m\} \;\; .$$

Notice that the join is across a finite set, since the G-`representation` of m is finite and unique and since L is finite. Then we replace each transition (p, μ_m^l, q) by a new path (we add new states) corresponding to the above expansion. More precisely, for each term $\mu_{g_1}^{l_1} \cdot \mu_{g_2}^{l_2} \cdot ... \cdot \mu_{g_t}^{l_t}$ in the above expansion, we add the set of transitions $(p, \mu_{g_1}^{l_1}, p_1)$, $(p_1, \mu_{g_2}^{l_2}, p_2)$..., $(p_{t-1}, \mu_{g_t}^{l_t}, q)$, where $p_1, ..., p_{t-1}$ are new states; followed by the removal of transition (p, μ_m^l, q). To this new automaton we perform a process of "determinization", considering each label (singleton) μ_g^l as a symbol. After this process we add transitions $(p, \mu_{1_M}^1, p)$ for all states p in $\tilde{A}$. One can check that this new fuzzy finite automaton can be extended to a fuzzy monoid action, due to the freeness of M (one may need to add a "sink" state to capture the singletons which are not considered in the above expansion). □

In the following we denote by FT-NFA(Σ^*) the family of fuzzy languages accepted by `nondeterministic finite automata with fuzzy transitions`, as defined in [17]. However, we do not restrict these automata to take a finite number of values in the unit interval (they may take values in an arbitrary lattice).

Theorem 6.3. *(a Kleene theorem for fuzzy sets) Let L be a finite, distributive lattice and Σ a finite alphabet. If $\nu \in L^{\Sigma^*}$ is a fuzzy set such that $\nu(1_M) \in \{0, 1\}$, then:*

$$\nu \in \tilde{R}at(\Sigma^*) \Leftrightarrow \nu \in \tilde{R}ec(\Sigma^*) \Leftrightarrow \nu \in FT\text{-}NFA(\Sigma^*).$$

Proof:
The monoid Σ^* is a finitely generated free monoid. Assume $\nu \in \tilde{R}at(\Sigma^*)$. Since Σ^* has no 1-divisors and $\nu(1_M) \in \{0, 1\}$ we have that ν can be realized by a restricted fuzzy finite automaton, by Lemma 4.3. Then, by Theorem 6.2 we deduce that $\nu \in \tilde{R}ec(\Sigma^*)$. Finally, a restricted fuzzy finite automaton for ν is an FT-NFA. The other direction is proven in a similar way. □

Let us denote by $\tilde{R}eg(\Sigma^*)$ the family of fuzzy regular languages over an alphabet Σ, as defined in [17]. For completeness, we give here the definition, adapted to our context (the family of "crisp" regular languages will be denoted by $Reg(\Sigma^*)$).

Definition 6.1. A fuzzy set $\nu \in L^{\Sigma^*}$ is a fuzzy regular language if $\mid \nu(\Sigma^*) \mid < \aleph_0$ and $\nu^{-1}(l) \in Reg(\Sigma^*)$, for all $l \in L$.

The following results give the relation between this family and the family of fuzzy rational languages.

Lemma 6.1.

$$\tilde{R}at(\Sigma^*) \supseteq \tilde{R}eg(\Sigma^*) \;\; .$$

Proof:
Consider an arbitrary fuzzy regular language $\nu \in \tilde{Reg}(\Sigma^*)$. By the definition, for any fuzzy value $l \in L$, the set $\nu^{-1}(l)$ is a "crisp" regular language in Σ^*. Assume $\nu(\Sigma^*) = \{l_1, ..., l_n\}$ (the set is finite by definition), and consider all regular languages $\{L_i = \nu^{-1}(l_i)\}_{i \in \{1,...,n\}}$. Then there exist n DFA $A_1, ..., A_n$ such that $L_i =\mid A_i \mid$ for all $i \in \{1, ..., n\}$. It is important to notice that the languages $L_1, ..., L_n$ are mutually disjoint. We transform each DFA A_i into a fuzzy finite automaton by augmenting the fuzzy value l_i to all its transitions. We obtain n fuzzy rational sets: $\tilde{L}_1, ..., \tilde{L}_n$ with

$$\tilde{L}_i = \{(u, l_i)/u \in L_i\} \cup \{(v, 0)/v \notin L_i\}$$

Since $\tilde{Rat}(\Sigma^*)$ is closed under joins and the languages $\{\tilde{L}_i\}_i$ are mutually disjoint, it follows that $\nu = \bigvee_{i \in \{1,...,n\}} \tilde{L}_i$, which proves that $\nu \in \tilde{Rat}(\Sigma^*)$. □

