Errata and Suggestions for Improvement
in the
Constructing Mathematics Textbook Series

by

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An earlier document, listing errors and suggestions for improvement in the *Mathematical Modeling* texts, was released by the Atlantic Provinces Council on the Sciences in November 2002 (see `http://www.math.mun.ca/~apics`). Provincial Ministries of Education received early drafts of that document, and have since circulated their own errata documents to teachers.

An early draft of this document, discussing the *Constructing Mathematics* texts, was sent to provincial Ministries of Education in February 2003.

Some teachers have suggested that the release of these documents has been "confrontational", and that some of our comments point to errors and omissions that are, surely, "obvious to all teachers."

Our responses:

- These documents were prepared after other avenues of communication had been exhausted.
- We know from experience that mathematics teachers are obliged to "state the obvious", frequently – something most of us forget to do, all too often. As we state on page 4 of this document,

> In most cases, goals and summaries can be found in the Teacher's Resource (TR) documents, which contain a lot of points that will be obvious to most teachers. This is *not* a fault. Such documents are obliged to provide details, if they are to be of assistance to busy teachers, some of whom are teaching outside their primary field of expertise. Indeed, we agree so strongly with this approach that we have followed the lead of the TR and indicated some points that, we are sure, many teachers will have recognized for themselves.

Three sub-documents have been copied from this document, for ease of reference by teachers. They cover, these three topics: "When are data Normal?", "Box and whisker plots", "Exponential functions", and can be found at `http://www.math.unb.ca/~maureen`. We will happily separate out other sub-documents, if teachers so request.

These various errata documents and sub-documents are temporary solutions to serious problems. Provincial Ministries have not announced plans for long-term solutions.

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1 Background to the Common Atlantic Mathematics Curriculum

In 1989, the National Council of Teachers of Mathematics (NCTM, a large North American group) produced a document presenting standards for mathematics curriculum and evaluation, for kindergarten through high school graduation. (The document is available at `http://standards.nctm.org/Previous/CurrEvStds/`). The introduction to the Standards lists "mathematical expectations for new employees in industry," including:

- the ability to see the applicability of mathematical ideas to common and complex problems;
- preparation for open problem situations, since most real problems are not well formulated;
- belief in the utility and value of mathematics.

Of course, employers value these skills, and educators should be aware of such expectations. Unfortunately, the new Atlantic Curriculum interprets the above to say: "All mathematics taught in the schools should be immediately applicable."

The NCTM Standards also make several references to the importance of "technology" in mathematics education. The Texas Instruments company has based a brilliant marketing campaign on these references, convincing K-12 educators across the continent that graphing calculators are essential for understanding mathematics.

Today's news media are full of statistics, and it is not surprising that these twin incentives (immediate relevance and technology) have led to an overemphasis of statistics in school curricula. Students should certainly be encouraged to watch news reports and read newspapers critically. (Does the report explain how data were gathered? Does it give the response rate for a survey? Does it quantify the doubt associated with numeric estimates?) Indeed, newspaper and magazine articles provide an ideal scenario in which educators can make connections between language and analytical skills.

If some of our students are eventually to become practicing statisticians, then they need to understand basic mathematics. The writers of this curriculum were not prepared to say: "Trust us. If you learn this today, you'll thank us later on." Multiplication tables may not look useful to a child in Grade 4 or 5, but the teacher knows that to do (for instance) arithmetic with fractions those multiplication facts will be essential. Of course, many "experts" in grade school education say that, with the advent of calculators, there is no need to learn arithmetic with fractions. Such statements demonstrate a basic ignorance of mathematics beyond grade school. Imagine trying to learn the rules of algebra if you don't know the rules for arithmetic with fractions. And so it goes on.

The theory of statistical inference is based on sophisticated mathematics. If students are to be shown the "black boxes" of statistical gimmicks available on a graphing calculator, then their teachers must be very much aware of when such boxes are appropriate and when they are inappropriate. Teachers must stress the importance of describing possible errors associated with conclusions based on data and, as with any subject, teachers must understand statistics at a level well beyond that which they expect of their students.

2 The *Constructing Mathematics* Series

In this document, our criticisms fall, roughly, into three categories:

1. Misinformation.

   (a) Errors in fact. Such errors occur primarily (but not exclusively) in the sections about statistics, finance, and geometry These topics are treated much less competently than algebra, and trigonometry.

   (b) Errors in explanation. These errors are usually the result of over-brief and over-simplified discussion of difficult topics.

   (c) Errors in emphasis. These include suggestions that trial-and-error is as good as an algebraic approach which can be adapted to handle more complicated problems; or basing algorithms / mathematical rules on observed patterns, in situations where students could understand a carefully-presented algebraic proof.

   Some sections take up a lot of student and teacher time while providing little new information.

2. Poorly constructed questions and problems.

   In many cases, introductions to supposed real-world problems lack information which would help students to understand the situation, thus causing unnecessary confusion. In other problems, the "input" (numbers, functions, etc.) are unrealistic. (These examples are pedagogically bad, because they damage the development of the students' number sense.) In some applications of mathematics to other disciplines, technical terms are used incorrectly; and in some cases "facts" are just plain wrong.

   Many questions are vague in the extreme. ("Discuss your confidence in your answer"). The TR[3] gives samples of correct answers, but rarely any suggestion of what would constitute a wrong answer. One may wonder whether a teacher could mark as wrong *any* answer to some of these questions.

   We by no means wish to discourage the asking of open-ended questions, which can be among the most valuable pedagogically. However, there is a distinction between questions to ponder and questions with "official" answers; and we suggest that both text and TR be more clear about such distinction.

[3] TR: Teacher's Resource.

3. Constructive criticisms which should be addressed when revising the texts.

   In some cases, these are suggestions for more lucid presentation of material. In others, they are examples of poor wording which one of us could not ignore. We have tried to include some background information about topics that have been treated particularly poorly.

As with the *Mathematical Modeling* series, these books are "lab manuals" rather than text books. They present a series of questions for investigation, without clearly stating goals and summarizing conclusions. Students who miss class will often need considerable help from the teacher in order to catch up. Since the texts rely heavily on facts and techniques to be assimilated in class, parents are effectively excluded from helping their children.

In most cases, goals and summaries can be found in the Teacher's Resource (TR) documents, which contain a lot of points that will be obvious to most teachers. This is *not* a fault. Such documents are obliged to provide details, if they are to be of assistance to busy teachers, some of whom are teaching outside their primary field of expertise. Indeed, we agree so strongly with this approach that we have followed the lead of the TR and indicated some points that, we are sure, many teachers will have recognized for themselves.

There are places where the TR should give clearer guidelines as to which results *must* be presented in class, and which are more "sideways enrichment". The TR should also state explicitly that specified key topics should be taught by traditional methods if time is not available for investigations and discoveries.

We do *not* claim to have found all errors in the *Constructing Mathematics* series.

2.1 Cultural sensitivity

We find it surprising that texts which purport to be relevant to students' every-day lives adhere, so exclusively, to metric measurement units and the strict Système Internationale protocol of grouping digits. APEF[4] appears to have taken upon itself the task of changing the mathematical culture of our region - innovations that go beyond anything mandated by law, and against what some future employers will expect.

While eliminating binary inches from the classrooms of tomorrow's carpenters may not be as offensive as punishing a child for speaking Gaelic or Mi'kmaq at recess, the underlying idea that the educational system can and should suppress an element of the students' culture is, at best, disturbing.

Units While the government has mandated that the majority of day-to-day measurement take place in metric units, there are still areas (carpentry, typesetting, sewing, and some engineering, for instance) in which inch-based units are standard. In some of

[4] Atlantic Provinces Education Foundation

these (notably carpentry and sewing) inches are not divided decimally but using binary fractions (halves, quarters, etc.) This is not just convention; it is done because the operation of halving (or finding a midpoint) is so ubiquitous.

Some recognition of (at least) binary and decimal inches, and points (seventy-secondths of an inch, used in both conventional and computer typography) should probably be given in sections on measurement. Conversion of inches and other "heritage units" into metric units would provide an excellent source of exercises on precision and significant figures at this level. (At an earlier level, conversion *between* heritage units would provide an interesting social context for multiplication and division problems, as well as making children realize *why* we now use the easier Système Internationale.)

Grouping of digits. It is true that the "pure" Système Internationale (SI) mandates the use of spaces, rather than commas (or periods), to separate groups of three digits. However, this usage is by no means universal among scientists in North America, and is more or less unknown among non-scientists. It might be better to compromise with common usage, as the book already does with time units (kilometres per hour, cents per minute: both excluded by strict SI usage), especially as most students using these books will not be majoring in science in university. Perhaps a sidebar feature would be appropriate, explaining the French, English, and SI conventions along with a warning that, when presented with a problem, students should check to see which convention is being used. The feature should mention also that year numbers are *never* broken.

2.2 Scientific notation

Scientific notation – while mandated as an outcome for high school in the APEF *Foundation for the Atlantic Canada Mathematics Curriculum* ("Outcomes", page 13) – has been almost completely omitted from *both* series.[5] Applications throughout all texts should occasionally use scientific notation (in appropriate contexts), and remind students to answer with reasonable implied precision (significant digits).

2.3 Reading level

We have heard many complaints that the *Mathematical Modeling* series requires too high a level of literacy. The *Constructing Mathematics* series is no easier to read, while the students are, generally, less academically inclined. Literacy is a prerequisite for numeracy. These texts breed anxiety about both.

[5] Problem 46 on page 115 of *Constructing Mathematics 3* may be the only reference to scientific notation in this series.

3 Constructing Mathematics 1

3.1 Chapter 1. Data Management

This chapter provides an extremely terse introduction to big statistical ideas. The authors demonstrate confusion about several concepts, including basic data types, outliers, precision, and fitting lines to data.

The distinction between discrete and continuous data is already needed at this level – for example, for a proper discussion of the mode. *The first section of Chapter 4 should be relocated in Chapter 1. As a stopgap measure, it could be used out of sequence.*

This chapter should introduce scientific notation.

Page 7, Think About... The *Mathematical Modeling* version is clearer.

Page 8, Measurement tools. The unit in which a scale is labeled is not a reliable guide to its precision; a centimeter ruler with tenths is as precise as a millimeter ruler without subdivisions. A vernier gauge may be marked in centimeters or inches but be precise to 0.01 cm or 0.005 inches; a micrometer may go still lower.

Page 9, Significant digits. The use of integers or decimals to indicate significant digits does not always work (there is no way to indicate 1000 ± 10, for instance [as noted in TR[6], page 11].) The only real solution is the use of scientific notation.

Page 9, Focus Question 2. Tools for measuring. "Why is it important to use the same tool to measure length and width when measuring area?" It is not important, though it is usually convenient. If the length and width are very different, you may get the best relative precision using different tools.

Page 10, Focus C. Precision. "... your answer will not be any more precise than your least-precise dimension." This is an oversimplification. For instance, 102m + 0.5m is accurate to 3 significant figures, not 1. The notes explain this better, but they belong in the main text.

