Nova Scotia

Math League

2010–2011

Game Three

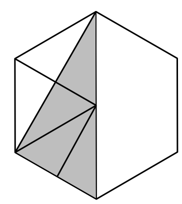
SOLUTIONS

Team Question Solutions

1. Let D, N, and Q be the total value of the dimes, nickels, and quarters. Then $D + N + Q = 7$ and $D : N : Q = 2 : 4 : 1$. Hence $D = \frac{2}{2+4+1} \cdot 7 = 2$. Similarly, $N = 4$ and $Q = 1$. Hence Billy has 20 dimes, 80 nickels, and 4 quarters, for a total of 104 coins.

   Alternative Solution: Let d, n, and q be the number of dimes, nickels, and quarters. Then we have the system $\{10d + 5n + 25q = 700,\ 10d = 50q,\ 10d = \frac{5}{2}n\}$. Solve this system to get $(d, n, q) = (20, 80, 4)$.

2. Divide one half of the hexagon into 6 congruent right-angled triangles, as shown below. Half the hexagon has area 6, and the shaded area includes 4 of the 6 small triangles. So the shaded area is $6 \cdot \frac{4}{6} = 4$.



3. The sum $|PR| + |QR|$ can be no smaller than the distance from P to Q, and $|PR| + |QR| = |PQ|$ precisely when R lies on the line segment joining P and Q. So we take R to be the point of intersection of the parabola $y = x^2$ and the line that passes through $P = (0, 4)$ and $Q = (7, 25)$. Thus $R = (x, y)$, where x and y satisfy the system $\{y = x^2, y = 4 + 3x\}$. Setting $x^2 = 4 + 3x$ gives $x = 4$, thus $y = x^2 = 16$ and $R = (4, 16)$.

4. Imagine that the envelopes that contain the red, blue, and green cards are labelled 1, 2, and 3, respectively. Then we can denote our guesses by rearrangements of the letters RBG. For instance, the arrangement BRG indicates that we guessed that envelopes 1, 2, and 3 contained the blue, red, and green cards, respectively.

   There are $3! = 6$ possible guesses, corresponding to the $3!$ rearrangements of RGB. Of these, notice that only BGR and GRB represent guesses that would result in a "win". So the desired probability is $\frac{2}{6} = \frac{1}{3}$.

   Note: This is a very special case of a classic problem in probability theory. The *Montmart Problem* is usually phrased as follows: If n men check their coats at a restaurant and, upon leaving, their coats are given back randomly, what is the probability that *no man* will be given his own coat? Surprisingly enough, as n gets larger and larger, this probability approaches $1/e$, where $e \approx 2.718$ is Euler's constant (i.e. the base of the natural logarithm).

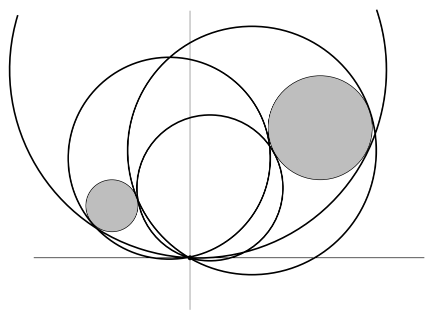
5. Let w be the number of widgets purchased, and let c be the original cost per widget. Then we know that $wc = 840$, and $(w + 4)(c - 7) = 840$. Thus $wc = (w + 4)(c - 7)$, from which we get

$4c - 7w = 28$. Now use $wc = 840$ to substitute $c = \frac{840}{w}$. This gives:

$$4 \cdot \frac{840}{w} - 7w = 28 \quad \Longrightarrow \frac{480}{w} - w = 4$$
$$\Longrightarrow w^2 + 4w - 480 = 0$$
$$\Longrightarrow (w-20)(w+24) = 0.$$

Thus $w = 20$ or $w = -24$. Since w must be positive, we have $w = 20$.