Theorem 6.4. If L is totally ordered then

$$\tilde{Rat}(\Sigma^*) = \tilde{Reg}(\Sigma^*) \ .$$

Proof:
Considering the result of Lemma 6.1, it suffices to prove that $\tilde{Rat}(\Sigma^*) \subseteq \tilde{Reg}(\Sigma^*)$. Let ν be an arbitrary fuzzy rational language. Consider a fuzzy finite automaton $\tilde{A} = (Q, S(L^{\Sigma^*}), E, I, T)$ which realizes ν. In order to prove that ν is fuzzy regular, it suffices to prove that ν takes a finite number of fuzzy values and that the preimage of any fuzzy value is a regular language. We have already proven in Proposition 4.2 that $\mid \nu \mid < \aleph_0$, since ν is rational. Then it remains to prove that $\nu^{-1}(l) \in Reg(\Sigma^*)$, for all $l \in L$.

Take an arbitrary $l \in L$. Since ν is rational, by Proposition 4.2 we infer that $\mid \nu(\Sigma^*) \mid < \aleph_0$, and let $\nu(\Sigma^*) = \{l_1, ..., l_n\}$ with $l_1 > ... > l_n$. If $l \notin \{l_1, ..., l_n\}$ then clearly $\nu^{-1}(l) = \emptyset \in Reg(\Sigma^*)$. It remains to consider the case when $l \in \{l_1, ..., l_n\}$.

Recall the notation $\nu^{\geq l}$, used in Proposition 4.3. We have that $\nu^{-1}(l_1) = supp(\nu^{\geq l_1})$ and for $i > 1$ we have $\nu^{-1}(l_i) = supp(\nu^{\geq l_i}) \setminus supp(\nu^{\geq l_{i-1}})$. Since $\nu \in \tilde{Rat}(\Sigma^*)$, by Proposition 4.3 we infer that $\nu^{\geq l_i} \in \tilde{Rat}(\Sigma^*)$. Then, by Corollary 4.1 (L has no 0-divisors, being totally ordered), we obtain $supp(\nu^{\geq l_i}) \in Rat(\Sigma^*) = Reg(\Sigma^*)$. Summing up, we have that $supp(\nu^{\geq l_i}) \in Reg(\Sigma^*)$ for all $i \in \{1, ..., n\}$ and $supp(\nu^{\geq l_i}) \setminus supp(\nu^{\geq l_{i-1}}) \in Reg(\Sigma^*)$, for all $i \in \{2, ..., n\}$, by the closure of regular languages under set difference. This implies that $\nu^{-1}(l_i) \in Reg(\Sigma^*)$, for all $i \in \{1, ..., n\}$. □

It is worth mentioning that the finiteness of L, in Theorem 6.1, 6.2 and 6.3 is not as restrictive as it seems (at least it can not be viewed as a restriction to the cases studied in the past). Most of the previous results concerning fuzzy regular languages and finite automata with fuzzy transitions are built upon the assumption that the set of fuzzy values is finite. For example, although a finite automaton with fuzzy transitions (or states) is said to take values in the unit interval, it actually takes a finite set of values which can be viewed as a finite, totally ordered lattice which contains all the transition values of the automaton. In a similar manner one can argue that the condition in Theorem 6.4, that L is totally ordered, is not a particularization of previous results (since the unit interval is already totally ordered).

We end this section by observing that if L is chosen to be the Boolean lattice $(\{0, 1\}, \leq, \wedge, \vee, 0, 1)$ with $0 < 1$, we obtain results pertinent to the crisp theory of rational and recognizable sets, and regular languages.

7. Application: Fuzzy Finite Transducers

In this section we intend to use the framework developed so far to a particular class of machines, namely finite transducers. For the sake of completeness, we first give a result concerning fuzzy recognizable relations, by proving a weak version of Mezei's characterization theorem. Then, we focus on fuzzy rational transductions (relations on words), fuzzy finite transducers and their properties. We start with a definition of fuzzy rational and recognizable relations.

