Nova Scotia

Math League

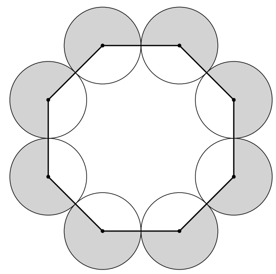
2018–2019

Game Two

Problems and Solutions

Team Questions

1. In the figure below, each side of the octagon is of length 1, the circles are centred at its vertices, and each circle is tangent to its neighbours. Find the area of the shaded region.



Solution: The circles have radius $\frac{1}{2}$ and area $\pi/4$. Note that $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$ of each circle is shaded, so the total shaded are is $8(\frac{5}{8})(\frac{\pi}{4}) = \frac{5\pi}{4}$. □

2. Subtracting one from the numerator of a fraction gives $1/4$, while subtracting one from its denominator gives $1/3$. Find the fraction.

Solution: Say a/b. Then $(a-1)/b = 1/4$, so $b = 4(a-1)$. Then $1/3 = a/(b-1) \implies 3a = b - 1 = 4a - 5$. Thus $a = 5$ and $b = 16$. □

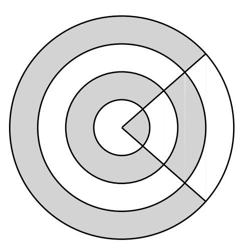
3. Let $S(n)$ denote the sum of the digits of n. For example, $S(132) = 1 + 3 + 2 = 6$.

   Define $a_1, a_2, a_3 \ldots$ by setting $a_1 = 1$ and taking $a_n = S(5a_{n-1})$ for $n > 1$.

   Find a_{2019}.

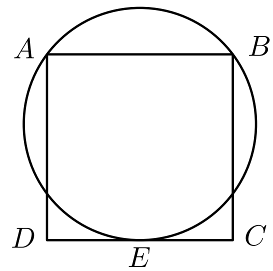
Solution: Iterate to get $(a_1, a_2, \ldots) = (1, 5, 7, 8, 4, 2, 1, 5, 7, \ldots)$. So we see that the sequence is periodic with period length 6. Since 2019 leaves remainder 3 upon division by 6, get $a_{2019} = a_3 = 7$. □

4. Find the shaded area in the figure below if the indicated angle is $80°$ and the concentric circles have radii 1, 2, 3, and 4.

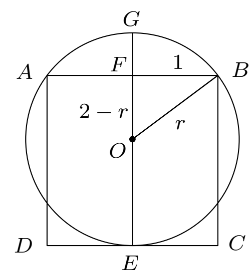


Solution: The full rings have areas $\pi((n+1)^2 - n^2) = \pi(2n+1)$ for $n = 0, 1, 2, 3$, that is $\pi, 3\pi, 5\pi$ and 7π. So the shaded area is $\frac{80}{360}(1+5)\pi + \frac{280}{360}(3+7)\pi = \frac{82}{9}\pi$. □

5. Vertices A and B of square $ABCD$ lie on a circle, and side CD of the square is tangent to the circle at its midpoint E. If the square has area 4, find the area of the circle.

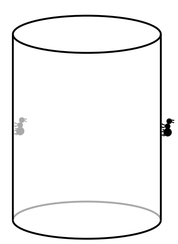


Solution: The square has side 2. Let r be the radius of the circle. Draw the diameter passing through E and the centre O of the circle. Say it meets AB at F. Then $\triangle OFB$ is right-angled, with sides $|FB| = \frac{1}{2}(2) = 1$, $|OF| = |EF| - |OE| = 2 - r$, and $|OB| = r$. Pythagorean theorem gives $1 + (2-r)^2 = r^2$, which simplifies to $5 - 4r = 0$, so $r = \frac{5}{4}$. Thus the desired area is $\pi(\frac{5}{4})^2 = 25\pi/16$.