6. There are 4 possible circles, as pictured below.



Note: This is a special case of the *Apollonius Problem*, which asks one to find all circles that are tangent to three given circles. There are always precisely 8 such tangential circles: Each of the three given circles can be chosen to be either *inside* or *outside* the tangential circle, leading to $2 \cdot 2 \cdot 2 = 8$ possibilities. In our case, one of the given circles has degenerated to a point, so "inside" and "outside" are synonymous, and we are left with only $2 \cdot 2 = 4$ tangential circles.
(See `http:\\mathworld.wolfram.com\ApolloniusCircle.html` for more information.)

7. For conciseness, let us refer to a side of the pentagon as an *edge*. Clearly every triangle contains either 0, 1, or 2 edges (that is, either 0, 1, or 2 sides of the triangle are edges of the pentagon). We now count these three classes of triangle:

   - There are 10 triangles containing no edges.
   - For any given edge, there are 4 triangles that contain that edge and no others. Since there are 5 edges, there are $5 \cdot 4 = 20$ triangles with exactly one edge.
   - If a triangle contains two edges, then these edges must be adjacent; and any two adjacent edges are contained in exactly one triangle. So there are 5 such triangles.

   Altogether, we have a total of $10 + 20 + 5 = 35$ triangles.

8. To say that the integer N ends with exactly k zeros is to say that 10^k is the highest power of 10 dividing N. So we wish to find the highest power of 10 that divides into $N = 25! \cdot 24! \cdots 2! \cdot 1!$. To do so, we notice that 10^k divides into N if and only if both 2^k and 5^k divide into N. So let us find

all factors of 5 inside the product that defines N:

$$\begin{aligned} N &= 25! \cdot 24! \cdot 23! \cdot \cdots \cdot 3! \cdot 2! \cdot 1! \\ &= 25 \cdot 24^2 \cdot 23^3 \cdots 3^{23} \cdot 2^{24} \cdot 1^{25} \\ &= 25 \cdots 20^6 \cdots 15^{11} \cdots 10^{16} \cdots 5^{21} \cdots 1 \\ &= 5^{2+6+11+16+21} \cdot (\text{a number not divisible by 5}). \end{aligned}$$

Thus 5^{56} is the highest power of 5 dividing N. Clearly applying the same trick will find at least 56 factors of 2 inside N (in fact, far more), so that 2^{56} also divides N. We conclude that 10^{56} is the highest power of 10 dividing N. Thus N terminates with 56 zeros.

9. By inspection we notice it is impossible to order 23 nuggets. However, it *is* possible to order 24, 25, 26, or 27 nuggets:

$$\begin{aligned} 24 &= 6 \cdot 4 + 0 \cdot 9 \\ 25 &= 4 \cdot 4 + 1 \cdot 9 \\ 26 &= 2 \cdot 4 + 2 \cdot 9 \\ 27 &= 0 \cdot 4 + 3 \cdot 9. \end{aligned}$$

But if you can order N nuggets, then surely you can order $N + 4k$ nuggets, for any $k \geq 0$. (Simply add k boxes of 4 nuggets on to your order!). Thus we can order any number N of nuggets of the form $N = 24 + 4k, 25 + 4k, 26 + 4k$, or $27 + 4k$. Since every integer greater than 23 is of this form, 23 must be the largest number of nuggets that we *cannot* order.

Note: This is a special case of a more general result, which states that if nuggets come in boxes of sizes a and b, and if the greatest common divisor of a and b is 1 (that is, they share no common factor), then the largest number of nuggets that cannot be ordered is $ab - a - b$. In our case, $a = 4$ and $b = 9$, so $ab - a - b = 23$.

10. First consider the horizontal matchsticks in a large pyramid of n levels: From top to bottom, we see rows containing $1, 3, 5, 7, \ldots, 2n - 1, 2n - 1$ horizontal sticks. Now consider the vertical matchsticks. From top to bottom, we see rows containing $2, 4, 6, 8, \ldots, 2n$ vertical sticks. So the total number of matchsticks in a pyramid of n levels is In total, there are

$$\begin{aligned} &(1 + 3 + 5 + \cdots + (2n - 1) + (2n - 1)) + (2 + 4 + 6 + \cdots + 2n) \\ &= (1 + 2 + 3 + \cdots + 2n) + (2n - 1) \\ &= \frac{2n(2n + 1)}{2} + (2n - 1) \\ &= 2n^2 + 3n - 1. \end{aligned}$$

Setting $n = 100$ gives 20299 matchsticks in total.