Definition 7.1. Let M, M' be arbitrary monoids and L be a c.d.c lattice. We denote

- $L^{M\times M'}$: the family of L-`fuzzy relations` on M and M';
- $\tilde{R}at(M \times M')$: the family of L-`fuzzy rational relations`;
- $\tilde{R}ec(M \times M')$: the family of L-`fuzzy recognizable relations`.

Let us define a Cartesian product of fuzzy recognizable sets as following:

$\times : L^M \times L^{M'} \to L^{M\times M'}$, given by

$$\forall \nu_1 \in L^M, \nu_2 \in L^{M'} : \nu_1 \times \nu_2 \in L^{M\times M'},\ (\nu_1 \times \nu_2)(m_1, m_2) = \nu_1(m_1) \wedge \nu_2(m_2)\ .$$

Theorem 7.1. (a Mezei representation for fuzzy recognizable relations) Let M, M' be monoids and L be a c.d.c. lattice. Then $\nu \in \tilde{R}ec(M \times M')$ only if it can be expressed as

$$\nu = \bigvee_{i=1}^{n} \varphi_i \times \xi_i\ ,$$

where n is a positive integer, $\varphi_i \in \tilde{R}ec(M)$ and $\xi_i \in \tilde{R}ec(M')$ $\ \forall 1 \le i \le n$.

Proof:
Since ν is a fuzzy recognizable set, there exists a finite monoid N, $P \subseteq N$ and a monoid morphism $h : S(L^{M\times M'}) \cup \{\mu_\emptyset\} \to N$ such that $\nu = \bigvee h^{-1}(P)$. Denote by "1" the unity of either M or M' (the choice will be established by the context). For any $n \in N$ we define two fuzzy sets $\varphi_n \in L^M$ and $\xi_n \in L^{M'}$ as following:

$$\varphi_n = \bigvee \{\mu^l_m / h(\mu^l_{(m,1)}) = n\},\ \ \xi_n = \bigvee \{\mu^l_m / h(\mu^l_{(1,m)}) = n\}\ .$$

Let us first observe that $\varphi_n \in \tilde{R}ec(L^M)$ and $\xi_n \in \tilde{R}ec(L^{M'})$. Indeed, consider the homomorphism $h' : S(L^M) \cup \{\mu_\emptyset\} \to N$, given by $h'(\mu^l_m) = h(\mu^l_{(m,1)})$ and $h'(\mu_\emptyset) = h(\mu_\emptyset)$. One can observe that $\varphi_n = \bigvee h'^{-1}(\{n\})$. By the definition of fuzzy recognizable sets, this proves that $\varphi_n \in \tilde{R}ec(L^M)$ and a similar argument may be used to show that $\xi_n \in \tilde{R}ec(L^{M'})$. Furthermore, for any $n_1, n_2 \in N$:

$$(\varphi_{n_1} \times \xi_{n_2})(m_1, m_2) = \varphi_{n_1}(m_1) \wedge \xi_{n_2}(m_2) =$$

$$= \Big(\bigvee \{l / h(\mu^l_{(m_1,1)}) = n_1\}\Big) \wedge \Big(\bigvee \{l / h(\mu^l_{(1,m_2)}) = n_2\}\Big) =$$

(here we use the transfinite distributivity law)

$$= \bigvee \left\{ l_1 \wedge l_2 / h(\mu^{l_1}_{(m_1,1)}) = n_1 \text{ and } h(\mu^{l_2}_{(1,m_2)}) = n_2 \right\} \ .$$

We now observe that for arbitrary $m_1 \in M, \ \ m_2 \in M'$ we have

$$\bigcup_{n_1 \cdot n_2 \in P} \left\{ l_1 \wedge l_2 / h(\mu^{l_1}_{(m_1,1)}) = n_1 \text{ and } h(\mu^{l_2}_{(1,m_2)}) = n_2 \right\} =$$
$$= \bigcup_{n_1 \cdot n_2 \in P} \left\{ l / h(\mu^{l}_{(m_1,m_2)}) = n_1 \cdot n_2 \right\} \ .$$

