On Implementing Recognizable Transductions [⋆]

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Abstract. Recognizable transductions constitute a proper subclass of rational transductions, characterized by the well-known Mezei's Theorem. We propose a family of transducers which reflect accurately this characterization, and we study their properties and their algorithmic aspects. We base our construct on the observation that there is a connection between recognizable transductions and languages consisting of edit strings. More specifically, we define a saturated transducer to be a transducer with the property that accepts all possible edit strings corresponding to each accepted pair of words, when viewed as an automaton over the edit alphabet. We revisit the theory behind recognizable transductions from the point of view of saturated transducers, and we give constructive proofs as well as descriptional complexities for their closure properties. For example, we give a novel and rather complex algorithm for constructing a saturated transducer for the concatenation of two saturated transductions. Finally, we discuss the natural role of these objects in edit distance problems. Perhaps the relevance of this work lies more in its point of view and initiative rather than in any particular result.

Keywords: Transducer; Recognizable transduction; Saturated transducer; Edit string; Edit language; Edit distance

1 Introduction

Recognizable transductions constitute a well known, proper subclass of rational transductions, the latter being the class of all binary relations of words realized by finite transducers. A well known characterization of recognizable transductions is given by Mezei's theorem [Eilenberg, 1974, note at p. 75]. Until now, however, there has been no characterization of recognizable transductions by a well-defined, special subclass of transducers. In this work we observe that there is an intimate connection between recognizable transductions and edit languages, that is, languages consisting of edit strings.

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An edit string (or string alignment) is a special word consisting of edit operations, and describes the sequence of changes (substitutions, insertions and deletions of symbols) that can transform a word into another word. Edit strings can be used to define formally concepts related to distances between words [Sankoff, Kruskal, 1999] and, in fact, recently ([Kari, Konstantinidis, 2002], [Kari et al., 2003]) there have been systematic treatments of edit languages (also called e-systems) in the sense of language theory. In the context of word and language distances, the main difference between a transduction and an edit language is that the latter describes the exact changes that are permitted in transforming words to words, whereas the former describes the result of these transformations.

This paper introduces the concept of saturated transducer and observes that this concept constitutes a natural point of connection between recognizable transductions and edit languages. More specifically, a saturated transducer is a transducer with the property that, for every pair of words that it realizes, the transducer, when viewed as an automaton over the edit alphabet, accepts all possible edit strings transforming the first word of the pair into the second one. The starting point of our study is marked by the observation that the class of recognizable transductions coincides with that of transductions realized by saturated transducers. Thus, saturated transducers become a convenient tool for implementing recognizable transductions. We provide several basic results on saturated transducers and discuss their use in edit distance problems. In the next paragraph we give a short overview of our paper.

The paper is organized as follows. In the next section, we provide the formal definitions about rational transducers, recognizable transductions and edit languages. Section 3 introduces saturated transducers and discusses several basic operations on these objects. These operations can be used to provide *constructive* proofs (by means of saturated transducers) of closure properties of recognizable transductions, such as intersection, composition and concatenation. Moreover, the descriptional complexity of these operations can be used to evaluate the time complexity of algorithms utilizing them. Section 4 contains the proof of the equivalence of saturated and recognizable transductions. This proof is constructive in the following sense. Given a tuple consisting of an even number of finite automata – according to Mezei's theorem such a tuple specifies a recognizable transduction – there is an effective construction of a saturated transducer realizing the transduction specified by the tuple. Moreover, there is a constructive proof for the converse problem. In Section 5 we elaborate on the use of saturated transducers in problems related to the edit distance of words and languages. The method here is not new, in the sense that certain examples of transducers have already been used for such problems, but we believe that the method is better understood with our systematic study of saturated transducers. Our observations in this context lead us to the question of whether the transduction consisting of all pairs of distinct words of some regular language is recognizable. We show that it is not recognizable in the case of infinite languages. Finally, Section 6 contains a few concluding remarks.

2 Preliminary Notions and Notations

We assume known basic notions of finite automata: DFA (deterministic finite automaton), NFA (nondeterministic finite automaton) and ε-NFA (NFA with ε-transitions) – a review of these terms can be found in [Hopcroft and Ullman, 1979] or [Yu, 1997]. We also assume known the basic notions of semigroup (monoid) theory [Howie, 1976] and of rational and recognizable sets in arbitrary monoids ([Eilenberg, 1974]). We recall that the class of monoids is closed under cartesian product.

Let $(M,\cdot,1_M)$ be a monoid which consists of a carrier set M equipped with a binary associative operation "$\cdot$" and an unit "1_M". By $Rat(M)$ we denote the family of rational subsets of M and by $Rec(M)$ we denote the family of recognizable subsets. To avoid cluttering the paper, we omit the folkloric results about rational and recognizable sets. The reader is encouraged to consult [Berstel, 1979] on this matter.

If X and Y are finite alphabets (nonempty sets of symbols), we denote by X^* and Y^* their freely generated monoids. Any element of X^* or Y^* is called a `word`, i.e., a finite string of symbols. We denote by ε the word with no symbols, i.e., `the empty word`. By $X^* \times Y^*$ we understand the direct product of the monoids X^* and Y^*, i.e., the monoid of word relations. We will use the terms "word relation" and "transduction" interchangeably. Notice that this monoid is finitely generated, in the sense that there exists a finite subset G, called a set of generators, such that $G^* = X^* \times Y^*$ (indeed, take $G = (X \times \{\lambda\}) \cup (\{\lambda\} \times Y)$). Notice also that $X^* \times Y^*$ is not necessarily a free monoid, in the sense that it may not exist a set of generators which generate each element of the monoid in a unique way (for example, G – above – may generate an element in more than one way: $(x,y) = (x,\lambda)\cdot(\lambda,y) = (\lambda,y)\cdot(x,\lambda)$. As a consequence of McKnight's theorem ([McKnight, 1964]) we have that

$$Rec(X^* \times Y^*) \subseteq Rat(X^* \times Y^*) \ ,$$

inclusion which is strict in general. For example, the transduction $\{(a^i,b^i) \mid i \geq 0\}$ can be proven to be rational without being recognizable.

In $X^* \times Y^*$, recognizable and rational sets may be specified by finite state machines. For example, each rational transduction τ is represented by some finite transducer $T = (Q,X,Y,\Delta,q_0,F)$, where

1. Q is a finite set of states;
2. $\Delta \subseteq Q \times X^* \times Y^* \times Q$ is a finite set of transitions; the `label` of a transition (p,x,y,q) is the pair (x,y);
3. q_0 is an initial state, $F \subseteq Q$ is a set of final states;
4. a `successful computation` of T is a sequence
$$c = (q_0,x_1,y_1,q_1),....,(q_{n-1},x_n,y_n,q_n) \ ,$$
where $(q_{i-1},x_i,y_i,q_i) \in \Delta$ for all $i \in \{1,...,n\}$, and $q_n \in F$. The `tag` of c, denoted by $\mid c \mid$ is the pair of words $(x_1...x_n, y_1...y_n)$;
5. $\tau = \mid T \mid = \{(u,v) \mid (u,v) = \mid c \mid$, for some successful computation $c\}$.