Page 12. Question 8. Precision. "Environment Canada stated yesterday that the temperature was 21.225° C." The statement is nonsense. Environment Canada would never make such a statement. (For the record, we checked on this. But think about it.)

Page 12-13. Questions 9, 10. See above on units and precision.

TR page 17, Optional Activity. This is an excellent idea.

Page 14, Question 17; TR page 18. To say that the last three are most reasonable because they are in a majority makes no sense. With five students trying to record the height of one bounce, even a precision of 1cm seems optimistic.

[6] TR: Teacher's resource

Page 14, Question 19 Unless Ming took unusual precautions, his measurements may not be more *accurate* than Rex's. A few minutes' activity with a tape measure would be instructive here.

Page 14, Note. "Significant digits are used only when the numbers involved are measurements". Not so - for instance, $\pi = 3.1416$ to five significant digits.

Page 15, Reaction time. TR page 21. A "master" for a scale that can be printed out and taped to a ruler, to yield a reaction-time measuring device calibrated in *seconds*, can be found on the last page of this report.

Page 15, A note about the mode. The mean and median describe location for continuous (measurement) data. Sets of continuous data often have no unique mode: we have to "round off" to see a mode (as in the income data on page 18 or height measurements on page 20). The mode is used as a measure of location for discrete data (including non-numeric categorical data).

These are important issues. University-aged students often find it difficult to distinguish between discrete and continuous data. They also have trouble identifying nominal versus ordinal data. An earlier course in this curriculum discuss different types of data, but the obvious "connections" have not been made in this text.

Page 16, Question 1(a). 50 **repetitions of an experiment.**

The answer in the Teacher's Resource (page 22) suggests confusion between data as time series and behaviour of cumulative means: "It is likely that after some initial variations the measurements will 'cluster' around some value."

Page 18, Question 5. "Which average better describes the central tendency?" "Better" depends on use. In a situation where the sum of the measurements is of interest (here, predicting spring runoff?) the mean would be better, even if the distribution were skewed.

Page 18, Question 7. If we permit a set to have more than one mode, then $\{1, 2, 3, 4, 5\}$ has five modes, not none. The only correct answers to Question 7(a) would be the empty set, or an infinite set in which the frequencies increased without attaining a bound. Neither of these is what is intended: part (a) should be dropped. Otherwise, this is an excellent exercise.

Page 20, Outliers. In this book, the treatment of outliers is so brief that students are unlikely to understand them. Students should be able to learn and use Tukey's boxplot method for labeling outliers (see below). If the concept is not important enough to justify doing it properly, then it should be dropped.

Pages 22-24. Boxplots. Since the concept of an outlier has been introduced, why not use full boxplots, which indicate outliers? What the book currently uses is the "skeletal boxplot." The standard among statisticians and experimenters is the "full boxplot," as invented by the late John Tukey, perhaps the greatest authority on exploratory data analysis. The full boxplot is not much more complicated and gives a standardized way of flagging outliers that is understood worldwide, as well as a quick and easy way to see the basic shape of the data set.

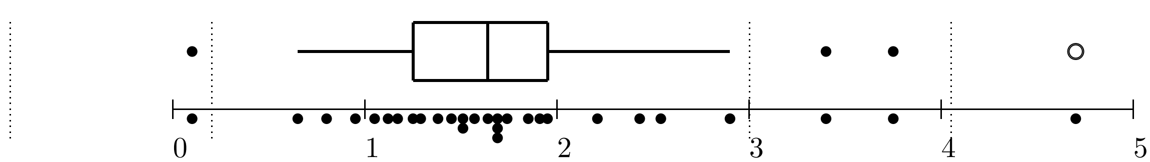
The box (extending from first to third quartiles) and the median line are drawn exactly as in the skeletal boxplot. The width of that box (technically called the "interquartile range," but "box width" will do in this context) is a nice descriptor of the variability displayed by the set of data. It is easier to calculate than the standard deviation, and easier to think about. A small box width means low variability; a large box width says high variability.

Tukey's insight, widely adopted by scientists, is that outliers are conveniently flagged as observations more than 1.5 box widths above the upper end or below the lower end of the box. We thus pencil in two lines 1.5 box widths above and below the box; these, called the"inner fences," are working lines that do not appear on the final plot. The whiskers extend to the most extreme data within these fences, not to the fences themselves. Every datum beyond the inner fences is an outlier, and is indicated by a dot.

Many researchers go further and draw a second pair of fences 1.5 box widths outside the inner fences (3 box widths beyond the box), and use a ring or bigger dot to indicate really unusual data that are even outside the outer fences.

Points which are isolated in this way by a boxplot are called "outliers" (or "extreme outliers," for the really unusual ones). In particular, in approximately normally distributed data, far fewer than 5% of observations are classified as outliers.

The diagram below illustrates construction of a boxplot. In order to demonstrate that a boxplot summarizes the location, spread and shape of the original data, we have displayed the raw data in a "dot plot" below the line, with the corresponding boxplot above the line. (The dotplot is *not* part of the construction of a boxplot. It is included here simply to compare two graphical descriptions of the same set of numbers.) The fences (dotted vertical lines) are part of the construction, but would be removed in the final plot.



Boxplots have two main uses:

1. Statisticians use them as a way to take a quick first look at large sets of data, looking for possible recording errors, unusual shapes, etc.

2. When plotted on the same scale, they provide quick informative comparisons of two or more sets of data. (e.g. male and female heights).

The boxplots on page 25 would give quick visual comparisons of relative bowling abilities *if* the same scale had been used for all five players. This demonstration misses the point of boxplots.

The series of boxplots printed at the bottom of page 136 in *Constructing Mathematics 2* demonstrate the use of boxplots for comparison of data sets. Note that *one* axis appears, below all boxes, and that no numbers are marked on the boxplots. (Most statistical packages would align the labels (a)–(c) on the left of the figure.) Boxplots can be drawn vertically, for equivalent visual effect.

Finally, any graphical display of data should "stand alone", with an informative title, clearly labeled axes, and footnotes giving the source of the data and sample size. (Look at any reputable newspaper or magazine.)

Page 24, Questions 10, 11. "Which group is more likely to have a student who has a lower reaction time?" This question concerns what is known in probability theory as the "stochastically below" relation. The properties of this relation are not elementary - in particular, it is intransitive (we can have group 1 $\succ$ group 2 $\succ$ group 3 $\succ$ group 1).

Certainly, this question is beyond the students' abilities (unless they undertake a brute-force enumeration of pairs). However, it does have a correct objective answer [the probability that the student from Group 1 has a lower reaction time is about 0.51, with about .02 probability of a draw]. There is no way in which this can be determined by casual inspection of boxplots or histograms, and students should not be encouraged to think that guessing on such a basis is a worthwhile activity.

Similarly, the correct answer to Question 11 is "there is not a significant difference". The question should take this into account: "Did one group have significantly better reaction times, and if so, which group?" - expecting the answer "No."

Page 25, Question 14. By standard definitions, neither group has any outliers. Outliers in the first case would start around 63; in the second, even higher. The lower inner fences would be below 0 in both cases. The implication in this exercise (reinforced by the TR) that an extreme value is an outlier (even in unusually short-tailed distributions such as those shown) is simply wrong.

Page 25, Side-by-side boxplots. Side-by-side boxplots should always use a common scale. In this book students never see this done correctly, but see it done incorrectly twice. (The corresponding chapter of *Mathematical Modeling* has some correct examples.)

Page 28, Question 21, Description of data. Here, students are asked to "Rate how your set of data is distributed using a 0 to 10 scale." This scale is *not* ordinal, though the use of numbers implies a natural ordering. A data set could be "moderately spread out" (Category 5) whether or not it had outliers (Category 7), and a data set could be "really spread out a lot" (Category 10) with no outliers at all.

In a chapter intended to introduce students to descriptions of data (in their many forms), the authors have used an inappropriate ordinal scale.

Page 32 ff. Line of best fit. How is this line to be determined? Later, on page 33 (F), the reader is directed to use the eye-and-ruler approach, which is fine. With the eye-and-ruler technique, a student could well fit different lines to identical scatter plots.

There is not sufficient emphasis placed on quantifying doubt, margin of error, whatever you want to call it. Teaching statistics without these ideas is dangerous, if not unethical.

Page 33, Investigation 4; Page 36, Question 7; and TR. On Page 33, students are meant to "recognize the dangers of" an extrapolation beyond the range [3,11] to the values 2 and 12 (an extrapolation through 12.5% of the range). On page 36, they are meant to be "fairly confident" of an extrapolation from the range [1920,1990] to the value 2020 (through 42.8% of the range). Even for ill-defined problems, these answers are inconsistent.

Page 35, Question 6. There is nothing wrong with making up data for exercises, but it should look natural. Students ought to suspect the "number of goals" figures, as 9 out of 13 data are divisible by 5. Too many round numbers is a standard give-away for made-up or estimated data.

(The most constructive way out of this dilemma would be to add a rider: "Mary Anne suspects that some of the players were making up or estimating their numbers of goals scored. Why might she think this?")

3.2 Chapter 2. Networks and Matrices

The unsuitability of this material for the high school curriculum was discussed in our document discussing the more advanced *Mathematical Modeling* textbook series. Everything said in that context applies even more strongly to this chapter. The only major difference between the two series is that *this* one has five extra "extension" topics, including tournament digraphs, adjacency matrices for polygons, and Huffman coding.

Page 50, Sidebar. Complete. This is not the standard meaning of "complete." The correct word is "Eulerian."

Page 51, Sidebar. Euler circuit. This is not what an Euler circuit is, either. An Eulerian graph *has* an Euler circuit.

Page 62, Question 12(a). Paths in a directed graph. The assumption that the routes are acyclic is implicit in the answers given in the teachers' manual and is contrary to the spirit of the matrix formalism.

Page 67 ff. Matrix multiplication. Matrices are used in many fields. Their entries may be positive or negative, and certainly need not be integers. A few problems requiring multiplication of matrices with fractional entries would be appropriate here, *and* a valuable opportunity to review basic arithmetic facts before the algebra of subsequent chapters.

A simple application of matrices, which convinces many students that matrices may indeed be useful, is this. Given a spread sheet of numbers (rows labeled by student names, columns labeled by test scores), show how to calculate each student's term mark (possibly a weighted average) as a problem requiring the multiplication of two matrices. etc.

3.3 Chapter 3. Patterns, Relations, Equations, and Predictions

As mentioned in the comments on the *Mathematical Modeling* series, the term "marginal cost" should be introduced and used. The substitution of the term "average cost" (as on pages 123-4) is wrong.

- By the "average cost" one would normally understand the *total* cost (including base fee) divided by the number of units. The cost corresponding to the slope of the graph is the *marginal* cost.
- The word "average" is redundant in the context of marginal costs anyway; the marginal cost of every hour is the same, so dividing the rise by the run gives the exact (marginal) cost of one hour of service.