Indeed, let us prove this equality by double inclusion. If we have $l = l_1 \wedge l_2$ with $h(\mu^{l_1}_{(m_1,1)}) = n_1$ and $h(\mu^{l_2}_{(1,m_2)}) = n_2$ with $n_1 \cdot n_2 \in P$ then $h(\mu^{l_1}_{(m_1,1)} \cdot \mu^{l_2}_{(1,m_2)}) = h(\mu^{l}_{(m_1,m_2)}) = n_1 \cdot n_2$. This proves the inclusion to the right ($\subseteq$). Conversely, let $h(\mu^{l}_{(m_1,m_2)}) = n_1 \cdot n_2 \in P$. Denote $t_1 = h(\mu^{1}_{(m_1,1)})$ and $t_2 = h(\mu^{l}_{(1,m_2)})$. Then $l = 1 \wedge l$, and $h(\mu^{1}_{(m_1,1)}) = t_1$, $h(\mu^{l}_{(1,m_2)}) = t_2$ and $t_1 \cdot t_2 \in P$ since $t_1 \cdot t_2 = n_1 \cdot n_2$. This proves the inclusion to the left ($\supseteq$).

We then have

$$\left[\bigvee_{n_1 \cdot n_2 \in P} \varphi_{n_1} \times \xi_{n_2} \right](m_1, m_2) = \bigvee_{n_1 \cdot n_2 \in P} \left[(\varphi_{n_1} \times \xi_{n_2})(m_1, m_2) \right] =$$
$$= \bigvee_{n_1 \cdot n_2 \in P} \left\{ \bigvee \left\{ l_1 \wedge l_2 / h(\mu^{l_1}_{(m_1,1)}) = n_1 \text{ and } h(\mu^{l_2}_{(1,m_2)}) = n_2 \right\} \right\} =$$

(here we use generalized commutativity in L)

$$= \bigvee_{n_1 \cdot n_2 \in P} \left\{ l_1 \wedge l_2 / h(\mu^{l_1}_{(m_1,1)}) = n_1 \text{ and } h(\mu^{l_2}_{(1,m_2)}) = n_2 \right\} =$$

(we use the previous observation)

$$= \bigvee_{n_1 \cdot n_2 \in P} \left\{ l / h(\mu^{l}_{(m_1,m_2)}) = n_1 \cdot n_2 \right\} = \bigvee \left\{ l / h(\mu^{l}_{(m_1,m_2)}) \in P \right\} =$$
$$= \bigvee \left\{ \mu(m_1, m_2) / h(\mu) \in P \right\} = \left[\bigvee h^{-1}(P) \right](m_1, m_2) \ ,$$

in other words, we have proven that

$$\bigvee_{n_1 \cdot n_2 \in P} \varphi_{n_1} \times \xi_{n_2} = \bigvee h^{-1}(P) = \nu \ .$$

Since N is finite, the join on the left side of the equality is a finite join, hence the conclusion follows. □

Remark 7.1. Unlike the case of "crisp" recognizable sets, so far we have no reason to believe that the converse of the above theorem holds. The difficulty of proving the converse reside in the fact that we have no results concerning the closure of fuzzy recognizable sets under inverse morphisms and under meets.

Yet again, let L be a c.d.c. lattice, and let X, Y be finite alphabets.

Definition 7.2. An L-`fuzzy rational transduction` is an element of $\tilde{R}at(X^* \times Y^*)$. An L-`fuzzy recognizable transduction` is an element of $\tilde{R}ec(X^* \times Y^*)$.

A direct consequence of Theorem 6.1 is the following corollary.

Corollary 7.1. If L is finite, any L-fuzzy recognizable transduction is rational.

Proof:
It follows from the fact that $X^* \times Y^*$ is finitely generated, which allows us to apply Theorem 6.1. □

For an arbitrary L-fuzzy transduction τ we use the notation $\tau : X^* \tilde{\rightarrow} Y^*$, which, in fact, implies the function $\tau : X^* \rightarrow L^{Y^*}$. Such a function can be viewed as a fuzzy relation $\tau' \in L^{X^* \times Y^*}$, with $\tau'(u, v) = \tau(u)(v)$. In the following we will not distinguish between τ and τ'.