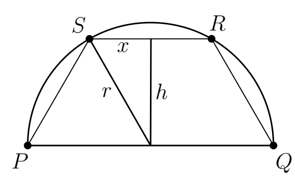
Pairs Relay Solutions

A. We quickly compute $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{2}{3}$, $a_3 = \frac{3}{5}$, and so on, until we get $a_8 = \frac{34}{55}$. (This comes very quickly if you notice that the numerators and denominators of the a_i are successive Fibonacci numbers.) Thus $\mathtt{A} = 55$

B. Since there are $\mathtt{A}$ girls in total, and 33 are brunette, there are $\mathtt{A} - 33$ blondes. Of these, 12 are blue-eyed, leaving $\mathtt{A} - 45$ brown-eyed blondes. But there are 35 brown-eyed girls in total, so there must be $35 - (\mathtt{A} - 45) = 80 - \mathtt{A}$ brown-eyed brunettes. Thus $\mathtt{B} = 80 - \mathtt{A}$, and with $\mathtt{A} = 55$ we get $\mathtt{B} = 25$.

C. Since $|ST| = \mathtt{B}$, and P divides ST in the ratio $2 : 3$, we have $|AP| = \frac{2}{5}\mathtt{B}$. Since Q divides ST in the ratio $3 : 4$ we have $|AQ| = \frac{3}{7}\mathtt{B}$. Thus $\mathtt{C} = |PQ| = |AQ| - |AP| = \frac{3}{7}\mathtt{B} - \frac{2}{5}\mathtt{B} = \frac{1}{35}\mathtt{B}$. Set $\mathtt{B} = 25$ to get $\mathtt{C} = \frac{25}{35} = \frac{5}{7}$.

D. Let x, h, and r be as indicated in the diagram below. Then $|PQ| = 2r$, $|RS| = 2x$, and $x^2 + h^2 = r^2$. We are given $|PQ| = 14$, so $r = 7$. We also know $|RS| = \mathtt{C}|PQ| = 7\mathtt{C}$ The area of trapezoid $PQRS$ is $\frac{1}{2}(|PQ| + |RS|)h$, where h is as indicated in the figure below.



Individual Relay Solutions

A. Let the legs of the triangle be of lengths x and y. Then we are given $x^2 + y^2 + \texttt{A}^2 = 128$, and Pythagorean Theorem gives $x^2 + y^2 = \texttt{A}^2$. Hence $2\texttt{A}^2 = 128$, which yields $\texttt{A} = 8$.

B. The greatest number of coins you could withdraw *without* getting at least 11 pennies or A dimes would be $10 + (\texttt{A} - 1) = \texttt{A} + 9$. (That is, 10 pennies and $\texttt{A} - 1$ dimes.) Thus $\texttt{B} = \texttt{A} + 10 = 18$.

C. Note that the radii of the semicircles are in the ratio $1 : 2 : 3$, so their areas are in the ratio $1 : 4 : 9$. Since the large semicircle has area B, the small and medium semicircles have areas $\frac{1}{9}\texttt{B}$ and $\frac{4}{9}\texttt{B}$, respectively. Thus the shaded area is $\texttt{C} = \frac{4}{9}\texttt{B} - \frac{1}{9}\texttt{B} = \frac{1}{3}\texttt{B}$. With $\texttt{B} = 18$, this gives $\texttt{C} = 6$.

D. The equation $|x + \texttt{C}| = 2|x - \texttt{C}|$ holds if and only if $x + \texttt{C} = 2(x - \texttt{C})$, or $x + CC = -2(x - \texttt{C})$. Solving these two equations in turn gives $x = 3\texttt{C}$ and $x = \frac{1}{3}\texttt{C}$. Thus $\texttt{D} = 3\texttt{C} + \frac{1}{3}\texttt{C} = \frac{10}{3}\texttt{C}$. With $\texttt{C} = 6$ we have $\texttt{D} = 20$.