The alphabet X is sometime called the input alphabet and Y the output alphabet. It has been shown that a transducer with labels in $X^* \times Y^*$ is equivalent with a transducer having labels only in $(X \cup \{\varepsilon\}) \times (Y \cup \{\varepsilon\})$. We bring this observation further, by noticing that one can eliminate all "null" transitions, i.e., transitions of the form $(\varepsilon,\varepsilon)$. However, for the sake of formalism, it is useful to consider all states having null loops, i.e., we have a transition $(p,\varepsilon,\varepsilon,p)$ for each state p of the transducer. Then we give the following definition:

Definition 1. *A transducer is in* `standard form` *if it has transitions with labels in* $(X\cup\{\varepsilon\})\times(Y\cup\{\varepsilon\})$ *and each state has an* $(\varepsilon,\varepsilon)$*-loop to itself.*

Then each rational transduction is realized by a transducer in standard form. Note that our definition is slightly different from that of "elementary machine" in [Elgot and Mezei, 1965] (where ε readings are not allowed) or normalized transducer mentioned in [Berstel, 1979, Corollary 6.2, p. 79] (in addition we require the presence of $(\varepsilon,\varepsilon)$-loops). This justifies our different nomenclature.

We define the `size` of a finite state machine M in general (hence of a transducer, in particular), as being the number of all its states together with all its transitions, and we denote it by size(M). In complexity matters we will consider the size of the input alphabet of M to be constant.

In the case of recognizable transductions, one can use Mezei's characterization (as in [Eilenberg, 1974, §3.12, Prop. 12.2 & note at p. 75]) to represent a transduction $\tau\in Rec(X^*\times Y^*)$ by a tuple of finite automata $(A_1,B_1,\ldots,A_n,B_n)$ such that

$$\tau=\bigcup_{i=1}^{n}\mathscr{L}(A_i)\times\mathscr{L}(B_i)\ ,$$

where by $\mathscr{L}(A)$ we understand the language accepted by the automaton A (automata A_i are over X and automata B_i are over Y). We say that any recognizable transduction is a finite union of blocks (a block is a direct product of two regular languages).

As a general observation, not much effort has been spent on the study of finite machines designed to precisely accept recognizable sets. Our paper addresses this issue and reveals the close connection between recognizable sets and edit languages – defined in the following.

Let E be the set consisting of all elements of the form (a/ε), (ε/b) and (a/b), where $a\in X$ and $b\in Y$. We treat the elements of E as symbols which denote the so-called `edit operations`: deletion, insertion and substitution (for example, the meaning of operation "(a/ε)" is "deletion of a"). Then, by E^* we denote the language of `edit strings`, i.e., the language of words over the alphabet E. The empty edit string over E will be denoted by $(\varepsilon/\varepsilon)$.

Edit strings can implement transductions as the following example shows: if $X=Y=\{a,b\}$ then each of the following edit strings defines the transduction $\{(aba,bab)\}$:

$$e=(a/b)(b/a)(a/b)$$

$$f=(a/\varepsilon)(b/b)(a/a)(\varepsilon/b)$$

$$g=(a/\varepsilon)(b/\varepsilon)(a/\varepsilon)(\varepsilon/b)(\varepsilon/a)(\varepsilon/b)$$

We say that each of the edit strings e, f and g "transforms the word *aba* into the word *bab*". Notation wise, we use the lowercase letters e, f, g to denote edit strings.

In this paper we are interested in sets of edit strings, i.e., in `edit languages`. Such languages are simply subsets of E^*.

3 Saturated Transducers: Definition and Basic Results

The notion of saturated transducer originates in the simple idea that a computation of a finite transducer in standard form defines both a pair of words and an unique edit string which transforms a word into another one.

Let X and Y be input and output alphabets and E be the alphabet of edit operations over X and Y. Across this paper we will frequently refer to the following monoid homomorphism:

$$h : E^* \to X^* \times Y^* \ ,$$

given by: $h(\varepsilon/\varepsilon) = (\varepsilon,\varepsilon)$, $h(a/\varepsilon) = (a,\varepsilon)$, $h(\varepsilon/b) = (\varepsilon,b)$, $h(a/b) = (a,b)$, for all $a \in X$ and $b \in Y$. Due to its importance to our matter, we name this morphism the `edit morphism` over X and Y. It should be clear that for any pair of words (u,v), $h^{-1}(\{(u,v)\})$ consists of all edit strings that transform u into v.

Let T be a transducer over X and Y, in standard form. By $h^{-1}(T)$ we denote the finite automaton over E, obtained from T by replacing each transition label (x,y) with the symbol $(x/y) \in E \cup \{(\varepsilon/\varepsilon)\}$. Then $h^{-1}(T)$ will be an ε/ε-NFA over E.

Conversely, given a finite automaton A over E, by $h(A)$ we understand the transducer over X and Y obtained from A by replacing each transition label (x/y) with the pair $(x,y) \in X^* \times Y^*$. Then $h(A)$ is in standard form, up to the missing $(\varepsilon,\varepsilon)$-loops for each state. For easing the formalism we assume that these loops are present and that $h(A)$ is readily in standard form.

In the previous section we have defined what is meant by a successful computation (and its tag) of a transducer $T = (Q,X,Y,\Delta,q_0,F)$. Let

$$c = (q_0,x_1,y_1,q_1),....,(q_{n-1},x_n,y_n,q_n)$$

be a successful computation in T. If the transducer T is in standard form, then all pairs (x_i,y_i) can be viewed as edit operations, or null operations, and we can define the edit string corresponding to c as $||c|| := (x_1/y_1)...(x_n/y_n)$.

Notice that we have $h(||c||) = |c|$, where h is the edit morphism from X to Y. Then a transducer T in standard form defines a transduction

$$\mid T \mid = \{(u,v) \mid (u,v) = |c|, \text{ where } c \text{ is a successful computation in } T\} \ ,$$

and an edit language

$$||T|| = \{e \in E^* \mid e = ||c||, \text{ where } c \text{ is a successful computation in T}\} \ ,$$

in other words $||T|| = \mathscr{L}(h^{-1}(T))$. In the next definition we use the meaning of h as a monoid homomorphism.

Definition 2. *A transducer T in standard form is* `saturated` *if and only if*

$$h^{-1}(\mid T \mid) = ||T|| \ .$$

In other words, T is saturated if and only if for any accepted pair of words $(u,v) \in X^* \times Y^*$, and for any edit string $e \in E^*$ which transforms u into v there exists a successful computation c in T such that $||c|| = e$.

The the property of saturation can be generalized to arbitrary transducers. Indeed, let T be an arbitrary transducer, and denote by $\overset{\circ}{T}$ the transducer obtained from T by removing all its transitions which don't have labels in $(X \cup \{\varepsilon\}) \times (Y \cup \{\varepsilon\})$. We call $\overset{\circ}{T}$ the `standard component` of T.

Definition 3. *An arbitrary finite transducer T is saturated if and only if*

$$\mid \overset{\circ}{T} \mid = \mid T \mid \ \ \textit{and}\ h^{-1}(\mid \overset{\circ}{T} \mid) = ||\overset{\circ}{T}|| \ .$$

In other words, a finite transducer is saturated if it is equivalent to its standard component, which is saturated.

We keep in mind that any saturated transducer is equivalent to a saturated transducer in standard form (its standard component): if T is an arbitrary saturated transducer, one can discard all transitions with labels not in $(X \cup \{\varepsilon\}) \times (Y \cup \{\varepsilon\})$ without changing the transduction realized by T.

Remark 1. The saturation of a transducer is not a trivial property, since there may exist a non-saturated transducer in standard form equivalent to a non-saturated transducer, as the following example shows.

Example 1. Consider the transduction τ, over $\{0,1\}$ and $\{a\}$, which contains all pairs (u,v) with the value of u, as a binary word, being odd and v an arbitrary word over $\{a\}^*$. Both transducers in Fig.1 are in standard form and realize τ; however, only the transducer in Fig.1 (b) is saturated.

We say that a transduction over X and Y is saturated if and only if there exists a saturated transducer T such that $\tau = \mid T \mid$. We denote by

$$Sat(X^* \times Y^*)$$

the family of saturated transductions. Then clearly $Sat(X^* \times Y^*) \subseteq Rat(X^* \times Y^*)$.