Remark 7.2. The above convention is in line with the fact that $L^{X^* \times Y^*}$, $(L^{Y^*})^{X^*}$ and $(L^{X^*})^{Y^*}$ are set-theoretic isomorphic.

If $R \subseteq X^*$ is an arbitrary language, the image of R through τ is the L-fuzzy set $\tau_R \subseteq L^{Y^*}$ given by

$$\tau_R(v) = \bigvee_{u \in R} \tau(u, v) \ ,$$

with the usual convention that $\bigvee \emptyset = 0$. We intend to prove that, as in the classical theory, a fuzzy rational transduction preserve rationality. In our context this translates to: the image of a regular language through a fuzzy rational transduction is a fuzzy rational language. This property will be proven later, by using fuzzy transducers (see Proposition 7.1).

Given two fuzzy transductions $\tau : X^* \tilde{\rightarrow} Y^*$ and $\tau' : Y^* \tilde{\rightarrow} Z^*$, we define their composition $\tau' \circ \tau : X^* \tilde{\rightarrow} Z^*$, given by

$$(\tau' \circ \tau)(u, w) = \bigvee_{v \in Y^*} \tau(u, v) \wedge \tau'(v, w) \ . \tag{3}$$

In Proposition 7.2 we will prove that the family of fuzzy rational transductions is closed under composition. The proof will be constructive, and will show that cascades of fuzzy transducers (the output tape of one transducer is the input tape of another) may be replaced by equivalent, compact fuzzy transducers. For that we must first define fuzzy transducers and prove that they represent exactly the family of fuzzy rational transductions.

Definition 7.3. A `fuzzy finite transducer` is a tuple $\tilde{T} = (Q, X, Y, E, q_0, F)$, where

1. X, Y are `finite alphabets`;
2. Q is a finite `set of states`;
3. $q_0 \in Q$ is an `initial state`, $F \subseteq Q$ is a set of `final states`; and
4. $E \subseteq Q \times X^* \times Y^* \times L \times Q$ is a finite `set of transitions`.

A computation in $\tilde{T}$ is a sequence in E^+ of the form

$$c = (p_1, u_1, v_1, l_1, p_2)(p_2, u_2, v_2, l_2, p_3)...(p_k, u_k, v_k, l_k, p_{k+1}) \ .$$

Then c is a successful computation if and only if $p_1 = q_0$ and $p_{k+1} \in F$. The computation c has two components:

1. the `label` of c, denoted by $\mid c \mid$, is the pair (u, v) where $u = u_1...u_k$ and $v = v_1...v_k$;

2. the `value` of c, denoted by $\sharp c$, is the fuzzy value $l = l_1 \wedge ... \wedge l_k$.

The fuzzy transduction realized by $\tilde{T}$ is $\mid \tilde{T} \mid : X^* \tilde{\to} Y^*$ given by

$\mid \tilde{T} \mid (u, v) = \bigvee \left\{ l / \exists c \text{ successful computation: } l = \sharp c \text{ and } (u, v) = \mid c \mid \right\}$,

or, in its other form,

$\mid \tilde{T} \mid (u) = \bigvee \left\{ \mu_v^l / \exists c \text{ successful computation: } l = \sharp c \text{ and } (u, v) = \mid c \mid \right\}$,

where by μ_v^l we denote, as usual, a singleton in L^{Y^*}.

Example 7.1. We present a generic, simple example which shows a possible use of fuzzy transducers, namely `classification` and `processing`. Suppose we want to classify an incoming traffic of words in Σ^*, with $\Sigma = \{a, b\}$, based on a given pattern/criterion. This pattern can be given by a "crisp language" L, in our case we use the simple case $L = a^*$. We are not interested in deciding whether an input word is or is not in L, but rather we want to asses "how far" from the pattern a^* an input word is. We have the following criteria, given by four levels of assessment:

1. "exact", for a word in L;

2. "close", for a word which contains only one "irregularity" from the pattern, i.e., it has exactly one occurrence of b's;

3. "far", for a word which contains two b's; and

4. "remote", for a word which has more than two irregularities.