Alternatively, intersecting chords GE and AB in the above diagram give $|GF| \cdot |FE| = |AF| \cdot |FB|$, so $(2r-2)(2) = (1)(1)$, whence $r = \frac{5}{4}$. $\square$

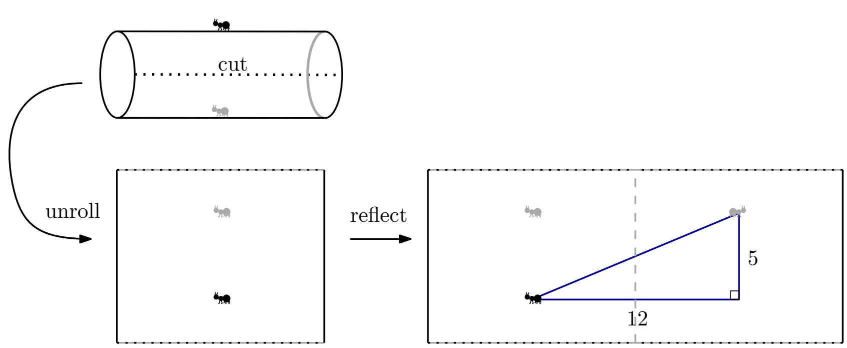
6. A toilet paper tube has the shape of a hollow cylinder with height 12cm and circumference 10cm. Two ants are on the surface: the first is midway up the outside of the tube and the other is on the inside diametrically opposite the first.



Find the length of the shortest path between the two ants.

(Note: The path must remain on the surface of the tube.)

Solution: Cut and unroll the tube, and then reflect to determine the straight-line path from outside to inside. This results in a right triangle with legs $6 + 6 = 12$ and $\frac{1}{2}(10) = 5$. Thus the desired distance is $\sqrt{5^2 + 12^2} = 13$.



$\square$

7. How many subsets of $\{1, 2, 3, 5, 8, 13, 21, 34, 55\}$ have an odd sum? (For example, $\{1, 2, 8\}$ is one such subset, since $1 + 2 + 8 = 11$ is odd.)

Solution: A subset will have an odd sum if and only if it contains an odd number of odd elements. So every odd-sum subset of S can be constructed by selecting any odd number of elements from $\{1, 3, 5, 13, 21, 55\}$, which can be done in $\binom{6}{1} + \binom{6}{3} + \binom{6}{5} = 6 + 20 + 6 = 32$ ways, and then adding any number of elements from $\{2, 8, 34\}$, which can be done in $2^3 = 8$ ways. So there are $32 \cdot 8 = 256$ odd-sum subsets of S.

Note: There are $2^9 = 256$ subsets of $\{1, 2, 3, 5, 8, 13, 21, 34, 55\}$ altogether, and we have found that exactly half of these have an odd sum. This is not a coincidence. More generally, if S is *any* finite set of integers containing at least one odd number, then then S has exactly $2^{|S|-1}$ odd-sum subsets. It's a good exercise to work through the proof of this claim. (Hint: Let x be any particular odd element of S. Given a subset A of S, define the subset A' of S by either adding x to A in the case $x \notin A$, or removing x from A in the case $x \in A$. Note that A has an odd-sum if and only if A' has an even sum. How does this prove the result?) $\square$

8. Sue randomly fills a Tic-Tac-Toe grid with five X's and four O's. Find the probability that there will be at least one line of three X's in a row (horizontal, vertical, or diagonal).

Solution: Note there are 8 possible lines of 3-in-a-row. We can create any configuration with at least one such line of X's as follows: Select any of the 8 possible lines, fill it with X's, and then fill any 2 of the remaining 6 cells with X's (the others are filled with O's). This can be done in $8 \cdot \binom{6}{2} = 120$ ways. However, this double counts configurations with *two* intersecting lines of X's. There are $\binom{8}{2} - 6 = 22$ of these, because all but 6 of the $\binom{8}{2}$ pairs of distinct lines intersect. Thus there are $120 - 22 = 98$ possibilities overall, and the desired probability is $98/\binom{9}{5} = \frac{7}{9}$.

Alternatively, it is not too difficult to directly count the configurations in which there is no line of three X's in a row. One possibility is to organize the counting by separately considering arrangements with k X's in the corner positions, for $k = 0, 1, 2, 3, 4$. $\square$

Pairs Relay

P-A. Consider the numbers $1^2, 2^2, 3^2, \ldots, 11^2$. Removing the entry A^2 from this list and replacing it with A decreases the average of the list by 10.