In this section we are interested in basic operations on saturated transducers with the aim of providing *constructive* proofs for the closure properties of saturated transductions. As it turns out, many known operations on ordinary automata and transducers result in saturated transducers with no extra effort when applied on saturated transducers. For example, the standard product constructions on finite automata (possibly with ε transitions) for union and intersection would result in saturated transducers when applied on saturated transducers. The same happens in the case of the product construction for the composition of transducers.

In the following operations, the operands A_1 and A_2 are arbitrary finite automata, possibly with ε transitions (unless specified otherwise), and the operands T_1 and T_2 are arbitrary finite transducers in standard form.

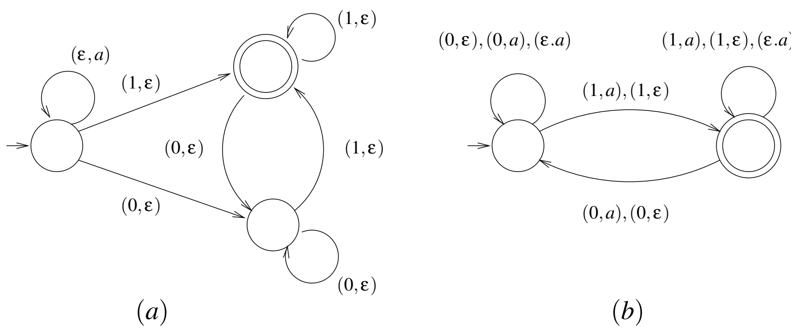

Fig. 1. Equivalent non-saturated and saturated transducers.

$\det(A_1)$**:** is the automaton obtained by determinization and completion of A_1.

$\overline{A_1}$, where A_1 is a DFA: the DFA that results when we complete A_1 and change its non-final states to final, and viceversa. It is well known that $\overline{A_1}$ accepts the complement of the language accepted by A_1 and that $\text{size}(\overline{A_1}) = O(\text{size}(A_1))$.

$A_1 \times A_2$**:** is a saturated transducer of size $O(\text{size}(A_1) \cdot \text{size}(A_2))$ such that

$$|A_1 \times A_2| = \mathscr{L}(A_1) \times \mathscr{L}(A_2).$$

The transducer $A_1 \times A_2$ consists of the transitions $((p_1,p_2),x_1,x_2,(q_1,q_2))$ for all transitions (p_1,x_1,q_1) of A_1 and (p_2,x_2,q_2) of A_2, where we assume that there is always an ε transition from each state to itself – see [Kari et al., 2003] for more details, where the notation "$A_1 \cap_E A_2$" is used instead of $A_1 \times A_2$.

$T_1 \cap T_2$**:** is the transducer in standard form that is obtained when we apply the standard product construction on automata for language intersection on the automata $h^{-1}(T_1)$ and $h^{-1}(T_2)$ over the edit alphabet E. The size of $T_1 \cap T_2$ is $O(\text{size}(T_1) \cdot \text{size}(T_2))$. Obviously, $|T_1 \cap T_2| = |T_1| \cap |T_2|$.

$T_1 \cup_\varepsilon T_2$**:** is the transducer in standard form that is obtained when we use a new start state s and two $(\varepsilon,\varepsilon)$-transitions form s to the start states of T_1 and T_2. Then

$$|T_1 \cup_\varepsilon T_2| = |T_1| \cup |T_2|$$

and $\text{size}(T_1 \cup_\varepsilon T_2) = O(\text{size}(T_1) + \text{size}(T_2))$.

$T_1 \cup T_2$**:** is the transducer in standard form that is obtained when we apply the standard product construction on automata for language union on the automata $h^{-1}(T_1)$ and $h^{-1}(T_2)$ over the edit alphabet E. The size of $T_1 \cup T_2$ is $O(\text{size}(T_1) \cdot \text{size}(T_2))$. Obviously, $|T_1 \cup T_2| = |T_1| \cup |T_2|$. The advantage of this construction over $T_1 \cup_\varepsilon T_2$ is that the automaton $h^{-1}(T_1 \cup T_2)$ is a DFA when both of $h^{-1}(T_1)$ and $h^{-1}(T_2)$ are DFAs.

$T_2 \circ T_1$**:** is the transducer in standard form that is obtained when we apply a product construction on the two given transducers for computing their composition (see for instance [Mohri, 2003]), hence,

$$|T_2 \circ T_1| = |T_2| \circ |T_1| \ .$$

Again, the size of $T_2 \circ T_1$ is $O(\text{size}(T_1) \cdot \text{size}(T_2))$. The transducer $T_2 \circ T_1$ consists of the transitions $((p_1, p_2), x, z, (q_1, q_2))$, for all pairs of transitions (p_1, x, y, q_1) in T_1 and (p_2, y, z, q_2) in T_2 and y in $Y \cup \{\varepsilon\}$ – recall here that the states of T_1 and T_2 have $(\varepsilon, \varepsilon)$-loops to themeselves, for being in standard form.

$\overline{T}_1$**:** is the transducer $h(\overline{\det(h^{-1}(T_1))})$ such that

$$|\overline{T}_1| = \overline{|T_1|} \ .$$

If $h^{-1}(T_1)$ is an NFA then the size of $\overline{T_1}$ could be exponential with respect to the size of T_1. On the other hand, if $h^{-1}(T_1)$ is a DFA then the size of $\overline{T_1}$ is $O(\text{size}(T_1))$.

Example 2. In Fig.2 we are given two automata A_1 and A_2, A_1 accepting all words which in binary have an odd value and A_2 accepting all words which have an even length. Following the above construction we obtain a saturated transducer for $A_1 \times A_2$.

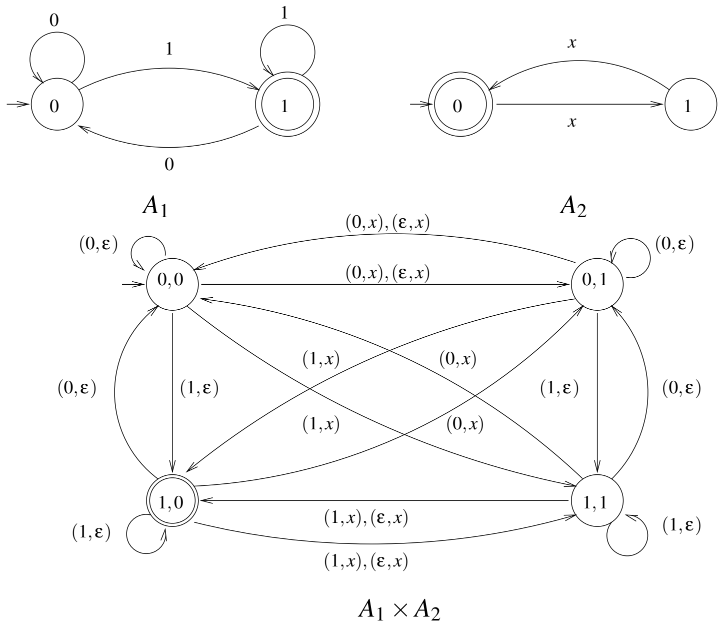

Fig. 2. The saturated transducer $A_1 \times A_2$.

Lemma 1. *If T_1 and T_2 are saturated transducers then $T_1 \cap T_2$, $T_1 \cup T_2$, $T_1 \cup_\varepsilon T_2$, $T_2 \circ T_1$ and $\overline{T}_1$ are saturated.*

Proof. We prove only that $T_2 \circ T_1$ is saturated.