Our levels of assessment are in total order (hence we have a finite lattice):

$$\text{exact} > \text{close} > \text{far} > \text{remote} \ .$$

We also want to perform some processing of the input, in our case we want to simply delete the "irregularities", i.e., the occurrences of b in the input. The fuzzy transducer in Figure 1 realizes these simple tasks.

Each transition is labeled by "i/o, l", where "i" is an input word, "o" is an output word and "l" is the associated fuzzy value. All states are final and the leftmost state is initial. For the input word $ababa$

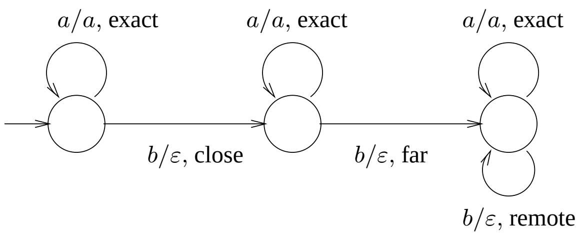


Figure 1. A simple example of a fuzzy finite transducer.

the transducer outputs the word aaa and classifies the input as "far". Notice that the transducer is input-deterministic and has no ε-input transitions. In practice the fuzzy transducers are more complex and the design of such transducers starting from a practical problem and a realistic set of criteria is an elaborated matter with no clear guidelines so far.

Aside from "classification and processing", fuzzy transducers may have two other uses/interpretations:

– "comparison, or pattern recognition": in this "mode", a fuzzy transducer is viewed as a two-tape fuzzy automaton. One tape holds a predefined pattern - which must be recognized, and the second tape holds an input sample. The automaton scans the tapes sequentially and at the end answers with a fuzzy value which evaluates to which extent the pattern has been "recognized" in the sample (it remains to investigate how learning capabilities may be added to these machines).

– "fuzzy automata switch": in this mode, a fuzzy transducer is viewed as a collection (possible infinite) of fuzzy rational sets. The input tape acts as a selector: if τ is the overall transduction realized by this machine, then an input word u (or a collection of input words) switches the machine to "simulate" a fuzzy automaton which realizes $\tau(u)$ (this view will be legitimated by Proposition 7.1).

It is important to note that these modes in which a fuzzy transducer may operate are indistinguishable from the theoretical point of view, and only specific application may reflect these uses (it is indeed a matter of interpretation).

Theorem 7.2. A fuzzy transduction over X and Y is rational if and only if it is realized by a fuzzy finite transducer.

Proof:
One can observe that a fuzzy finite transducer is exactly a fuzzy finite automaton over the monoid $X^* \times Y^*$. □

Proposition 7.1. Let $\tau \in \tilde{R}at(X^* \times Y^*)$ and $R \in Reg(X^*)$. Then $\tau_R \in \tilde{R}at(Y^*)$.

Proof:
In other words, we prove that the image of a regular language through a fuzzy rational transduction is a fuzzy rational language. By definition we have that $\tau_R \in L^{Y^*}$, given by

$$\tau_R(v) = \bigvee_{u \in R} \tau(u, v) \quad .$$

Let R be accepted by a DFA $A = (Q, X, \delta, q_0, F)$ and τ be represented by the fuzzy transducer $\tilde{T} = (Q', X, Y, E, p_0, F')$. Without loss of generality, we can assume that the transitions of $\tilde{T}$ have as input label either a symbol, or ε. We construct a fuzzy finite automaton $\tilde{B}$ such that $\mid \tilde{B} \mid = \tau_R$. The set of states of B consists of pair of states, in $Q' \times Q$, with initial state (p_0, q_0). In an initial stage, a transition in B will be of the form

$$\Big((p,q), a, y, l, (p', q')\Big), \text{ if } \delta(q,a) = q' \text{ and } (p, a, y, l, p') \in E$$

and

$$\Big((p,q), \varepsilon, y, l, (p', q)\Big), \text{ if } (p, \varepsilon, y, l, p') \in E \ .$$

The final states of $\tilde{B}$ will be $F' \times F$. One can observe that at this stage $\mid \tilde{B} \mid$ is a restriction of $\mid \tilde{T} \mid$ to the domain $R \times Y^*$; nevertheless it is still a fuzzy rational transduction. We now take all transitions of $\tilde{B}$ and we erase their input labels: for example a transition $((p,q), a, y, l, (p', q'))$ becomes $((p,q), y, l, (p', q'))$. It can be checked that after this transformation $\tilde{B}$ becomes a fuzzy automaton and $\mid \tilde{B} \mid = \tau_R$. □

Remark 7.3. Notice that although in the above proposition we could perform the product of a transducer and an automaton, this is not always possible in the case of two transducers. Indeed, if it were, rational sets would be closed under intersection, fact which is known to be not true. However, in the proof of the next proposition we show how a "semi"-product of transducers may be constructed.