As in other chapters, a few practical examples are overused. There are several problems with this lack of variety, including the following.

- It is likely to reduce the interest of the book for most students.
- It gives the students less practice in skill transfer. The main point of this chapter is linear equations (which students will need to apply in a multitude of circumstances), not Internet fees.
- Without a greater variety of applications, examiners may feel obliged to use "Internet provider questions" on exams, providing an artificial hint ("if this is the Internet, it must be linear equations").

The opening lines of Section 3.3 (page 114) ask the all-important question about the two internet companies: "For what number of hours of Internet use are the costs the same?" This is an important question because the answer (20 hours) describes a critical number of hours of monthly internet use: below 20 hours one should use Company C; above 20 hours one should use Company A. We cannot find discussion of this crucial idea anywhere in the chapter.

The other major conceptual error is with the unusual (and inaccurate) introduction to quadratic functions in Focus K. The errors made here are not corrected elsewhere in the text.

Other problems with Chapter 3:

Page 97, B. "Does it make sense to join the points on the graph with a line or a curve?"

This is intended as a yes-or-no question (as TR, page 113, makes clear). In *Mathematical Modeling 1*, the hint was given that the answer involves the discrete nature of the data, giving the student some chance of figuring out the answer.

Here, the student has no reason not to suppose the question is an either-or question, with the apparently correct answer "a line".

Students using this book are obviously meant to be able to use the concepts "discrete" and "continuous" to answer questions. These are important concepts, especially for the APEF curriculum objective of choosing appropriate data displays. They are not too advanced for students who can understand the rest of this text. They are introduced in Chapter 3 of *Mathematical Modeling 1*: for some reason, the definitions are left to Chapter 4 of this book, long after the students have started to use them.

Page 98, Note. A student who needs to be told that "only quadrant 1 is used in this problem" will probably also need to be told or reminded what quadrant 1 is. A suggested rephrasing: "In this problem, negative numbers of cubes or faces don't make sense. The pairs (x, y) in which both x and y are positive or 0 make up quadrant 1."

Page 98, Questions 4, 5. (b) "Is a graph the best way to do this?" If the graph is drawn at all carefully then (since the answer to 5(a) will be a positive integer) the graph is as good as any other method. The TR comment that the answer obtained from a graph "may not be exact but should be close" seems strange unless the discrete nature of the variables is ignored.

4(c), 5(c) "How confident are you that your answer is correct?" In Chapter 4, little attention is paid to quantifying doubt. In a situation where there is nothing random about the underlying model, the authors ask about doubt.

Page 100, Sidebar. "An answer that you find using a graph is only an estimate." The authors are confusing two kinds of scatter plots

- those that arise from observations which vary randomly about an underlying model;
- those that arise from plotting points (x, y) where y is a function of x.

Such confusion is more blatant in Chapter 4. In this particular problem, any tidy graph will lead to correct answers, since students understand that the answer will be a positive integer.

Page 102, Question 15. Unrealistic data. If Frau could find a way to measure gas consumption at 1-hour intervals, then the data would *not* fit perfectly on a straight line.

Page 103, Sidebar. The definition is provided for an "equation" in one variable, x. Students have already been deriving equations involving two variables.

Page 112, Focus Question 11. "Explain why constructing a table of values and / or using a graph would not be good strategies for getting an exact solution for this problem." Yes, it is best to resort to algebra when the numbers get complicated; this point should be made earlier and more emphatically!

Page 128, Focus H. Students would (and should) understand this repayment scheme to involve monthly lump sum payments. Thus, the graph is not a straight line but a "staircase" function. It may be represented as a discrete graph as shown; but such a graph is not a line in the usual sense.

Page 129, Question 13(a). As asked, this question requires a 131-entry table (we note that the TR leaves off after 4 entries). The relationship does not describe "the number of kilograms of potatoes still in the food bank after each family has received potatoes" (as written), but the number of kilograms of potatoes still in the food bank *as a function of* the number of families who *have* received potatoes.

Page 129, Question 13(b). There is absolutely no reason for the students to "estimate" the y - intercept.

TR page 148, Sidebar. This graph is uncomfortably steep. As the axes are dimensionally different there is no need to try to make the intervals the same.

Page 129, Question 14. Extrapolation. Students are asked to fit a straight line (a topic from Chapter 4) to data which surely requires a different model. They are then asked to extrapolate way beyond the range of observed data (recall Investigation 4H on page 33).

Page 132, Focus I. The "appropriate technology" for graphing a function of the form $y = mx + b$ is a pencil, a straight-edge, and a sheet of paper. This is especially true for students who may not have a graphing calculator or computer available to them in the workplace once they leave school.

It is particularly appropriate technology for graphing equations of the form $ax+by=c$, partly because you do *not* need to convert the whole equation before graphing. Instead, set x to 0 to find the x intercept c/b, set y to 0 to find the y intercept c/a, and then draw the line through the two points $(0, c/b)$ and $(c/a, 0)$

Use of pencil, straight-edge, and paper helps to reinforce the idea that the straight line consists of *many* points (x, y), and every one of those points satisfies the equation.

Page 133. Investigation 7. The pattern is difficult to follow: three of the four squares have black in the top left position. For an obvious pattern (add a row on the bottom and on the right) reverse the colours in the second square.

Page 134. Question 1(d). The word "accuracy" has multiple meanings, which are not well distinguished in this book. Sometimes it is used in the sense of "how close an estimate or approximation is to the correct answer". Here it is redundant. "How might you check your answer?" would be better.

Page 137. Focus K. The crucial idea behind using factoring to find the x-intercepts of a quadratic function is this:

> If two real numbers w and z satisfy the equation $wz = 0$, then at least one of them is zero.

Page 138. Focus K. Julia's light-bulb idea is illogical. *If* I know that a function is of the form $f(x) = ax^2 + bx + c$ and if I also know that $f(1) = 0$ and $f(3) = 0$, then I can certainly prove that $f(x) = a(x-1)(x-3)$. *However*, there are many *other* types of function g for which $g(1) = g(3) = 0$.

For example, the graph of $y = \sin x$ (with x measured in degrees) crosses the x-axis at $0, 180, 360, \ldots$, but multiplying $(x-0)(x-180)(x-360)\cdots$ gives no insight into the relationship between x and y.

Page 138. Sidebar note This should say that the product ab is 0 if *and only if* at least one of them is 0.

3.4 Chapter 4. Modeling Functional Relationships

This chapter considers two very different sorts of graphs: those that arise from plotting points (x, y) which satisfy some kind of mathematical equation (e.g. Focus C has several equations relating earnings to hours worked; problem 6 on page 191 describes GIC account balance as a function of time), and those that arise from plotting empirical data (e.g. Manatee data on page 179). With the first type of graph, there is nothing random, no "doubt." There is all kinds of doubt (also called "error") when lines and curves are fitted to empirical data. There is no discussion of the inherent differences between these two sorts of graphs and, consequently, the material is confusing and disjointed.

Both situations (mathematical equation, empirical data) can lead to plots which are not the graphs of functions. The text seems to imply that non-functions are associated primarily with empirical data, omitting discussion of classic mathematical equations whose graphs are not the graphs of functions. The text omits explanation that least squares models force a functional relationship to a scatter plot.

The haste to introduce data-driven examples has confused introduction of the basic mathematical concept of function. For the purpose of this text, a function f is a *rule* which associates a unique real number $f(x)$ with each real number x in its domain. We often write $y = f(x)$ and describe the function f as a set of ordered pairs (x, y), graphing the set of points on coordinate axes. When we use the word "function," the rule should be clearly stated in the form $f(x) = \ldots$ or $y = \ldots$.

Page 165, Question 21(b). At first glance it appears that the order of dependent and independent variable are interchanged, that the numbers are very unrealistic, and that the correct answer should not be a function. However, the most likely explanation is sloppy wording: the independent variable is not the number of boxes of cargo that the truck *can carry*, but the number that it *is carrying*. The masses are not the unloaded masses of trucks with different capabilities, but loaded masses for one truck with different loads.

Pages 167-9 Investigation 2. "Draw the line or curve that best fits your data." Neither the text nor the TR suggest how this should be done. We suggest a *line*, drawn using straight-edge or a piece of string (pulled tightly).

Pages 169-170, Questions 29,30 and TR. The graphs on TR page 193 and 194 both show strong but not perfect linear trends ($r^2 = 0.995,\ 0.987$). No good explanation is given as to why the relations should be described differently (the first as "linear", and the second as "nonlinear"); and certainly no criterion that made that distinction could be reliably checked "just by examining the table", as suggested on TR page 192.

Much weaker trends are described as "linear" in Section 1.3 (for instance, the manatee data set has $r^2 = 0.905$). Students should *never* be encouraged to "see" a complicated pattern when a simpler model adequately describes the basic pattern in a set of data.

Page 176, Sidebar. The vertical line test has already been introduced on the previous page. The sidebar note here ("imagine a vertical line crossing each graph at more than one point") is not only badly phrased, but unnecessary.

Page 177, Question 9 (c)(d). Has the convention that an empty circle represents a missing point been introduced in a previous course? If not, it should be made explicit here.

Page 181, Question 3(b); page 186, Sidebar; page 189 E. What mathematicians (more usually, statisticians) call a correlation is *not* a subjective rating on a 0-to-10 scale. There is nothing wrong with introducing the word "correlation" and using it when describing the relative strengths of trends in two different scatter plots. We see no advantage to presenting correlation as a number that can be obtained by pushing buttons on a calculator (with no indication of how it might be calculated), or as a subjective guess.

Page 181, Note. This is a serious oversimplification, and in some cases simply wrong. In particular, the least-squares line is most influenced by the points that lie a long way away from it.

All that can usefully be said at this level is that when the points are very close to lying on a straight line, the exact choice of line does not matter much and an "eyeball fit" will be useful for most purposes.

TR, Page 203, "Technology". "Note that the slope and intercept of the equation have been corrected to ... the least number of significant digits in the data." This is meaningless. Firstly, as with the average of a large set of data, it is possible for the slope and intercept of a regression line to have *more* valid significant figures than the data. More importantly, though, the major sources of variation in such data are not represented by the precision with which the data are stated. Here, for instance, the lack of a perfect linear fit is *not* explained by "rounding to the nearest manatee".

The relevant measures of error are the "standard errors of the coefficients", an important diagnostic not given by the TI-83 calculator. For the manatee data, these are 6.8 for the constant term and 0.011 for the slope. The slope is thus (with 95% confidence) between about 0.11 and 0.15, and the y-intercept between -51 and -37.

Some students will balk at the physical interpretation of a negative intercept – a nice opportunity to discuss the dangers of extrapolation.

Page 188, r^2. The r^2 value is used for several reasons, but not because "some parts may be sloping up and others sloping down". Even with linear regression models, r^2 is used as a measure of how much of the variation is explained by the model; the specialized use of r to describe slope as well is not appropriate in a multivariate or nonlinear model, whether the slope changes sign or not.