We need to show that for any pair (x,z) in $|T_2 \circ T_1|$ and for any edit string e with $h(e) = (x,z)$, it is the case that e is in $||T_2 \circ T_1||$. Suppose that

$$e = (x_1/z_1)\cdots(x_n/z_n),$$

where each (x_i/z_i) is an edit operation. There is a computation c' of $T_2 \circ T_1$ such that $||c'||$ is some edit string $(x'_1/z'_1)\cdots(x'_m/z'_m)$ and $h(||c'||) = (x,z)$. By the definition of $T_2 \circ T_1$, there are successful computations c'_1 and c'_2 of T_1 and T_2, respectively, such that the edit strings $||c'_1||$ and $||c'_2||$ are of the form $(x'_1/y'_1)\cdots(x'_m/y'_m)$ and $(y'_1/z'_1)\cdots(y'_m/z'_m)$, respectively. Let y be the word $y'_1\cdots y'_n$. We continue by distinguishing two cases.

Firstly, suppose that $m \le n$. Let $y_j = y'_j$ for $j \le m$, and $y_j = \varepsilon$ for $j = m+1,\ldots,n$. Consider the edit strings

$$e_1 = (x_1/y_1)\cdots(x_n/y_n) \text{ and } e_2 = (y_1/z_1)\cdots(y_n/z_n).$$

As T_1 and T_2 are saturated, and $h(e_1) = (x,y)$ and $h(e_2) = (y,z)$, there are successful computations c_1 and c_2 of T_1 and T_2, respectively, such that $||c_1|| = e_1$ and $||c_2|| = e_2$. Then, by definition of the transducer $T_2 \circ T_1$, there is a computation c of this transducer such that $||c|| = e$, as required.

Secondly, suppose that $m > n$. The proof of this case is similar to the first one and is left to the reader. □

Example 3. Let τ_1, τ_2 be transductions given by

$\tau_1 = \{(u,v) \mid \sharp_2 u \text{ is odd }, v \in \{a\}^*\}$,
$\tau_2 = \{(u,v) \mid u \in \{a\}^*, \sharp_2 v \text{ is even}\}$

where by $\sharp_2 u$ we understand the value of u as a binary number. The first two saturated transducers in Fig.3 realize them. Then, using the above construction we obtain a saturated transducer(shown also in Figure 3) which realizes the transduction

$$\{(u,v) \mid \sharp_2 u \text{ is odd, and } \sharp_2 v \text{ is even}\} \ ,$$

which is their composition.

A natural question that arises here is whether saturated transductions are closed under the Kleene-star operation and concatenation. The first operation is discussed in the next section. For the second one consider two transducers T_1 and T_2 and the standard construction that connects each final state of T_1 with the start state of T_2 using an $(\varepsilon,\varepsilon)$-transition, such that the new transducer realizes $|T_1| \cdot |T_2|$. Unfortunately, however, this transducer is not necessarily saturated when both T_1 and T_2 are saturated. For example, if we connect a saturated transducer for $\{(a,ab)\}$ with a saturated transducer for $\{(ab,b)\}$, we obtain a transducer T such that $h(T)$ does not accept the edit string $(a/a)(a/b)(b/b)$ – hence T is not saturated. A new construction for saturated transducers for the concatenation operation is presented in the following.

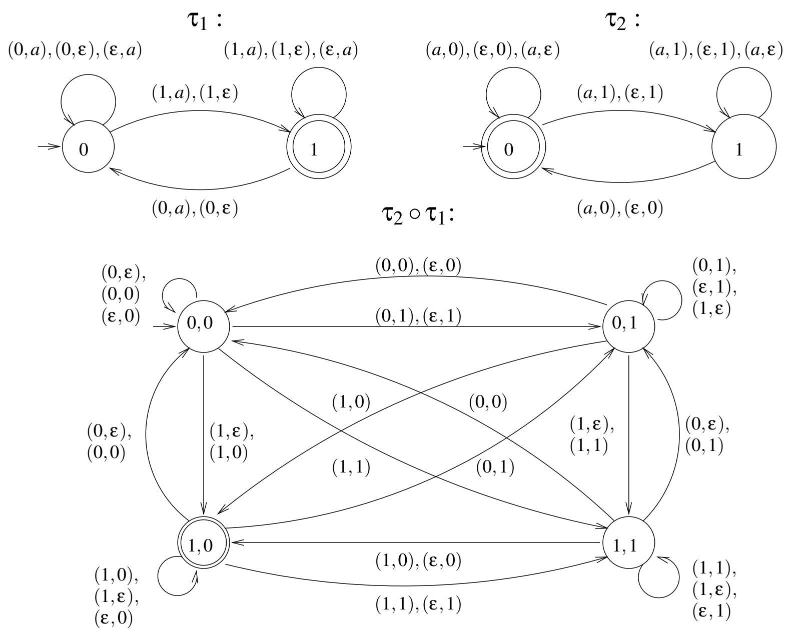

Fig. 3. Composition of saturated transducers.

For any two edit strings f and g of the form

$$f = (x_1/\varepsilon)\cdots(x_n/\varepsilon) \text{ and } g = (\varepsilon/y_1)\cdots(\varepsilon/y_n),$$

where each x_i is in $X\cup\{\varepsilon\}$ and each y_i is in $Y\cup\{\varepsilon\}$, we define the `left` and `right merge` operations '$\lhd$' and '$\rhd$' such that

$$f \lhd g = g \rhd f = (x_1/y_1)\cdots(x_n/y_n).$$

Lemma 2.

1. *For any edit strings f and g of the form shown above, we have that $h(f\lhd g) = h(g\rhd f) = h(fg)$. Also, $(\varepsilon/\varepsilon) = (\varepsilon/\varepsilon)\lhd(\varepsilon/\varepsilon) = (\varepsilon/\varepsilon)\rhd(\varepsilon/\varepsilon)$.*
2. *If τ_1 and τ_2 are transductions and e is any edit string with $h(e)\in\tau_1\cdot\tau_2$, then e can be written as $e_1e_2e_3$ such that e_2 is of the form $f_2\lhd g_2$, or $f_2\rhd g_2$, and $h(e_1f_2)\in\tau_1$ and $h(g_2e_3)\in\tau_2$.*

Proof. The first statement follows easily from the definition of the operations $\lhd$ and $\rhd$. For the second statement, first note that there are (x_1,y_1) in τ_1 and (x_2,y_2) in τ_2 such that $h(e) = (x_1x_2, y_1y_2)$. Notation wise, if $\alpha = (u,v)$ is a pair of words, then we denote $\pi_1(\alpha) = u$ and $\pi_2(\alpha) = v$. We distinguish the following factors of e:

- Let e_1 be the shortest prefix of e such that either $x_1 = \pi_1(h(e_1))$, or $y_1 = \pi_2(h(e_1))$.

- Let e_2 be the edit string such that e_1e_2 is the shortest prefix of e such that either $y_1 = \pi_2(h(e_1e_2))$, or $x_1 = \pi_1(h(e_1))$, respectively.
- Finally, let e_3 be such that $e = e_1e_2e_3$.

By looking in detail at the edit operations comprising e, one can verify that there are edit strings f_2 and g_2 such that $e_2 = f_2 \triangleright g_2$, or $e_2 = f_2 \triangleleft g_2$, respectively, and $h(e_1f_2) \in \tau_1$ and $h(g_2e_3) \in \tau_2$, as required. □

`Construction of` $T_1 \cdot T_2$:

`input:` Two saturated transducers $T_1 = (Q_1, X_1, Y_1, \Delta_1, s_1, F_1)$ and $T_2 = (Q_2, X_2, Y_2, \Delta_2, s_2, F_2)$ in standard form. We shall assume that T_1 is already trim, that is, each state can be reached from s_1 and can reach a final state in F_1.