Find A. $\boxed{\text{Pass on A}}$

Solution: Let x be the initial average. Then $(11x - \mathsf{A}^2 + \mathsf{A})/11 = x - 10$, or $\mathsf{A}^2 - \mathsf{A} - 110 = 0$. Thus $(\mathsf{A} - 11)(\mathsf{A} + 10) = 0$, giving $\mathsf{A} = 11$. □

P-B. You will receive A.

Triangle T has vertices $(5, 1)$, $(\mathsf{A} - 2, 5)$ and $(\mathsf{A}, 1)$. The midpoints of the sides of T form the vertices of a smaller triangle.

Let B be the area of this smaller triangle. $\boxed{\text{Pass on B}}$

Solution: The area of the small triangle must be one quarter that of T. Thus $BB = \frac{1}{4} \cdot \frac{1}{2}(\mathsf{A} - 5)(4) = \frac{1}{2}(\mathsf{A} - 5) = 3$. □

P-C. You will receive B.

Find the value of C for which the point $(0, \mathsf{C})$ is equidistant from the points $(3, \mathsf{B} - 1)$ and $(7, \mathsf{B} + 3)$.

$\boxed{\text{Pass on C}}$

Solution: The segment joining $(3, \mathsf{B} - 1)$ and $(7, \mathsf{B} + 3)$ has slope 1 and midpoint $(5, \mathsf{B} + 1)$. Thus the perpendicular bisector of this segment has equation $y - (\mathsf{B} + 1) = -(x - 5)$. Since $(0, \mathsf{C})$ lies on this line, we get $\mathsf{C} = \mathsf{B} + 6 = 9$ □

P-D. You will receive C.

Let D be the unique digit such that the number DCD is divisible by 7.

$\boxed{\text{Done!}}$

Solution: Modulo 7 we have $\mathsf{DCD} \equiv 101\mathsf{D} + 10\mathsf{C} \equiv 3\mathsf{D} + 3\mathsf{C} \equiv 3(\mathsf{D} + \mathsf{C})$. Thus DCD is divisible by 7 if and only if $\mathsf{D} + \mathsf{C}$ is divisible by 7. With $\mathsf{C} = 9$ get $\mathsf{D} = 5$. □

Individual Relay

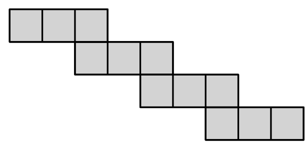
I-A. In the morning Jason bikes to work at 25 km/h, and in the evening he bikes home along the same route at 36 km/h. Jason's journey to work is 11 minutes longer than his journey home.

Let A be the distance (in km) between his home and work. $\boxed{\text{Pass on A}}$

Solution: We have $\frac{A}{25} - \frac{A}{36} = \frac{11}{60}$, so $A = \frac{25 \cdot 36}{60} = 15$. □

I-B. You will receive A.

Billy doodles on a sheet of 1×1 grid paper, filling in a number of squares to create the figure shown below.



Billy continues this pattern until he has filled a total of 3A squares.

Let B be the **perimeter** of the resulting shape. $\boxed{\text{Pass on B}}$

Solution: Start with 3 squares having perimeter 8. Each additional group of 3 squares adds 6 to the perimeter, so $B = 6(A - 1) + 8 = 92$. □

I-C. You will receive B.

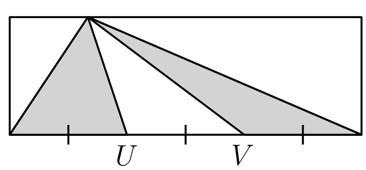
Let C be the sum of all integers between 1 and 100 whose last (rightmost) digit is the same as that of B.

$\boxed{\text{Pass on C}}$

Solution: Let b be the last digit of B. If $b \neq 0$ then we have $C = 10(1 + 2 + \cdots + 9) + 10b = 450 + 10b$, and if $b = 0$ then $C = 550$. With $B = 92$ get $b = 2$ and $C = 470$. □

I-D. You will receive C.

In the figure below, points U and V divide one side of the rectangle into three equal segments, and the shaded region has area C.



Let D be the area of the unshaded region. $\boxed{\text{Done!}}$

Solution: The shaded region is $\frac{1}{3}$ the entire rectangle, so the unshaded region is twice the area. Thus $D = 2C = 940$. □