It should be made explicit that a low correlation may represent either randomness in the data or nonlinearity. The illustrations in the text don't make this important point.

Page 189, Polynomial regression. Polynomial regression is a technique that professional statisticians use rarely and with caution. Using it on a small data set (as here) for extrapolation (as here) is foolhardy. Intelligent use of polynomial regression is outside the scope of this course; the topic should be dropped.

As a simple illustration of the pitfalls of this technique, note that the graph on page 214 of the TR suggests that cod shrink during the first two years of their lives. (But see the note on Question 2 below.)

Page 189, Investigation Question 1; TR page 215. Any bivariate data that are not perfectly correlated will give a higher r^2 for a quadratic fit, and will usually lie at least as close to the quadratic curve as to the linear curve. The problem here is that the improvements may not be significant (either statistically or practically). The cost in model simplicity may not be justified; the best model is usually not polynomial.

Page 189, Investigation Question 2; TR page P215. "14.03 kg is a more reasonable prediction than 28.71 kg". Why? This response supposes that after putting on 4.4 kg in its 8th year, a cod's growth suddenly slows so that it puts on only 3.6 kg over the next 4 years.

If instead it continued to grow at the same rate, the cod would weigh about $11.4 + 4 \times 4.4 = 29$ kg.

Indeed, if one examines the logarithms of the data, we see that an exponential growth model fits the data excellently ($r^2 = 0.994$). By *this* model the cod would weigh about 48 kg after 12 years.

Which model is correct? We cannot tell on the strength of the data. This is why extrapolation – in the absence of a model that gives some reason to suppose that trends will (or won't) continue – is usually foolish, and students should not be encouraged to engage in it.

Page 190, Question 4. The depth of the water decreases faster in hot weather; that is not the same as saying that it is *less* in hot weather. If the bath is filled each morning, then water level will be higher in the heat of the day than later, in the cool of the evening.

3.5 Chapter 5. How Far? How Tall? How Steep?

This chapter is technically correct. Most of the comments below refer to inaccuracies in wording, and lack of connection to other chapters.

It might be good to start with a review of a few facts used in the chapter: a straight angle is 180°; the sum of the angles in a triangle is 180°; formulae for calculating area of a triangle.

Page 207. Sidebar, second note. The definition given is of a similarity, not a dilatation.

Page 209, Sidebar, second note This should read "When a ratio *of measurements* is calculated..." In many problems the lengths given will be exact.

Page 211, Focus C. The statement that "compass bearings are based on magnetic north" is not true in general. The orienteering context should be emphasized: "In orienteering, compass bearings ..."

Page 218–219. The symbols A and B have different meanings on these two pages.

Page 219, Sidebar. Proof. Where in the text is “proof” defined or explained?

Page 220, “Did You Know?” This is a beautiful problem, but requires a knowledge of modular arithmetic. The level of difficulty is about right for a provincial-level competition or a university discrete math assignment. At the very least, the TR ought to have the outline of the solution. Moreover, the throwaway “Why do you think this is so?” is highly misleading. The proof is not a simple conceptual one; it involves several steps, and nobody – even a professional mathematician – has any right to “think” it's so without doing the work. Any attempt to guess a one-line outline of a proof (“Because $3 \times 4 \times 5 = 60$.”) will be wrong.

Without giving the answer away completely, the following hints may help.

1. This problem involves *modular arithmetic*. At the very least, the solver must realize that remainders from division by any base can be added, subtracted, or multiplied.
2. While the full Chinese Remainder Theorem is not needed, the solver must know (*e.g.*) that a number is divisible by 12 if and only if it is divisible by both 3 and $4 = 2^2$.
3. For two of the factors, the search is based on enumerating solutions to Pythagoras' equation modulo those factors.
4. To show divisibility by 4, first show that either all three numbers are divisible by 2 or only one is; in the first case we are done. In the second case, write the two odd numbers as $2p + 1$ and $2q + 1$ and show that the even square must be the difference, not the sum, of their squares. Now show that the even square is divisible by 8, and conclude (using either a variation on the classical proof that $\sqrt{2}$ is irrational, or facts about prime factorizations) that it is in fact divisible by 16. (Another approach, probably more time-consuming, is to work modulo 16 throughout.)

Page 222, Introduction to square roots. The preceding chapter spent considerable time on quadratic functions, yet there is no reference to the graph of $y = x^2$ when the “principal square root” is introduced—nor elsewhere in the chapter.

Using a carefully drawn graph, it is easy to demonstrate that, for example, $\sqrt{9+4} \neq \sqrt{9} + \sqrt{4}$.

“The square root of 16 is 4 or −4.” This may only be a grammatical mistake, but it's confusing. At least at first, it is better to stick to the usages “the principle [or ‘positive'] square root”, “the negative square root” and “the two square roots” unless the context restricts the range to the non-negative numbers. It is also appropriate to say: “There are two solutions to the equation $x^2 = 16$, 4 and -4 . . . .”

Page 222. Think About... The sidebar omits the most important question: *for which* x is $\sqrt{x} > x$? For which x is $\sqrt{x} = x$?

Page 223 or Page 224 Two other good problems to put in here would be:

- Make a list of all the whole numbers from 1 to 50 that are the areas of squares with corners on the grid. Which of these squares are slanting? Can you find any patterns? Do numbers which are areas of slanting squares seem to get rarer or more common as you keep going?
- Is there a square with corners on the grid that has an area that is not a whole number? Why or why not?

Page 226, Question 8. This problem is illogical. For example, $\sqrt{72}$ could be the length of the longest side of a flower bed similar to any one of the three given triangles: dilations of 6, $\sqrt{14.4}$, $\sqrt{7.2}$, going clockwise from top left. (The authors of these texts have a disturbing tendency to assume that numbers are integers, without saying so.)

Page 226, Question 9. As the figures from which Lydia calculated are also measurements, both lengths are approximate. Indeed, we know that Elijah measured to within 1 cm; we do not have any idea to what precision Lydia's data were measured. By the way, it is not clear how Lydia made her calculation; note that $5\sqrt{3}$ is not the hypotenuse of any right triangle with integer sides ($(5\sqrt{3})^2 = 75 \equiv 3 \bmod 4$.) The utility of the two representations (not "lengths") depends on the intended purpose. For some purposes (e.g., laying out further geometric features in the garden) Lydia's representation might be more useful.

Page 228 ff. Investigation 4. This investigation should refer back to Investigation 1, which is remarkably similar.

Page 230. Figure. This simple figure should have appeared much earlier in the chapter.

Page 231. Sidebar. Definitions. "$\tan X$—a constant value based on the ratio of the side opposite to angle X to the side adjacent to angle X in a triangle." If tan were constant, then all values in the last column of the table on page 233 would be the same (and we wouldn't need tan buttons on calculators).

Of course, the authors are trying to say that the size of angle X is all that matters; that, no matter what their side lengths, *all* right triangles with angle X will yield the same value for $\tan X$.

The phrase "based on" is needlessly vague here. Similar corrections are required for the definitions of sin and cos. Wording of definitions must be clear and precise.

Page 231, Sidebar diagram. This diagram suggests that $\tan 45°$ is approximately 6. The ruler should be renumbered.

Page 232 ff. Check Your Understanding. Question 16 from page 237 of *Mathematical Modeling 1*, or something similar (on the classic $30° - 60° - 90°$ triangle), should be included here. This triangle and the classic isosceles right triangle should be featured in a sidebar of the chapter.

Page 233, Trig table. It would be appropriate to include a note about interpolation, for angle measures that are not whole numbers. This is of historical interest, is still useful in a pinch, and should make the students appreciate their calculators (and their ancestors).

An interesting questions to ask: "About how many pages long would the table have to be to get values, without interpolation, at intervals of 1°? 0.1°? 0.01°? For each angle that your calculator can display?"

Page 239, Question 8. The story should clearly state that the first pole is directly in front of Jesse.

Page 245, Case Study 2. "• A section showing how $\sqrt{58}$ relates visually to 58."

This is not an easy question to answer well. The problem is that both $\sqrt{58}$ and 58 are numbers, and dimensionless. A solution that shows them in a comparable fashion (e.g., as line lengths) would ignore the obvious visual interpretation of the square root. A solution that shows one as an edge length and the other as a square may not be depicting "58". Possible solutions would be a square with edge length about 7.6 units, divided into 49 unit squares, 14 rectangles, and a small square; or following Euclid by indicating the ratio $1 : \sqrt{58} :: \sqrt{58} : 58$ using similar rectangles. It is to be feared that many responses will merely indicate that $\sqrt{58}$ is quite a bit smaller than 58, or put arbitrary labels on the center and side of a square.

3.6 Chapter 6. The Geometry of Packaging

This chapter is well written, and uses techniques learned in the preceding chapter. There is a lot of packaging for a relatively small amount of geometry. As in other chapters, teachers need to be told which material is crucial for understanding of subsequent sections of the chapter, and which material can be skipped.

Answers that are irrational numbers must not be given as integers or terminating decimals. We have not recorded all instances of this mistake in the texts.

Page 254. Shipping costs are a major consideration of package designers.

Page 255. Second set of figures. All symbols appearing in formulae should also appear on the diagrams: w, d, r are missing.

Page 263. Question 3. Results from this problem are so fundamental that they should be summarized somewhere in the chapter. (They are needed on the next page.)

Page 264, Sidebar. Rotational symmetry is defined as "the property of a shape where . . . ." This is incorrect grammar.

Page 269, Question 20(c). The term "limiting shape" is sloppy. (One should say " . . . as the number of sides goes to ∞ . . . .") The term "maximum shape" is simply wrong.

Page 270, Question 25. Unrealistic numbers. Rectangular lumps of fudge (as illustrated) of any reasonable size will not come close to packing such a box (base not square, barely 3" across, half that in height). In particular, the implied precision of "3.8 cm" is unrealistic.

Page 282, Economy Rate. Students should be taught that economy rates have dimension, equal to length (measured in cm on page 282), and are not a dimensionless quantity as suggested.

Page 286, Cube-root. How many parents will rush out to try and buy a calculator with a cube-root button? See also pages 287, 288: "Use a calculator to find out." See our introductory comments on the organization of *Mathematical Modeling* 1, and our expanded comments about organization of material in Chapter 3 of *Constructing Mathematics 3* (page 55 of this document).

Page 286, Question 8. The intent is not clear here. Perhaps the following was intended (in the spirit of Chapter 1): "You are handed a cube which has (although you are not told this) volume 100cm^3. How would you measure the edge length of such an object?"

Perhaps this meaning was intended: "You are handed a cube, and told that its volume is 100cm^3. Knowing the volume, how would you calculate the edge length of such an object?"

Page 289, Question 1. Volume of a snow house. Most Canadian children would wonder what was expected here: the walls of an igloo are thick; the *usable* inside volume would be much less than that calculated from the overall diameter.