`step 1:` Let U_{10} be the set of states p_1 in Q_1 such that there is a successful computation of T_1, from p_1, with tag (ε, v), for some v in Y_1^*. Let U_{01} be the set of states q_1 in Q_1 such that there is a successful computation of T_1, from q_1, with tag (u, ε), for some u in X_1^*.

`step 2:` Define the set Q consisting of the following states.

- All states r_1 in Q_1. Such an r_1 means that the automaton $h^{-1}(T_1 \cdot T_2)$ corresponding to the intended transducer $T_1 \cdot T_2$ has read an edit string e which is also the tag of some computation of $h^{-1}(T_1)$ from s_1 to r_1. This implies that, at state r_1, the machine $T_1 \cdot T_2$ has read some input (x_1', y_1') for which there is (x_1, y_1) in $|T_1|$ with x_1' and y_1' being prefixes of x_1 and y_1, respectively.

- All states $(q_1, q_2, 01)$ with $q_1 \in U_{01}$ and $q_2 \in Q_2$. Such a state means that $h^{-1}(T_1 \cdot T_2)$ has read an edit string e_1e_2 such that e_2 is of the form $f_2 \triangleleft g_2$ and there is a computation of $h^{-1}(T_1)$ from s_1 to q_1 with tag e_1f_2, and a computation of $h^{-1}(T_2)$ from s_2 to q_2 with tag g_2. This implies that, at state $(q_1, q_2, 01)$, $T_1 \cdot T_2$ has read some input of the form (x_1', y_1y_2') for which x_1' is a prefix of some x_1 with $(x_1, y_1) \in |T_1|$ and y_2' is a prefix of some y_2, with (x_2, y_2) in $|T_2|$ for some x_2. The "flag" 01 above reminds us that $T_1 \cdot T_2$ has completed reading only the second component of (x_1, y_1) and that no part of x_2 can be read before completing x_1.

- All states $(p_1, p_2, 10)$ with $p_1 \in U_{10}$ and $p_2 \in Q_2$. Such a state means that $h^{-1}(T_1 \cdot T_2)$ has read an edit string e_1e_2 such that e_2 is of the form $f_2 \triangleright g_2$ and there is a computation of $h^{-1}(T_1)$ from s_1 to p_1 with tag e_1f_2, and a computation of $h^{-1}(T_2)$ from s_2 to p_2 with tag g_2. This implies that, at state $(p_1, p_2, 10)$, $T_1 \cdot T_2$ has read some input of the form (x_1x_2', y_1') for which y_1' is a prefix of some y_1 with $(x_1, y_1) \in |T_1|$ and x_2' is a prefix of some x_2, with (x_2, y_2) in $|T_2|$ for some y_2.

- All states r_2 in Q_2. Such an r_2 means that $h^{-1}(T_1 \cdot T_2)$ has read an edit string $e_1e_2e_3$ such that e_2 is of the form $f_2 \triangleright g_2$, or $f_2 \triangleleft g_2$, and there is a computation of $h^{-1}(T_1)$ from s_1 to F_1 with tag e_1f_2, and a computation of $h^{-1}(T_2)$ from s_2 to r_2 with tag g_2e_3. This implies that, at state r_2, $T_1 \cdot T_2$ has read some input of the form (x_1x_2', y_1y_2') for which (x_1, y_1) is in $|T_1|$ and there is (x_2, y_2) in $|T_2|$ such that x_2' and y_2' are prefixes of x_2 and y_2, respectively.

`step 3`: Define the set Δ consisting of the transitions of $T_1 \cdot T_2$ in such a way that the meaning of the states in Q is preserved. More specifically we have that Δ consists of the following transitions.

- All transitions in Δ_1.
- All transitions of the forms $\langle p_1, \varepsilon, \varepsilon, (p_1, s_2, 10)\rangle$, with p_1 in U_{10}, and $\langle q_1, \varepsilon, \varepsilon, (q_1, s_2, 01)\rangle$, with q_1 in U_{01}.
- All transitions of the form $\langle (p_1, p_2, 10), a, b, (p_1', p_2', 10)\rangle$, with $p_1, p_1' \in U_{10}$, $p_2, p_2' \in Q_2$, and $(p_1, \varepsilon, b, p_1')$ in Δ_1, and $(p_2, a, \varepsilon, p_2') \in \Delta_2$.
- All transitions of the form $\langle (q_1, q_2, 01), a, b, (q_1', q_2', 01)\rangle$, with $q_1, q_1' \in U_{01}$, $q_2, q_2' \in Q_2$, and $(q_1, a, \varepsilon, q_1')$ in Δ_1, and $(q_2, \varepsilon, b, q_2') \in \Delta_2$.
- All transitions of the forms $\langle (p_1, p_2, 10), \varepsilon, \varepsilon, p_2\rangle$, with p_1 in F_1 and $p_2 \in Q_2$, and $\langle (q_1, q_2, 01), \varepsilon, \varepsilon, q_2\rangle$, with q_1 in F_1 and q_2 in Q_2.
- All transitions in Δ_2.

`output`: The transducer $T_1 \cdot T_2 = (Q, X_1 \cup X_2, Y_1 \cup Y_2, \Delta, s_1, F_2)$.

Theorem 1. *For any saturated transducers T_1 and T_2, the transducer $T_1 \cdot T_2$ is saturated and realizes the transduction $|T_1| \cdot |T_2|$. Moreover,* $\text{size}(T_1 \cdot T_2) = O(\text{size}(T_1) \cdot \text{size}(T_2))$.

Proof. The statement about the size of $T_1 \cdot T_2$ follows easily from its construction. For the first statement, it is sufficient to prove that $|T_1 \cdot T_2| \subseteq |T_1| \cdot |T_2|$ and that, for any edit string e with $h(e) \in |T_1| \cdot |T_2|$, we have that $e \in h^{-1}(T_1 \cdot T_2)$. Let (x,y) be any element in $|T_1 \cdot T_2|$. There is a computation of $T_1 \cdot T_2$ with tag (x,y) and a corresponding computation of $h^{-1}(T_1 \cdot T_2)$ with some tag e, with $h(e) = (x,y)$. By the definition of the final states of $T_1 \cdot T_2$, e is of the form $e_1 e_2 e_3$ with $e_2 = f_2 \triangleleft g_2$ – the case $e_2 = f_2 \triangleright g_2$ is symmetric – and $h^{-1}(T_1)$ accepts $e_1 f_2$, and $h^{-1}(T_2)$ accepts $g_2 e_3$. This implies that

$$(x,y) = h(e_1)h(f_2 g_2)h(e_3) = h(e_1 f_2)h(g_2 e_3) \in |T_1| \cdot |T_2|.$$

Now consider any edit string e such that $h(e) \in |T_1| \cdot |T_2|$. We shall use the notation in the preceding construction. The string e can be written as $e_1 e_2 e_3$ such that e_2 is of the form $f_2 \triangleleft g_2$ – the case $f_2 \triangleright g_2$ is symmetric – and $h(e_1 f_2) \in |T_1|$ and $h(g_2 e_3) \in |T_2|$. This implies that there is a computation of $h^{-1}(T_1)$ from s_1 to some $q_1 \in U_{01}$ with tag e_1, and a computation of $h^{-1}(T_1)$ from q_1 to some state $q_1' \in U_{01}$ with tag f_2. Moreover there is a computation of $h^{-1}(T_2)$ from s_2 to some $q_2 \in Q_2$ with tag g_2, and a computation of $h^{-1}(T_2)$ from q_2 to some state $q_2' \in F_2$ with tag e_3. Using the transitions of $T_1 \cdot T_2$ one can verify that there is a successful computation of $h^{-1}(T_1 \cdot T_2)$ with tag $e_1 e_2 e_3$, as required. □

We close this section by noting that the construction of $T_1 \cdot T_2$ can be carried out in time $O(\text{size}(T_1) \cdot \text{size}(T_2))$ – recall here the size of a transducer is the number of states plus the number of transitions in the transducer. This is clear in steps 2 and 3. In Step 3, the computation of U_{10} can be done in time $O(\text{size}(T_1))$ as follows. Let G_1 be the (directed) graph obtained by adding in the graph of T_1 a new state N and $(\varepsilon,\varepsilon)$-transitions from all final states of T_1 to N. Consider the graph G_2 obtained if we keep only the transitions of G_1 of the form (ε,a) and reverse the direction of these transitions. Then the set U_{01} consists of all the states in G_2, other than N, that can be reached from the state N. This traversal can be performed in time linear with respect to the size of G_2. The computation of U_{01} is analogous.