Proposition 7.2. $\tilde{R}at(X^* \times Y^*)$ is closed under composition.

Proof:
The composition of fuzzy rational transductions has been defined by the relation (3). It can be shown that any fuzzy rational transduction can be realized by a transducer with transitions of the form $(p, \varepsilon, b, l, q)$ or $(p, a, \varepsilon, l, q)$, or $(p, \varepsilon, \varepsilon, l, q)$, with $a \in X \cup \{\varepsilon\}$ and $b \in Y \cup \{\varepsilon\}$. We assume by convention that such transducer has loops of the form $(p, \varepsilon, \varepsilon, 1, p)$ for all their states.

Let $\tilde{T}_1 = (Q, X, Y, E, p_0, F)$ and $\tilde{T}_2 = (Q', Y, Z, E', q_0, F')$ be transducers as defined above. We construct a transducer $\tilde{B}$ such that $\mid \tilde{B} \mid = \mid \tilde{T}_1 \mid \circ \mid \tilde{T}_2 \mid$. The set of states of $\tilde{B}$ is $Q \times Q'$, its final states are $F \times F'$ and its transitions are formed as following. For any pair of transitions

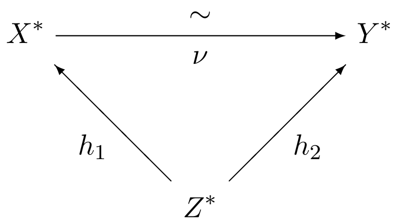
(p, x, y, l, q) in E, with $x \in X \cup \{\varepsilon\}$ and $y \in Y \cup \{\varepsilon\}$,

(p', y, z, l', q') in E', with $z \in Z \cup \{\varepsilon\}$,

we add the transition $\big((p,p'), x, z, l \wedge l', (q, q')\big)$ in $\tilde{B}$. The initial state of $\tilde{B}$ is (p_0, q_0) and the set of final states of $\tilde{B}$ is $F \times F'$. It is easy to check that indeed $\tilde{B}$ realizes the composition of $\mid \tilde{T}_1 \mid$ and $\mid \tilde{T}_2 \mid$, by invoking the (generalized) associativity and commutativity laws in L. □

Theorem 7.3. (a Nivat representation of fuzzy rational transductions)
A fuzzy transduction $\nu \in L^{X^* \times Y^*}$ is rational only if there exist an alphabet Z, a fuzzy set $\tilde{R} \in \tilde{R}at(Z^*)$ and two monoid morphisms $h_1 : Z^* \to X^*$ and $h_2 : Z^* \to Y^*$ such that

$$\nu(u,v) = \bigvee_{z \in h_2^{-1}(v)} \Big[\big(h_1^{-1}(u) \times L\big) \cap \tilde{R}\Big](z) \ .$$



$$\tilde{R} \in \tilde{R}at(Z^*)$$

Figure 2. A Nivat representation of fuzzy rational transductions.

Proof:
We give an informal proof - the details are straightforward. Diagram 2 is useful for this proof. We assume that X and Y are disjoint (otherwise, we can apply a coloring). If ν is rational, there exists a fuzzy transducer realizing the function $(u, v) \to \nu(u, v)$. Since X and Y are disjoint, we can view this transducer as a fuzzy automaton with transitions labeled by words in $(X \cup Y)^*$ (we concatenate the input label with the output label on each transition). Denote $Z = X \cup Y$, h_1 is the projection of Z^* into X^* and h_2 is the projection of Z^* into Y^*. This fuzzy automaton accepts a fuzzy rational set $\tilde{R} \in \tilde{R}at(Z^*)$. It now suffices to observe that $\left(h_1^{-1}(u) \times L\right) \cap \tilde{R}$ represents all the computations (and their fuzzy values) of the initial transducer when u is read from the input tape. Notice that, as expected, this expression denotes a fuzzy set. Since many of these computations give a same output, we select all computations with a same output and perform their join: $\bigvee_{z \in h_2^{-1}(v)} \Big[\dots \Big](z)$ represents the join of all computations which give the same output v. $\Box$