Page 289, Question 2. Realism. Most Canadian students know that a snow house with square base and flat roof would be impossible to build.

4 Constructing Mathematics 2

4.1 Chapter 1. Making Choices – Linear programming

Chapter 1 of *Constructing Mathematics 2* covers the same topics as Chapter 7 of *Mathematical Modeling 1*. Examples, emphasis and page lay-outs are similar, but not quite the same. The chapter in *Mathematical Modeling 1* covers 31 pages, while the corresponding chapter in *Constructing Mathematics 2* covers 44 pages. The extra pages are used to provide more complete discussions of topics which are rushed in the corresponding *Mathematical Modeling* chapter. There are more worked examples, some "Notes" indicate key problems which must be worked before going on, and there are some answers in the back. Such aids are missing in the corresponding chapter of *Mathematical Modeling 1*.

On the other hand, the *Mathematical Modeling* series (Book 1, page 321) does a better job of justifying **why** an objective function achieves its maximum or minimum value at a vertex of the feasible region. The *Constructing Mathematics* TR (page 47, Question 5) alludes to such justification, without showing any graphs.

Many of our previous comments on Chapter 7 of *Mathematical Modeling 1* apply to Chapter 1 of *Constructing Mathematics 2*. In particular, instructions are often too vague – a recurring editorial problem with these texts. (e.g. Page 16, problem 2(b). How many students will puzzle over how to represent the last sentence by an inequality?)

The key idea in linear programming is this: if the feasible region is bounded by straight line segments in the xy plane, then any objective function of the form $f(x, y) = ax + by + c$ will be minimized / maximized at a vertex of the feasible region.

Page 13, Question 30. Incomplete explanation in TR page 27. The "relationship" is between the time spent walking and jogging. In the TR solution, x represents the number of hours that Mark walks in any week, and y represents the number of hours that he jogs in the same week.

Page 16, Question 3(c). Problem more difficult than authors realize. The solution in TR (page 34) is incorrect, and must cause confusion. *If* a graph with an axis labeled "Day number" were appropriate for this problem, then surely such an axis would be relevant to Heather's problem?

Correct solution: Let x represent the number of servings of fruit a teenager eats in a day, and let y represent the number of servings of vegetables. Then the required inequalit<u>ies</u> are $x + y \geq 5$ and $x + y \leq 10$. Of course, a better solution would use variables with easy-to-remember names, such as f and v.

Page 20, Question 10. TR page 41. Notation. In problems such as these, teachers should be careful to label the rows "Number of Morning Glory packs", "Number of jars of jam", etc. We note that, in most places, the authors have paid careful attention to these important details.

Page 22, Table in A. Continuing the comment on notation. In this table, the intent is to summarize production and profit information for each hat and each visor. So, it would be better to omit the symbols h and v. Alternatively, label these rows "1 Hat" and "1 Visor" then add two more rows labeled: "h Hats" and "v Visors".

Page 24, Sidebar. Inappropriate confidence scale. (See comments on Book 1, and our report on the *Mathematical Modeling* series.)

Page 27, Graph. "It looks like the point of intersection is (18, 12)." The graph does not look like this. A better guess for the graph shown would be (15,12) or (15,13).

Page 28 ff. Focus F and G The TR suggests a total of 65 - 90 minutes for the important topic of solving systems of equations algebraically (with and without fractions). If this is the first time students have seen the procedure, then the topic will need to be revisited several times beyond this 90 minute limit.

Page 28 – 29, worked example in Focus F. TR Page 51. Approximate solutions. The TR states: "The exact solution to this problem is (17.78, 11.33)". When students resort to calculators to approximate answers, they should be taught to indicate the approximation. In this problem, the solutions should be written $x \doteq 17.78$ and $y \doteq 11.33$. Such conventions help students to understand (eventually) that $\frac{4}{0.225}$ is one point on the number line, and 17.78 is a different point, very close to it.

Any *practical* solution to this problem would *not* be that Heather should cut 17.78 chair bundles and 11.33 couch bundles.

Page 29, Sidebar. Check "verify the point of intersection" is a roundabout way to say "check your answer". Students should be encouraged to check their answers to *all* problems. It would be simpler, and more to the point, if the note simply said: "Check your answer: substitute your values of x and y into the original equations, and simplify to see whether both equations are true."

Page 35 and TR pages 59 – 60. Errors. In the first paragraph of the text, x is defined to be the number of minutes that music is played during a 30-minute radio show, and y is defined to be the number of minutes of commercials during the same show. But part (a) states "Let x represent the length of a song and y represent the length of a commercial." This second definition of x and y is incorrect. The solution assumes the first definition. The TR gives the correct answer for part (a), but the feasible region is incorrect.

The correct feasible region is bounded by the points $(5, 5)$, $(25, 5)$, $(18, 12)$, $(12, 12)$. The solutions to (c) and (d) are correct, with the maximum number of listeners occurring when $x = 25$ and $y = 5$ (a point that is not part of the feasible region sketched in the TR).

The extension to the problem involves a cost function which is constant on one edge of the feasible region (the answer to part (a) is correct). Cost is $9000 for all (x, y) on the line segment joining $(0,0)$ and $(12,12)$, and this is the minimum cost over the feasible region. The TR should stop to discuss this phenomenon: if the objective function takes the same value at two vertices, then it takes that same value at every point on the line segment joining those two vertices. (Ideas discussed on page 321 of *Mathematical Modeling 1* help to explain this fact.)

Finally, question (c) of the extension does not make sense as stated. If the word "minimum" were omitted, the the answer would be: substitute $x = 25$ and $y = 5$ into the cost function to obtain $11,000.00 (the answer given in the TR). A more interesting question would be: "What is the maximum number of listeners that can be attracted for this minimum cost?" The answer is: "Let $x = 12$ and $y = 12$, and there will be 3000 listeners."

Page 38. Misprint. h should be n in the last inequality.

Page 42. Question 1. Language. The word "graph" has more than one meaning in this problem. Perhaps part (b) should read: "Find three points on the graph paper that satisfy the inequality $y \leq 2x - 1$."

4.2 Chapter 2. Mathematics – Check it Out!

Most of this chapter is contained in *Mathematical Modeling 2*, Chapter 2, and the reader is referred to our earlier document. The suggested topics vary greatly in difficulty. The TR gives no "back up" help or additional information for teachers, but lists additional topics.

4.3 Chapter 3. Decision Making in Consumer Situations

Much of the material in Sections 3.1 and 3.2 is excellent, and involves the sort of mathematics (pay stubs, loans, deductions, etc) that even the least academically-inclined student will need in the real world.

However, Sections 3.3 and 3.4 venture into much more difficult types of problem, involving payments at different times. When looked at over time, all money must have interest attached to it (either interest paid out on a loan, or interest earned on savings). It is a standard axiom of financial mathematics that to compare payments at different times, one must determine their values at a common time, based on an appropriate rate of interest and/or inflation. Thus, with a 10% annual rate of interest, a $100,000 payment now and a $110,000 payment in a year's time are equivalent. Applying this axiom to a mortgage or annuity, for instance, yields rather complicated formulae, often involving indexed sums of exponential functions. Such formulae would be quite beyond the scope of this course, and we agree with the authors' decision to omit them.

The authors have chosen to omit all present-value / future-value corrections when comparing payment schemes. From a business-math point of view this would be very bad practice. However, it is actually quite appropriate in many situations faced by a modern high-school student. In these days of low inflation, it is not uncommon for an individual to keep significant savings in a non-interest-bearing current account.

Consider, for example, "Challenge yourself" on page 96, where Monica has saved \$300 per month for two years. The calculations given in the story imply that no interest was earned on that money. Monica will neither profit from delaying payment of a bill until the last possible date, nor lose by paying a bill early. Present-value corrections are irrelevant.

It is not clear whether the decision to attach no interest to money viewed over time was deliberate or accidental. Nevertheless, the text should clearly state that a simplifying assumption has been made, *and* alert students to situations in which such simplification would be inappropriate.

The assumption that any money not explicitly borrowed or invested in the exercise attracts no interest does make some (otherwise inaccessible) problems accessible to less-mathematical students; and the relevance to their own personal finances is actually increased. Teachers *must* be aware that the simplified solutions provided in this chapter can be used when:

- *The consumer's excess cash does not earn significant interest over the time involved.*
- *The decisions made do not affect the amounts that the consumer borrows or invests.*

In particular, in Investigation 7, Sasha cannot use the simplified model: in the time before the purchase he is investing his savings in a GIC, earning interest.

Students *must* realize that, when the sum involved is large enough that savings will be invested productively or used to reduce debt (while losses will reduce investment or increase debt), more sophisticated computations must be used to make comparisons across time. Common situations of this type include saving for retirement, using a mortgage to finance a house purchase, and dealing with serious credit-card debt problems. Twenty-five years ago, when interest rates were high, these considerations could not be ignored, even for savings accounts. Perhaps interest rates will rise again within the next twenty years. Thus, we are obliged to make students aware of this simple fact: *when looked at over time, all money must have interest attached to it.*

In a revised edition, material in Sections 3.3 and 3.4 might be split into two parts. In the first part, the simplified model should be explicitly introduced, and used to study different investment and repayment schemes. In the second part, compound interest and major financing decisions should be considered. Financial calculators should be introduced, and used (either the TI 83, or a good web site). Then Section 3.3 of Book 3 should "make connections" with the material in this chapter.

Investigation 7 (pages 89 – 91) introduces a classic comparison of simple interest and compound interest. The story compares a 5-year GIC that pays out simple interest at the end of each year and a GIC that reinvests (i.e. compounds) interest at the end of each year. The point of the exercise (Question 19, and TR) is that the graph of total interest earned versus time is a straight line for the first savings plan, and a curve for the second. The point that is *not* made in either the text or the TR is this: so long as 5% is a reasonable interest rate over the five-year period, the two plans are equally fair to the investor. Under the first plan, the investor has quick access to earned interest. Under the other plan, access to earned interest is delayed, but the investor is compensated with appropriate interest. If the GIC had a penalty for early withdrawal, and if interest rates were to go up during the five-year period, the simple interest GIC would be a better deal.

Investigation 9 (pages 94 and 95) compares two very different financing schemes to purchase a $20,000 car. (All taxes seem to have been included in the price.) Discussion of Step D in the TR (page 169) gives teachers all the information they need to make a sensible comparison of the two financing plans, but stops short of emphasizing the all-important point that the naive answer given for Step A is misleading, since it does *not* factor in interest (saved or paid).