4 Saturation and Recognizability

Let us recall a few facts mentioned in the preliminaries of this paper. We know that a recognizable subset of $X^* \times Y^*$ is rational, therefore there exists a finite transducer which realizes it. The opposite does not hold: there exist quite simple rational transductions which are not recognizable, for example the identity over X^*. We also know a characterization of recognizable transductions as finite unions of blocks. There exist another two definitions of recognizable sets in arbitrary monoids: a morphism-based definition and a definition based on monoid actions on finite sets. We recall here the later one.

Definition 4. *Let Q be a finite set and $(M,\cdot,1_M)$ an arbitrary monoid.*

(i) An `action of M on` *Q is a function $f : M \times Q \to Q$ which satisfy the following two properties: $f(q,1_M) = q$ and $f(f(q,m),m') = f(q,mm')$, for all $q \in Q$ and $m,m' \in M$.*

(ii) A subset D of M is recognizable if there exists a such finite set Q and action f, and there exists $F \subseteq Q$ and $q \in Q$ such that $D = \{m \in M \mid f(q,m) \in F\}$.

In this section we give a fourth characterization of recognizable transductions by proving that the appropriate machines which realize them are saturated transducers. We start by giving two useful constructions.

`Construction #1`

`input`: We are given a saturated transducer T, which we put in standard form, if it is not already.

`step 1`: We construct the finite automaton $h^{-1}(T)$ by interpreting the labels of transitions of T as edit operation symbols. The automaton $h^{-1}(T)$ is over the alphabet E (and has been described in details at the beginning of Section 3).

`step 2`: We determinize and minimize the automaton $h^{-1}(T)$, obtaining a minimal, complete DFA B. Denote $B = (Q,E,\delta,q_0,F)$.

`step 3`: For each state q of B we construct a corresponding automaton C_q as follows:

(a) C_q has the same set of states as B, the same initial state, and it has $\{q\}$ as the set of final states;
(b) for each transition in B of type $(p,(a/\varepsilon),p')$ with $a \in X$ we assign a transition (p,a,p') in C_q.

`step 4`: For each state q of B we construct a corresponding automaton D_q as follows:
(a) D_q has the same set of states as B, the same set of final states, and it has q as initial state;
(b) for each transition in B of type $(p,(\varepsilon/b),p')$ with $b \in Y$ we assign a transition (p,b,p') in D_q.

`output`: Let $Q' := \{q \in Q \mid \mathscr{L}(C_q) \neq \emptyset \text{ and } \mathscr{L}(D_q) \neq \emptyset\}$. The algorithm ends by delivering $\{C_q\}_{q\in Q'}$ and $\{D_q\}_{q\in Q'}$.

Lemma 3. *The above construction ensures the following properties:*

(i) $\mid T \mid = \bigcup_{q\in Q'} \mathscr{L}(C_q) \times \mathscr{L}(D_q)$.
(ii) The languages $\{\mathscr{L}(C_q)\}_{q\in Q'}$ *are disjoint. The languages* $\{\mathscr{L}(D_q)\}_{q\in Q'}$ *are distinct.*
(iii) The transition function of the automaton $h(B)$ *can be extended to a monoid action of* $X^* \times Y^*$ *on* Q.
(iv) If $h^{-1}(T)$ *is deterministic then*

$$\sum_{q\in Q'} (\text{size}(C_q) + \text{size}(D_q)) = O(\text{size}(T)^2) \ .$$

Proof. We analyze each step of the above construction. The automaton $h^{-1}(T)$ found in `step 1` has the following property:

$$\forall e \in \mathscr{L}(h^{-1}(T)),\ \forall e' \in E^* :\ \ h(e') = h(e) \Rightarrow e' \in \mathscr{L}(h^{-1}(T)) \ , \tag{1}$$

given by the saturation of T. In other words, if $h^{-1}(T)$ accepts some edit string e, it will necessarily accept all edit strings which express the same word transformation as e. In algebraic terms, we say that the congruence induced by h – let us call it $\equiv_h$ – saturates $\mathscr{L}(h^{-1}(T))$.

Since B found at `step 2` is the minimization of $h^{-1}(T)$, it will preserve the above property. The automaton B has the following additional property:

$$\forall e, e' \in E^* :\ \ h(e) = h(e') \Rightarrow \delta(q_0, e) = \delta(q_0, e') \ ,$$

in other words:

$$\equiv_h \ \subseteq \ \equiv_{\mathscr{L}(B)} \ , \tag{2}$$

where by $\equiv_{\mathscr{L}(B)}$ we denoted the Myhill-Nerode equivalence of $\mathscr{L}(B)$. We justify this property by the following:
Let $h(e) = h(e')$ and denote $p = \delta(q_0, e)$ and $q = \delta(q_0, e')$. Assume by contradiction that $p \neq q$. Then, since B is minimal, it follows that there exists $e'' \in E^*$ such that

$\delta(p,e'')$ is a final state in B and $\delta(q,e'')$ is not. But then, $ee'' \in L$ and is easy to see that $h(ee'') = h(e'e'')$. By the property expressed in relation (1) we infer that $e'e''$ must be accepted - a contradiction.

Let a pair of words $(u,v) \in X^* \times Y^*$ be accepted by the given transducer T. Consider that $u = u_1u_2...u_m$, $v = v_1v_2...v_n$, with $u_1,...,u_m \in X$ and $v_1,...,v_n \in Y$. An edit string which transforms u into v is

$$e = (u_1/\varepsilon)...(u_m/\varepsilon)(\varepsilon/v_1)...(\varepsilon/v_n) \ ,$$

and denote $e = e_1e_2$, with $e_1 = (u_1/\varepsilon)...(u_m/\varepsilon)$. Since $(u,v) \in | T |$, we have that $e \in \mathscr{L}(B)$, hence $\delta(q_0,e_1e_2) \in F$ in B. Denote $q = \delta(q_0,e_1)$ and observe that $u \in \mathscr{L}(C_q)$ and $v \in \mathscr{L}(D_q)$. Since the reciprocal also holds, we have that

$$(u,v) \in | T | \Leftrightarrow u \in C_q \text{ and } v \in D_q \text{ for some } q \in Q' \ ,$$

which proves Property (i) of the lemma.

By the fact that B is deterministic, it follows that $\{C_q\}_{q\in Q}$ are disjoint. For the second part of Property (ii), we use yet another property of the automaton B, that is,

$$\forall q \in Q, \forall e,e' \in E^* \text{ such that } h(e) = h(e') : \ \ \delta(q,e) \in F \Rightarrow \delta(q,e') \in F \ , \qquad (3)$$

which can easily be verified (invoking the saturation of T). Since B is minimal, and by the above property, we conclude that $\mathscr{L}(D_p) \neq \mathscr{L}(D_q)$ for any two distinct states $p,q \in Q$, as long as either $\mathscr{L}(D_p)$ or $\mathscr{L}(D_q)$ is not empty. This completes the proof of Property (ii).