Remark 7.4. In proving the converse of the above theorem we encounter the following difficulty. One can observe that $\left(h_1^{-1}(u) \times L\right) \cap \tilde{R}$ is the restriction of $\tilde{R}$ to $h^{-1}(u)$ – which is a regular language – hence it is a fuzzy rational set. Let us denote it by τ_u, being dependent on u. Denote also $R_v = h_2^{-1}(v)$, which is a regular language. In order to prove the converse of Theorem 7.3 one should prove that the transduction $\nu : X^* \times Y^*$ given by

$$\nu(u, v) = \tau_u(R_v)$$

is rational, with τ_u rational fuzzy sets and R_v regular languages. The conditions in which this property holds are left for further investigation.

Most of the classical theory on finite transducers can readily be ported to fuzzy finite transducers. For example, there exists a normal form (already mentioned) and a matrix representation of fuzzy transducers as well.

8. Conclusion and Further Work

In this paper we proposed a different approach for the study of fuzzy sequential machines. Unlike previous attempts, we have defined and studied fuzzy rational and recognizable sets in arbitrary monoids, and in doing so, we relied on completely distributive complete lattices. Beside outlining an alternative framework for the study of fuzzy sequential machines, we investigated these families (of fuzzy rational and recognizable sets), per se, from the theoretical point of view. For future work, we have left unaddressed a number of questions which require further investigation. A few of them are presented in the following.

In our discussion about fuzzy rational sets, we have proven that the image of such sets must be finite. However, we have not addressed properties of the preimage of particular subsets of the lattice. More precisely, let $f \in \tilde{R}at(M)$ and $A \subseteq L$. What can we say about $f^{-1}(A)$ when, for example, A is finite, or A is a sublattice of L, or L is totally ordered, finite, etc. . A similar question can be asked for fuzzy recognizable sets. For fuzzy recognizable sets we have not produced any results concerning the finiteness of their image.

Proposition 4.3 shows that any step fuzzy set of a fuzzy rational set is rational when the lattice is totally ordered. Does the proposition hold when the order in L is partial? Does its converse hold? If not, in what particular circumstances does it hold?

Another matter for further work is to investigate possible closure properties of fuzzy recognizable sets under meets, inverse fuzzy morphisms, complement and difference. For complement, we must require that L be complemented (i.e., $\forall l \in L, \exists l' \in L : l \vee l' = 1, l \wedge l' = 0$; consequently the complement is unique) which implies that also L^M is complemented.

The results of Section 6 hold for particular lattices. We have found that at this stage, these results suffice to make the point that well established results can be obtained in our framework. However, it is worth investigating whether these results hold for more general lattices.

In Remark 5.3 we mentioned fuzzy transition monoids. We believe that these monoids together with fuzzy syntactic monoids (which can be defined similarly) deserve further attention. Furthermore, we mentioned that it is not clear whether we can replace $S(L^M) \cup \{\mu_\emptyset\}$ with $FS(L^M)$ in either Definition 5.1 or Definition 5.3, a fact that may very well lead to a different family of recognizable sets. This matter deserves further attention as well.

At the beginning of Section 7 we gave a representation of fuzzy recognizable relations. In what circumstances does the converse of Theorem 7.1 hold? At the end of the same section we gave a representation of fuzzy rational transductions. It also remains to investigate in what circumstances the converse of Theorem 7.3 holds: see Remark 7.4.

We believe that there is still a great deal to explore within our framework. For example, we have not accommodated yet the family of fuzzy sets realized by FS-NFA (NFA with fuzzy states) or FT-DFA (DFA with fuzzy transitions) or FS-DFA. As another example, we believe that algebraic sets in arbitrary monoids, which have not received the deserved attention in the past, are equally suited to a similar process of fuzzification.

Finally, as an application of our framework, it is worth exploring fuzzy finite transducers from the perspective proposed in this paper.

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