The easiest way to compare the two financing plans of Investigation 9 involves a comparison of *true interest rates*, also called effective interest rates, i.e. the cost of borrowing money. The TR (page 169) correctly calculates the true interest rate for traditional financing to be 9.76%. (Store 9.7567 in a calculator.) Look at the situation at the end of 3 years, just before Monica could pay off the car under Wiselease. *If* Wiselease were also charging 9.76% interest, then

- The accumulated value of Monica's 36 monthly payments (with interest) would be $14,361.54 (using a financial calculator).
- The accumulated interest on her security deposit would be $350(1 + 0.097567/12)^{36} - 350$, or $118.46 .
- The accumulated value of a $19,000 investment at 9.76% interest would be $25,430.68. (Think of this as what the money would be worth to the bank if Monica had not borrowed it)

Thus, Monica would owe: $25,430.68 - $118.46 - $14,361.54 = $10,950. Since Monica is required to pay only $10,000, she should conclude that the true interest rate is lower with Wiselease. That is to say, Wiselease offers a better deal.

A formula that students should see at this point (used to calculate both the interest earned on the security deposit and the accumulated value $25,430.68):

Accumulated value under compound interest =

$$\text{Principal } \times (1 + \text{ interest rate per compounding period})^{\text{number of compounding periods}}.$$

We cannot find the above formula anywhere in the chapter. Students must wait until Section 3.3 of *Constructing Mathematics 3* in order to see it. Surely this is a disservice.

A formula that teachers should see at this point (used to calculate the amount $14,361.54 above):

$$\text{Future value of an annuity} = \text{payment} \times \frac{(1 + \text{interest rate per time period})^{\text{number of time periods}} - 1}{\text{interest rate per time period}}$$

This second formula is readily derived, with careful bookkeeping and the formula for the sum of a geometric progression (as outlined in Section 3.3 of Book 3).

The TR points teachers to financial calculators (some models of the TI calculator, and the Internet), but falls short of stressing the importance of using such tools when comparing both financing plans and investment plans. If the goal is to help students understand money, then there is no choice here. (Some of the Internet sites given in these books are no longer running, but search engines readily find new sites.)

In the same investigation, Question 2(b), page 95, is good. Our complaint is that big ideas are not clearly explained, and that practice making *appropriate* comparisons seems to be presented as an optional extra.

The worked Example 3 on page 106 is more difficult than any problem we would give in first year university. The *method* of solution is incorrect. The bank loan is indeed a better deal, but not for the reasons given. Once again, the correct comparison uses true interest rates. The bank financing plan (plan (b)) has interest rate 9.503% p.a. with monthly compounding (calculated on a TI-83). The computer store lends clients \$1,455.05, with no payments required for the first six months. *If* the store were also charging 9.503% p.a. (with monthly compounding), then the value of that loan would be \$1,525.57 after six months (using the above formula for calculating accumulated value of an investment). Then use a financial calculator to find what the payments should be *if* the interest rate were 9.503%: \$48.87. Since the store requires a larger payment than this, it must be charging *more* than 9.503% interest. Therefore the bank is a better deal.

Some students may want to know how to calculate the true interest rate for plan (a). Let the unknown rate be $r100\%$ p.a. After six months, the accumulated debt is:

$$(1500 - 44.95)(1 + \frac{r}{12})^6.$$

That debt is paid off after another 36 months. So, the "future value" of the 36 month annuity is set equal to what the accumulated debt *would* have been 36 months later:

$$(1500 - 44.95)(1 + \frac{r}{12})^{(6+36)} = 54.61 \frac{(1 + \frac{r}{12})^{36} - 1}{\frac{r}{12}}.$$

Students could then rearrange this equation and use the "trace" option on a graphing calculator to solve the equation. (Answer: plan (a) charges \$15.26% p.a.)

The authors have stumbled upon the correct answer to Example 3 (page 106). Suppose that the bank were to raise its interest rate to 14% p.a. Then, under plan (b), the 48 monthly payments would increase to \$47.14, and the total payments (ignoring interest) would be \$2,262.33 (correcting for rounding in the stated payment amount). The method presented in the text would imply that plan (a) is a better deal, when it is not.

There are probably other places in this chapter where students are misled about optimal financing. These two examples illustrate the problem. Note that we are actively encouraging the use of technology, so that students can answer these interesting, and practical, problems correctly.

Teachers should *not* use the approximate formula for true interest rate taught under the old curriculum.

The practice problems at the end of the chapter are, generally, much more difficult than those in the body of the chapter, and the TR provides no insight.

We wonder why some of this material is not taught to students in the more advanced mathematics stream. Sections 3.1 and 3.2 are relevant for all students. As we have indicated, the mathematics of interest payments and loans is not so obvious, and provides examples of the applicability of geometric series and exponential functions.

Some teachers may say that our criticisms of this chapter are too harsh, that students have every opportunity to address issues about interest in class-room discussions. There is certainly a lot of time dedicated to discussions and "brain-storming". Perhaps mathematics and sociology teachers could cooperate, so that some discussion of consumer-related issues could be moved to another class, freeing up more time for mathematics in the mathematics class.

Many of the comments below underscore the fact that financial mathematics is a difficult topic; that many of the problems in Chapter 3 are too difficult for students studying from the *Constructing Mathematics* series; and that the text must clearly state when simplifying assumptions are being made. We suggest that most of these criticisms can be addressed by careful modification (simplification) of problems.

Page 70, Question 16. "versus". The tradition in mathematics is "y versus x": vertical axis versus horizontal axis.

Page 72, Challenge yourself (b). Inappropriate method. The point of this challenge is simple: Tax deduction tables describe functions whose graphs are *not* straight lines. The point is best demonstrated by having students:

- plot tax deduction versus gross monthly income for any province;
- use a piece of string or spaghetti to demonstrate nonlinearity.

The least-squares regression line has no sensible interpretation here and should not be computed. Deviations from any line are due to real nonlinearity, not random variation. Moreover, the line obtained depends on the (arbitrary) cutoff at $ 10,000; if the chart had been truncated at $ 8,000 or extended to $ 20,000, different lines would have been obtained. Both these criticisms apply - even more strongly - to the computation of r. This confusion between nonlinearity and random variation was observed also in Book 1, Chapter 4.

Continuing on to part (d), the correct answer to this question is not (as the TR claims) "because the equations give negative taxes for values below \$ 1000". It is simply "because the line doesn't fit". This is not a minor quibble! The question in the textbook implies that the linear model *does* accurately predict taxes for other incomes (which it doesn't, except for values close to \$1500 or close to \$7500 per month), and the suggested answer in the TR implies that the prediction is "accurate" provided the sign is correct.

Page 73, Question 25 This is an oversimplification of the usual "flat tax" proposals, which generally involve a personal deduction. Between the deduction and low rates, flat tax schemes often claim to lower the tax paid by each taxpayer, whatever his or her income. Any discussion as to whether this is a good thing must take into account the consequent massive reduction in government programs.

Page 77, Question 11 and TR. The problem states that Monica is a "heavier–than–average" smoker; the figure of 12.7 cig/day given is for an *average* smoker. While her "miscellaneous" expense item of \$150/month (see table on page 76) gives an upper bound of a bit less than a pack a day, we really have no idea where the correct figure lies. The TR confuses matters further by presenting a "correct" answer based on the assumption that Monica is an *average* smoker.

Page 83, Focus E, and following. Clearly state when taxes are included. Question 1(a) asks students to add sales tax to the cash price. But the instalment price (Question 1(b)) includes taxes. Perhaps there should be a clear explanation at the beginning of the chapter that such is always the case: cash price does not include tax, instalment price does include tax.

Page 91, Question 21(a). Saving to buy. Sasha is saving to buy a stereo system costing \$2,354.00. If he saves for 5 years (or even for 1 year), the list price of the stereo will no longer be \$2,354.00. This is not a serious problem, but it is always good practice to acknowledge an assumption, realistic or not: "Assume that the price does not change."

Page 109, Question 11. "If \$4000 is invested ... it will be worth about \$36,000." Why not give the exact value, \$35,817.21. Better yet, have students calculate the exact value.

Page 109, Question 12. TR, Page 182. Paying off part of a credit card bill. Most credit card companies do not charge interest until **after** the first billing period. Thus, the solution given in the TR assumes that the full balance of \$4000 has been owing since last month's bill was paid. In that case, the bill would clearly state that \$4040 was owed. Below are the correct calculations for a more realistic scenario.

Suppose that the unpaid balance at the end of last month was \$1200, and that a further \$2,788 was charged to the credit card during the most recent billing period. Then the total bill would be:

\$1200 outstanding plus \$12 interest plus \$2,788 new charges = \$4,000.
If the credit-card holder makes a payment of \$3,000, then that payment *includes* the \$12 interest charge noted above. The unpaid balance is \$1,000, on which a further \$10 interest will be charged next month.

Some credit card companies might charge *more than* one month's interest on that unpaid balance of \$1200 carried over from the previous month: the interest charge could be calculated by going back to the actual date(s) at which the last \$1200 were credited, leading to an interest charge greater than \$12. Similarly, interest charged next month on the new outstanding credit of \$1000 could be something greater than \$10.

Credit card calculations can become quite complicated. We suggest that credit card problems focus on following a series of consecutive monthly bills (under varying scenarios), and calculating how much of each bill is an interest payment.

Page 110, Question 13. The problem should clearly state that "no interest is charged for the first year". We believe that part (b) is simply asking students to add up tax and 24 payments, which comes to \$763.67 (a little different from the answer in the TR).

A far more interesting problem (manageable with a TI calculator) would be this. Calculate the monthly payments two ways: assuming that interest is charged for that first 12 months (\$31.823), then assuming that interest is not charged for the first 12 months (\$28.242).

Page 110, Question 14. Most financial plans use compounding plans and payment plans with the *same* time units. This problem is unnecessarily difficult. (We assume that "14% " means "14% per annum".) The use of simple interest makes this problem highly non-standard. The correct calculations are as follows.

Loan principal: $1.07 \times 12200 - 5000 = 8054$.

Future value of \$8054, invested at 14% p.a. simple interest for 4 years:
$8054 \times (1 + 4(0.14)) = 8054(1.56) = 12564.24$.

Future value of 48 monthly payments of $\$x$, each earning 14% p.a. simple interest:

$$x(1 + 47\frac{0.14}{12}) + x(1 + 46\frac{0.14}{12}) + \ldots x(1 + \frac{0.14}{12}) + x$$

$$= 48x + x\frac{0.14}{12}(47 + 46 + \ldots + 1)$$

$$= 48x + x\frac{0.14}{12}(1128) = 61.16x.$$

Solve for x: $61.16x = 12564.24$, to obtain $x = 205.432$. That is, the monthly payments should be \$205.43 .

We doubt that the authors understood the complexity level of this problem. At first we thought the solution given in the TR might have solved the following problem: Find $x = \frac{y}{12}$, where y is the *annual* payment on a loan of \$8054, borrowed at 14% p.a. *compounded annually*, and paid back in 4 years. The TI calculator gives $y = 2764.1713$ and $x = 230.35$.