Let us consider the transducer $h(B)$, which is obtained from B by replacing the transition labels(symbols) of the form x/y with the corresponding pairs (x,y). Clearly, $| T |=| h(B) |$. If we denote f to be the transition function of $h(B)$ (it is a partial function due to the determinism of B) it is enough to show that we can extend f to $X^* \times Y^*$ such that it verifies the properties of an action. For any $(u,v) \in X^* \times Y^*$, let $e_{u,v}$ be a chosen edit string such that $h(e_{u,v}) = (u,v)$. We set $f(p,(u,v)) := \delta(p,e_{u,v})$ and $f(p,(\varepsilon,\varepsilon)) := p$, for all states in Q. It can readily be checked that the definition is independent of the choice of $e_{u,v}$, that is, f is a function

$$f : (X^* \times Y^*) \times Q \to Q \ ,$$

and that

1. $f(p,(\varepsilon,\varepsilon)) = p, \forall p \in Q$,
2. $f(f(p_1,(u_1,v_1)),(u_2,v_2)) = f(p,(u_1u_2,v_1v_2))$.

Finally we have that $(u,v) \in | T | \Leftrightarrow f(p_0,(u,v))$ is a final state in $h(B)$ (where p_0 is the initial state of $h(B)$). □

Remark 2. Notice that Property (i) of the above lemma does not depend on the minimality and completeness of B. Indeed, if we eliminate `step 2` of the above construction, and we consider $h^{-1}(T)$ instead of B in the subsequent steps, we would still obtain Property (i) of the lemma.

Corollary 1.

$$Sat(X^* \times Y^*) \subseteq Rec(X^* \times Y^*) \ .$$

Proof. By Mezei's characterization of recognizable transductions, we observe that the transduction realized by a saturated transducer is a finite union of blocks, hence it is recognizable. □

We now turn our attention to a possible reciprocal of the above corollary, and we are aiming, as usual, at a constructive proof.

`Construction #2`

`input`: We have a transduction $\tau \in Rec(X^* \times Y^*)$ effectively given as a tuple $(A_1, B_1, ..., A_n, B_n)$ of finite automata. That is, we know that

$$\tau = \bigcup_{i=1}^{n} \mathscr{L}(A_i) \times \mathscr{L}(B_i) \ .$$

`step 1`: For each $i \in \{1, ..., n\}$ we construct a saturated transducer T_i such that $\mid T_i \mid = \mathscr{L}(A_i) \times \mathscr{L}(B_i)$ (the construction has been presented in Section 3).

`step2`: Since all T_i are saturated, we construct in $n-1$ iterations the transducer $T^{\cup} = T_1 \cup_{\varepsilon} ... \cup_{\varepsilon} T_n$ which realizes the transduction $\mid T_1 \mid \cup ... \cup \mid T_n \mid$ (this construction has also been presented in Section 3).

`output`: The algorithm delivers $T^{\cup}$.

Lemma 4. *The above construction ensures that*

$$\mid T^{\cup} \mid = \tau \ .$$

Moreover, $T^{\cup}$ *is saturated and* $\mathrm{size}(T^{\cup}) = \sum_{i=1}^{n}(\mathrm{size}(A_i) \cdot \mathrm{size}(B_i))$.

Proof. The correctness and finiteness of each step has been proven in Lemma 1. □

Corollary 2.

$$Rec(X^* \times Y^*) \subseteq Sat(X^* \times Y^*) \ .$$

Remark 3. This corollary can also be proven by using the closure properties of recognizable sets, as follows:

Proof. Let τ be a recognizable transduction and consider the edit morphism over X and Y,

$$h : E^* \to X^* \times Y^* \ .$$

Since h is a morphism and τ is recognizable in $X^* \times Y^*$ we have that $h^{-1}(\tau)$ is recognizable in E^* (by the fact that recognizable sets are closed under inverse morphism). Then, by Kleene's theorem we have that $h^{-1}(\tau)$ is a regular language, hence there exists a finite automaton A over E which accepts $h^{-1}(\tau)$. Assume that A is a complete DFA. It now suffices to observe that the transducer $h(A)$ is saturated, in standard form, and it realizes τ. □

Summing up, we have the following characterization of recognizable transductions.

Theorem 2. *A transduction is recognizable if and only if it is realized by a saturated transducer.*

Proof. It is a direct consequence of Corollary 1 and Corollary 2. Notice that the previous two constructions give a constructive proof of this theorem. □

It is important to note a consequence of this result : there exist saturated transducers whose transition table can not be extended to a monoid action; however, the theorem implies that even these transducers realize recognizable transductions.

Remark 4. We can now give an elegant argument for the fact proven in Lemma 1, using Mezei's theorem. Indeed, if T_1 and T_2 are saturated transducers, then by the theorem we have that $\mid T_1 \mid$ and $\mid T_2 \mid$ are recognizable, hence by Mezei's theorem we have that

$$\mid T_1 \mid = \bigcup_{i=1}^{m} A_i \times B_i \text{ and } \mid T_2 \mid = \bigcup_{j=1}^{n} C_j \times D_j \ ,$$

where we expressed the transductions as union of blocks. Then it suffices to observe that

$$\mid T_1 \mid \circ \mid T_2 \mid = \bigcup_{1\le i\le m,\ 1\le j\le n} G_{i,j} \ \text{, with } G_{i,j} = \begin{cases} \emptyset, & if\ B_i \cap C_j = \emptyset; \\ A_i \times D_j, & otherwise. \end{cases}$$

Consequently, $\mid T_1 \mid \circ \mid T_2 \mid$ is recognizable, therefore realizable by a saturated transducer $T_1 \diamond T_2$, which can effectively be constructed. Notice that $T_1 \diamond T_2$ may have a structure different than that of $T_2 \circ T_1$ which was proposed in Lemma 1.

Remark 5. We have seen in Theorem 1 that given two saturated transducers T_1 and T_2, one can construct a size $O(\text{size}(T_1) \cdot \text{size}(T_2))$ transducer $T_1 \cdot T_2$ which realizes $\mid T_1 \mid \cdot \mid T_2 \mid$. That construction can stand as an alternative proof that recognizable transductions are closed under concatenation (the other proof makes use of Mezei's theorem).

Remark 6. We can now explain why in Section 3 we have not mentioned anything about the "star" operation on a saturated transducer. The reason is that saturated transductions are not closed under iteration, as the following classical example shows: $\{(a,b)\}$ is a saturated transduction, being finite; however, $\{(a,b)\}^*$ is not recognizable, hence can not be realized by a saturated transducer.

Remark 7. It is worth noticing that, given a finite transducer T over alphabets with at least two letters, it is undecidable whether there exists a saturated transducer equivalent with T. Indeed, this follows from the known fact that is undecidable whether a finite transducer over alphabets with at least two letters realizes a recognizable transduction ([Berstel, 1979, §III.8, p. 90]).