In fact, the solution in the TR takes an extremely simplistic approach. The answer is simply $\frac{1}{48}$ times the future value of of \$8054, invested at 14% p.a. simple interest for 4 years, $\frac{12564.24}{48} = 261.76$. As can be seen from the preceding calculations, the *effective* interest rate for this payment scheme is considerably more than 14% p.a. Consumers should be wary of such a scheme.

Page 110, Question 15. "Find the total cost of the van." Winston paid a deposit of \$2500 plus a one-time payment of \$1949.33. So, over the course of the year, he paid out a total of \$4449.33, the answer given in the TR. Once again, teachers should help students to see that, allowing for interest, the "cost" in terms of dollar value at the time of delivery is something less than \$4449.33, since Winston had the use of that \$1949.33 for 12 months.

Page 110, Question 16, and TR. Money saved by walking to work. The solution in the TR deducts the total ownership cost.

It is true that some insurance companies charge a reduced rate if a vehicle is *not* used to commute to work, and depreciation will be reduced (because of a lower odometer reading). However, it is *not* reasonable to deduct all ownership costs.

Page 110, Question 18(b). Once again, the true "cost" to the consumer is something less than the total of 60 payments, \$66,000, since he/she has the use of most of that money for some time before payment.

We repeat: an oversimplified approach to money can do more harm than good.

4.4 Chapter 4. Statistics

It is difficult to convey the big ideas behind statistics to young people whose experience of the world is (necessarily) limited. This chapter shares some of the errors previously noted for Chapter 5, in *Mathematical Modeling 2*.

This chapter requires students to think about difficult open-ended questions. See, for example the first four focus questions on page 113. "Brainstorming" is a nice idea, but only 20 hours have been allocated for the entire chapter. Students with below-average reading and writing ability will find this chapter extremely difficult. Teachers may save time, and help students to stay focused, if they concentrate discussion around criticism of one or two well-chosen newspaper or magazine articles, rather than ask students to construct their own questionnaires.

Problem 20 on page 118 introduces the concept of a 95% confidence interval for a population proportion, without definition. Investigations in Section 4.2 reinforce the idea that the sample estimate of a proportion need not be exactly equal to the population proportion, and Question 22(b)-(c) on page 126 requires students to construct confidence intervals for a proportion. But both the text and the TR fall short of presenting the classic formula for a 95% confidence interval for a population proportion, $\hat{p} \pm 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ represents the sample estimate of the population proportion and n denotes the size of the *random* sample.

Use of this formula, with reference to a newspaper article describing results of a poll, would certainly help students to understand some of the reports they hear in the media. We are surprised that the authors recommend TI "black boxes" to generate random data and to calculate least squares lines (often inappropriately), yet do not empower students with this simple formula which is, in fact, in common use in today's media reports. Use of this formula would also provide valuable practice with order of operations when using a calculator. As with the *Mathematical Modeling* series, we recommend that the above formula be presented without proof, but that simulations be used to demonstrate that it "makes sense".

The word "significant" and its variations are used throughout the chapter. The authors do not distinguish adequately between statistical and practical significance. If data hold a practically significant message, then that message can be readily conveyed with an appropriate diagram.

In summary, this chapter needs major revisions. At this stage in their education, students should focus on:

- obtaining appropriate graphical displays of data;
- describing the pattern (or lack of pattern) they see in such graphical displays.

Students should gain some experience with data before moving on to more difficult material. Critical reading of newspaper articles is valuable at this stage.

Page 113, Sidebar, Population. Technically, the "population" is the *group*, not the set of data. Many different measurements can be taken from any one population (e.g. heights, eye-colours, score on a test). Similarly for "sample", although statisticians themselves are more sloppy with this word.

Page 114, Census or sample? These are difficult questions. In question 8, the TR (page 203) pays no attention to the difficulty of obtaining a *random* sample.

The sidebar on page 115 of the text mentions "bias" in a sample. The phrase "not truly representative" is sloppy, and possibly misleads students and teachers.

The important idea is this: basic statistical techniques assume that the sample has been selected *randomly*, that *each individual in the population was equally likely to be* *selected.* If a survey is administered in a high school or a skate board park, then young people are far more likely to be selected than are old people. Therefore the sample is *biased* in favour of young people.

In particular, a *biased sample* is not a random sample: it has been selected in such a way that some individuals were more likely to be included in the sample than were others.

It is extremely difficult to select truly random samples from any population. If students gather data from human subjects then samples will be biased towards selection of their own friends and relatives, and students should be cautioned against making unrealistic generalizations.

Page 116, Questionnaire construction. It is as difficult to construct a good questionnaire as it is to select a random sample. The texts should be praised for making this point. However, with a view to saving time, teachers should note that it is far easier to find fault with someone else's questions than to construct good ones. Newspapers and magazines often supply copies of the questions asked in surveys.

Pages 116, 117. A *proportion* is a number between 0 and 1. A *percentage* is a number between 0 and 100. Some students will worry about whether their answer is supposed to be a number between 0 and 1, or a number between 0 and 100. For this reason, statistics teachers are careful to use the word "proportion" when quoting a number in the 0 – 1 range, and "percentage" for numbers between 0 and 100. In order to avoid confusion, it is best to avoid saying "the population proportion was approximately 8%." It is acceptable to say "The population proportion was approximately 0.08, or 8%."

Page 117, Sidebar. Given popular jokes about how to lie with statistics, the phrase "that has the desired result" is an unfortunate choice here. A better wording: " that has the characteristic of interest".

Page 121, Investigation 2. Spinners. Spinners have certain pedagogical advantages, in that the proportion/probability is directly visible at the time of the experiment. The conceptual tie-in with the pie chart is also a strong point in their favour. Moreover, they represent sampling via a continuous, rather than discrete, process - an important distinction.

However, a hastily-made spinner is likely to demonstrate significant bias if the pin is not vertical. A sheet of cardboard may not be thick enough to keep a pin vertical, especially after repeated impacts from the same direction.

Teachers may want to look into other solutions. A school shop might be able to turn out a number of permanent spinner mounts that can be used with several different cards; or inexpensive dice can be colored with permanent markers to provide several different probabilities. A useful set would consist of one standard cubical die and one octahedral gaming die, each with two sides colored red and one blue.

Page 122, 123, Investigation 3. Sampling without replacement. The formula for a confidence interval for a population proportion, calculated from a random sample of size n, assumes sampling *with replacement*: any individual *could* be selected more than once. When sample size is small compared with population size, then the assumption can be ignored, and the confidence interval formula (given above) works well. But all sorts of technical problems arise when sample size is comparable to population size: read about the "finite population correction." The same problems arise when samples are used to estimate means for continuous data (such as heights), for small populations.

Investigation 3 requires students to select samples, without replacement, from small populations. In particular, results from Step E will lead students to *underestimate* the variability of sample estimates of a population mean calculated from samples of size 20.

These technical problems cannot be avoided by asking students to sample *with* replacement, since they will (quite rightly) object that such a sampling procedure doesn't make sense to them.

Steps D and F are, of course, impossible without combining classes.

Page 123, 124 and TR. Confidence intervals and prediction intervals. Investigations 2 and 3 ask students to describe the distributions of sample proportions and averages, which is a good exercise.

However, in Investigation 4, the students are asked to calculate a crude *prediction interval* for the proportion of coloured squares in one more sample, rather than a (much narrower) *confidence interval* for the underlying population proportion. The prediction interval is the correct answer to the question; however, it is an unusual technique, easily confused with the confidence interval (as the authors demonstrate repeatedly in Section 4.7).

The prediction interval gives an answer to the question "What is my next sample probably going to be like?" A confidence interval answers the question "What might the makeup of the population be?". Prediction intervals do not get much smaller as the number of data increases, because much of the uncertainty is in the hypothetical single *next* sample. Confidence intervals do get narrower: if you take 10,000 samples of size 10, you will have a fair probability of picking the exact right number of shaded squares.

An experiment that might get this across to most students would be to get all the students to sample from one bag, create the prediction interval *and* the standard confidence interval for proportions; then pool everyone's data and do it again. Overall, the prediction interval won't get any tighter. The confidence interval will.

Another way to explain the difference between a confidence interval (for some underlying population parameter) and a prediction interval (for one more observation, or set of observations) is as follows. If I toss a coin *many* times, then I can estimate the probability of it landing heads with considerable accuracy. In particular, if the coin is a tad biased towards head, such bias can be detected with perseverance. But, so long as the probability of landing heads is anything close to 0.5, if I want to be reasonably sure of winning a bet on the next toss, I'd have to bet on *both* heads and tails (a prediction interval).

It is arguable that the need of students in this stream for exposure to the prediction interval is not great enough to justify the potential for confusion. In any case, the confidence interval should be introduced first; the distinction should be stressed repeatedly; and the authors *must* get it right themselves.

See comments on pages 160 – 162, below.

Page 123, Investigation 4. Sampling without replacement. Once again, students are asked to sample without replacement from a small population, this time to estimate a proportion. See the above comments.

There is a gap in the description of the story behind this investigation. Jacob wants to estimate the total number of moose in a 50 km by 50 km region, which he has divided into 5 km by 5 km squares. In order for the simulation exercise to be relevant, Jacob must be assuming that he will find *no more than one* moose in any one of those smaller squares.

Page 123, and TR. Think about. Appropriate Sample Sizes. The TR gives no insight to teachers as to how answers were obtained. For large populations, when estimating a **proportion**, a sample size of 1100 leads to a 95% confidence interval of the form $\hat{p}\pm$ error bound, where the error bound is no more than 0.03 (or "three percentage points").

However, when estimating the population *mean* for a continuous *measurement*, appropriate sample size varies greatly according to the innate subject-to-subject variability of the population (population standard deviation), and the required accuracy of the estimate (error bound).

Amateurs should definitely *not* consider sample-size questions for small populations, as in part (a).

Are the authors aware of these issues? If so, why did they ask such a difficult question without providing help for the teachers?

Page 125, Question 18. Many Atlantic students would be aware that the first boats back to port may not be a *random* sample from the (small) population of boats using the wharf. (Note for hangashores: compare the first students to hand in an exam!)

Page 125, Question 18,19, data. The first step in analyzing any data set such as these should be to plot it, as in Chapter 1 of the previous book. (This is a good example of the NCTM's "connections standard", and a missed opportunity). Both a boxplot and a histogram should be created.

Once this is done, the students may notice that there is evidence of positive skewness. This should not cause great concern about the validity of the t interval, but should cause the question to be asked "Is the mean an appropriate summary of these data?" Here, the answer is "yes" because total lobster catch is estimated as (average per boat) $\times$ (total number of boats), and total money spent in the supermarket is estimated as (average per customer) $\times$ (total number of customers).

Page 125, Questions 18c, 19c; TR page 220 While getting students to speculate on the answer to a problem in advance of a rigorous treatment is often valuable, this must be done with caution when the solution is nonobvious. At the end of the exercise, students they should be left knowing and understanding the right answer. In question 18c, no understanding of the correct answer is given; in question 19c, the reason given is fallacious.