5 Edit Distance and the non-Recognizability of $(L \times L)_{\neq}$

Edit strings and edit languages constitute natural tools for dealing with problems related to the edit distance between words and languages. In this context, the weight of an edit string

$$e = (x_1/y_1)\cdots(x_n/y_n)$$

is the number of edit operations (x_i/y_i) in e with $x_i \neq y_i$, and is denoted by weight(e). For example, the weight of the edit string f in Section 2 is 2. Then the edit distance between two words u and v is the minimum of the weights of the edit strings transforming u into v, that is,

$$\mathrm{dist}(u,v) = \min\{\mathrm{weight}(e)/e \in h^{-1}(\{(u,v)\})\}.$$

If we construct automata A_u and A_v accepting $\{u\}$ and $\{v\}$, respectively, then the saturated transducer $A_u \times A_v$ accepts all edit strings e with $e \in h^{-1}(\{(u,v)\})$. Hence, the quantity dist(u,v) is the weight of the smallest-weight path (computation) in the graph corresponding to $A_u \times A_v$ – here the weights on the transitions are in $\{0,1\}$ such that the weight of a transition $(p,(x/y),q)$ is 1 if and only if $x \neq y$. This simple idea can be generalized for any pair of automata A_1 and A_2 and for more general types of distances – see [Kari et al., 2003] and [Mohri, 2003] for details.

The problem of computing the (inner) edit distance of a language L is more difficult, however. This quantity is the minimum edit distance between any pair of *distinct* words of L. Suppose that A is an automaton accepting L. The difficulty here lies in the fact that the saturated transducer $A \times A$ accepts edit strings e corresponding to pairs of equal words. Therefore, one would like to have a saturated transducer for the transduction

$$(L \times L)_{\neq} = \{(u,v)/u,v \in L \,\mathrm{and}\, u \neq v\}.$$

Although one can construct an ordinary transducer for this transduction, we show next that there exists no corresponding saturated transducer, that is, $(L \times L)_{\neq}$ is not recognizable when L is infinite. For the sake of completeness we mention that the problem of computing the inner edit distance is solved in [Konstantinidis, 2005] by observing that (i) this quantity is always realized by two words differing at some position bounded by j_A, for some index that depends on the automaton A accepting L; and (ii) for any index j, there is a transducer T_j (which turns to be saturated, in our terminology) realizing all pairs of words that differ at position j.

Given an arbitrary set P, by $(P \times P)_{\neq}$ we understand the set of all pairs of different elements of P. In other words, $(P \times P)_{\neq} = (P \times P) \setminus id(P)$.

Proposition 1. *Let P be an arbitrary, infinite set. The set equation*

$$(P \times P)_{\neq} = \bigcup_{i=1}^{n} X_i \times Y_i$$

has no solution $(n, \{X_i, Y_i\}_{i=1}^{n})$. *(note: n is viewed as a variable in the equation)*

Proof. Assume, by contradiction, that there exists $(n, \{X_i, Y_i\}_{i=1}^n)$ – a solution of the above equation. Notice first that necessarily $X_i \cap Y_i = \emptyset$ for all $i \in \{1, ..., n\}$. Since P is infinite, there exist 2^{n+1} different elements in P. Denote by $U_1 := \{u_1, ..., u_{2^{n+1}}\}$ a set of such elements.

Consider the triplet U_1, X_1 and Y_1. We can write

$$U_1 = (U_1 \cap X_1) \cup (U_1 \cap Y_1) \cup \big(U_1 \setminus (X_1 \cup Y_1)\big) \ ,$$

since X_1 and Y_1 are disjoint. Let us assume, without loss of generality that $| U_1 \cap X_1 | \geq | U_1 \cap Y_1 |$, and let us denote $U_2 := U_1 \setminus Y_1$.

We first prove that U_2 has at least 2^n elements. We have that $| U_1 \cap X_1 | + | U_1 \cap Y_1 | \leq 2^{n+1}$ and that $| U_1 \cap X_1 | \geq | U_1 \cap Y_1 |$. This implies that $| U_1 \cap Y_1 | \leq 2^n$, by the fact that $U_1 \cap X_1$ and $U_1 \cap Y_1$ are disjoint. Then clearly $| U_1 \setminus Y_1 | \geq 2^n$, hence $| U_2 | \geq 2^n$. We may also observe that the pairs of different elements in U_2 can not appear in $X_1 \times Y_1$. Indeed, we can not have $(u, v) \in X_1 \times Y_1$ and $u, v \in U_2$, since $U_2 = U_1 \setminus Y_1$.

We repeat the above argument for the triplet U_2, X_2 and Y_2. We obtain a set $U_3 \subseteq U_1$ with $| U_3 | \geq 2^{n-1}$ and no pair of elements in U_3 can be found in $X_2 \times Y_2$.

Then, we repeat this argument till we obtain $U_{n+1} \subseteq U_2$ with $| U_{n+1} | \geq 2$ and no pair of elements in U_{n+1} can be found in $X_n \times Y_n$.

Take two different elements $u, v \in U_{n+1}$. Since we have $U_{n+1} \subseteq U_n \subseteq ... \subseteq U_1$, we conclude that the pair (u, v) does not belong to any $X_i \times Y_i$, for $1 \leq i \leq n$.

But this contradicts the fact that $U_1 \subseteq P$. □

Corollary 3. *Let $L \in X^*$ be an infinite regular language. The transduction $(L \times L)_{\neq}$ can not be realized by a saturated transducer over X.*

Proof. In order to have a saturated transducer for $(L \times L)_{\neq}$, this set must be recognizable, by Theorem 2. However, Proposition 1 shows that it can not be written as a finite union of blocks, hence it is not recognizable, by Mezei's characterization. □

6 Final Comments and Future Work

In this paper we have achieved the following. We have revealed the relation between edit languages, recognizable transductions and saturated transducers. We have shown that operations with saturated transducers can efficiently be implemented, and we outlined methods to construct and manipulate saturated transducers. We have shown how one can use saturated transducers for computing the edit distance between words and languages. Finally, we have studied situations when our framework can not be used, due to the non-recognizability of various rational relations.

Our starting point in this study was the homomorphic relationship between the free monoid of edit strings and the Cartesian product monoid of word pairs (as reflected in the proof of Remark 3 at page 16). This homomorphism has inspired the idea of a saturated transducer, as a natural machine meant to realize recognizable relations. However, saturated transducers are interesting machines in their own right, and we studied them accordingly. For example, we would like to emphasize the construction of a saturated transducer realizing the concatenation of two saturated transductions (page 11):

although it is known that recognizable relations (thus, saturated transductions, by Theorem 2) are closed under concatenation, this construction in itself seems rather ingenious, and certainly non-trivial. Several complexity issues related to operations on saturated transductions have been investigated as well (Section 3). Purposely, we provided effective proofs and constructions, to emphasize the algorithmic side of our framework. Finally, we have exploited the connection between these machines and edit operations. We revealed some of the potential and limitations of using transducers in computing the edit distance between regular languages. The combinatorial Proposition 1 has a rather surprising connection (Corollary 3) with the non-recognizability of a natural and simple word relation: the set of pairs of distinct words of an infinite regular set. Thus, we feel that beside the technical aspects of the paper, we delievered a vantage point of view meant to embrace three distinct areas, that of recognizability in the algebraic sense, saturated transducers (the machinistic approach) and edit operations. It is worth noticing that our entire framework still holds when is restricted to the use of only two edit operations: insertion and deletion (for this case, one defines "restricted saturated transducers"). This restriction may be of importance in applications where only these two edit operations are of interest (e.g., [Levenshtein, 1966]).

Left for further analysis are a few matters which have not been tackled yet. For example, it is worth investigating algorithms to efficiently compute saturated transducers for given finite transductions; in particular, for finite identities.

It is interesting to notice that the notion of minimal saturated transducer for a recognizable transduction makes sense, since it is given by the minimal corresponding DFA over the edit alphabet. Size-complexity matters may be investigated in this aspect.

Finally, we have left for study the comparison of two representations (characterizations) of recognizable transductions: one using saturated transducers the other using tuples of automata.

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