The comment in the TR that "in statistics, there is often no absolutely correct ... nor ... absolutely incorrect answer" is a dangerous half-truth. It is true that the combined process of sampling and performing an inference may lead to different answers when done correctly (every sample is different); and it is true that there are legitimate reasons why different researchers may make different choices of confidence level, leading to different interval widths. There is even room for different *informed* opinions about the appropriate methods to use.

However, the processes by which these numbers are obtained are fixed, exact, and not a matter of opinion. The interval $[92.9, 155.0]$ was obtained as a 95% confidence interval - that is, an interval generated by a technique that will yield an interval containing the population mean 95% of the time. The authors have not given a "reasonable and adequate" justification of their answer by saying that it contains about 2/3 of the values (once again confusing the two kinds of intervals, prediction and confidence). Their justification is unacceptable. In particular, anybody attempting to solve a problem with a different sample size by this "rule" would get an answer which was not the standard 95% CI.

Statistical inference is not a "feel good" exercise in which everybody gets to be right. While correct answers may vary, the methods by which those answers are reached are correspondingly important. If the authors judge that the correct explanation of a t interval is too advanced for this course, we will not argue. But this does not mean that it is appropriate to substitute a wrong explanation.

Page 126, Question 20. The nonresponse rate here is high enough to cast considerable doubt on the validity of the survey. It might be instructive to add the following questions:

(e) Suppose that the students who failed to respond were more apathetic than the others, and only 25% of them went to the opening. How many students would attend then?

(f) What are the largest and smallest numbers that might attend, assuming that those who did respond told the truth?

Page 127, Questions 23,24. We reiterate that it is pedagogically useless to ask students to do something that they do not know how to do (construct an interval estimate), give them no instruction on how to do it, and give them no feedback on whether they got it right. To suggest that whatever they answer *is* right is worse than useless.

Page 128, Random samples and Representative samples. The sidebar asks: "Why might a sample chosen at random from the audience at a high school hockey game not be representative of the school population?" This issue may be easier to handle the other way around. As mentioned earlier, it is very difficult for students to obtain any kind of sample. *If* a student decides to conduct a survey at a hockey game, then ask "For which subpopulation of the school do you think these results can be generalized?" Depending on the types of questions asked, the appropriate subpopulation may be "students who like hockey". (We are assuming that only *students* at the game will be questioned.)

It is best to avoid the phrase "representative sample", since some people interpret this to mean "judgement sample": the surveyor hand picks a group whom he/she considers representative of the population. It is easy to demonstrate that judgement samples are fraught with bias.

Judgement samples, convenience samples and self-selected samples all have inherent bias. The problem for the statistician is this: he/she has no way to gauge the bias. All survey data reported by pollsters arises from self-selected samples, since people are not obliged to answer the questions. An honest pollster will *always report the response rate*.

Page 129, Systematic and cluster samples. In many cases, a systematic sample is as good as a random sample. Unfortunately, some systematic samples have inherent bias, and statisticians never know for sure whether they are dealing with biased systematic samples.

The point *must* be made that statistical procedures for interpreting data collected from cluster samples are much more complicated than for data collected from simple random samples.

Page 132, Boxplots. Full boxplots (see comments on Book 1) would indicate outliers in both groups.

How were these boxplots constructed? In particular, consider the third quartile for non-hockey players. Book 1 defines the third quartile to be the median of the top half of the data. The median of the fourth column of data in the table is 17, the average of 15 and 19. Another common rule for calculating the third quartile is to find (using linear interpolation) observation number $\frac{3}{4}(n+1)$, where sample size n is 40 in this example. For the non-hockey players, this calculation gives observation number $30\frac{3}{4} = 15 + \frac{3}{4}(19-15) = 18$. The boxplot marks the third quartile at 16.5, which is neither of the above values.

Page 133, Histograms. Good students will be confused by this example. The text correctly describes the accepted convention that "borderline" observations are counted in the interval on the right. *However*, it is always best to avoid having to make such an arbitrary decision. For this problem, choose intervals with *midpoints* at 5, 10, 15, etc. Then the issue does not arise. This is an important point.

These comments apply to all histograms in the chapter.

Page 134, Pie chart. Pie charts are mostly used for discrete data with no natural ordering. The most common use of pie diagrams is to describe budgets: proportion of income from (or expenditures on) various sources (or projects). A (weak) case can be made for using pie charts for dichotomous data (even where there is a natural ordering), or dichotomized measurement data (pass/fail, above/below 0°C); but dichotomous data are often treated anomalously.

A pie chart is never appropriate for describing ordinary measurement data, because its circular shape confuses the ordering on the outcomes. For the same reason, it would not be an appropriate choice for ordered discrete data, such as letter grades; A should be at the other end from F, not next to it! Straightening a pie out yields a divided bar, which is useful and accepted.

Representations of numeric data using histograms or boxplots clearly convey the relative sizes of the various measurements. The "balance point" of the histogram tells our eyes the numeric value of the sample average. The box and median line of the boxplot tell us typical values. All such information is lost in a pie diagram.

For problem 2(a) on page 136, a pie chart is *not* appropriate, since there is an obvious ordering to the age categories.

For part (c) of the same problem, two boxplots (one describing times for males and one describing times for females), drawn above the same axis, would give a very nice description of the data.

Page 135, questionnaire The first question on this questionnaire is an example of the (often dubious) practice of grouping numerical data unnecessarily. Information is lost when this is done; a simple question "What grade are you in?" will do. The practice has some validity with information that is seen as sensitive or that the respondent may not know exactly, such as income; but in our experience students (and others) overuse it, perhaps because they feel it "looks professional". It should be made clear what 'other" means.

Page 136, Question 3. These plots are not absolutely impossible, but they are extremely implausible. For plot e to be as shown, either some samples would have to be highly non-representative of their populations or there would have to be a very large group - larger than the four groups represented by $a - d$ put together - spending amounts in the \$22-\$28 range. The same point could have been made with more realistic plots (in which the whiskers and box of b and the box of e would have been much longer.)

Simulated data, like simulated biological specimens, have a place in the classroom. But to be pedagogically useful, they must be realistic.

Page 137, Question 4(a), TR Page 235. Bar graphs. This type of bar graph to describe averages, though readily available from a common software package, is inappropriate. In bar graphs and histograms, the height (or area) of the bar is supposed to convey information about proportion or percentage. In order to compare location of the two sets of data, draw boxplots on the same axis (as displayed in the TR), and compare positions of the median lines and middle boxes. Or draw histograms, one above the other, with identical horizontal scales, and compare "balance points".

With reference to discussion in the TR, the two boxplots show little difference in location *or* in variability, and it is wrong to suggest otherwise.

Page 137 Sidebar, TR Page 235. Sample Selection. Stratified sampling is not mentioned as an option in Section 4.3. With stratified sampling, sample sizes are rarely the same size for any two strata.

A stratified sample is one in which subjects are selected on the basis of membership in various groups, and the data are then pooled and analyzed as a single sample. Freda has sensibly kept the data separate and compared them; she has thus done a *two-sample survey* (a very standard and easy-to-analyze design), not a stratified one-sample survey (extremely laborious to collect or analyze correctly).

Pages 137,138, Questions 6 and 7, TR Page 237. Since these data arise from situations which do not require a finite population correction, the histograms should convey the correct trends. The TR should empower teachers with the following information (a consequence of the Central Limit Theorem).

Counting *total* number of heads in a sample:
For any fixed population proportion, overall spread should increase approximately as $\sqrt{n}$, where n is sample size. In other words, the spread (and, in particular, the standard deviation) of the histogram for $n = 48$ should be about $\sqrt{4} = 2$ times that of the histogram for $n = 12$.

Calculating *proportion* of heads in a sample:
For any fixed population proportion, a histogram showing sample proportions for samples of size $n = 12$ will show about twice as much variability as a histogram showing sample proportions for samples of size $n = 48$, since $\frac{\sqrt{12}}{12} = 2\frac{\sqrt{48}}{48}$.

Page 139, Investigation 6, TR Page 241. Prices of skate boards are, effectively, continuous measurements, with dollar and cent values. (The recorded data have been rounded to the nearest dollar.) It is, therefore, inappropriate to calculate the mode. Mean and median describe central tendency for continuous measurements and for discrete data that take many numeric values. The mode is appropriate for all other discrete data.

Page 140, Think about; Page 141, Question 6, TR page 241,3. The sample median provides a good *one-number* description of "typical" value. The range of the box on a boxplot provides a good *range of values* to describe "typical", the middle half of the data. We cannot fathom the explanation in the TR.

Continuing on (Question 7, page 141), if data are symmetric, then the mean and median will be very similar and will provide equally good one-number descriptions of "typical". If the data are highly skewed, then the mean and median will be quite different. Consider income data, for example. Typically, a histogram will be "bunched up" on the left, but trail off to the right (indicating positive- or right- skewness). In such cases, average income is quite a bit higher than median income. The average is *not* typical. (Listen carefully next time your hear incomes reported in the media.)

Page 141, Graphs These graphs are horrible examples of poor graphing practice. There is no need to use two two-dimensional graphs to present ten univariate data; a simple pair of lists or double stemplot would have been adequate. Moreover, the practice (apparently borrowed from the weekly newsmagazines) of using large "icons," instead of dots, should be avoided. They add no information and make the graph harder to read. (Usually one would read from the center; in this context, one might guess that the bottom of the icon represented the jump height, but who knows?)

Page 141, Bottom of page and Sidebar Note Alex had one jump significantly higher than one of Brendan's; the other four matched closely. Neither the descriptions in the text ("... Brendan tended to jump to about the same height each time. The height of Alex's jumps were quite different each time") nor in the sidebar ("Brendan's jumps showed a high degree of uniformity, while Alex's jumps showed a high degree of dispersion") is an accurate description of the data shown. The data simply do not establish any significant difference in dispersion between the two jumpers. (Both the F test and Levene's test give P-values close to 0.4.)

Pedagogically, this exaggeration of differences is counterproductive. The main lesson that students at this level should be learning is that not every difference is significant, but these comments tend to lead students in precisely the opposite direction. Focus Questions 10 and 12, suggest that one should make quite different inferences from two very similar sets of data.

Page 142 and TR page 244-6. Standard deviation. It is good to see the conceptually simpler "long method" has been chosen to calculate standard deviation. However, the use of n rather than $n-1$ is an unnecessary simplification. The texts repeat a conceptual error from the *Mathematical Modeling* texts, and the following paragraph is adapted from the critique of that series.

A big idea in statistics is that of "residual," the difference between an observation and the predicted value under some prescribed model. (In this example, the predicted value is the average, 1.14). The calculation of standard deviation is a student's first exposure to residuals, even if the term is not used. One important feature of a residual is its sign, positive if the observed value is greater than the model value, negative otherwise. The signs in column three of the table on page 142 are precisely the opposite of what they should be. While calculations of values in the next column are correct, graphical description (using dots on a number line) would make much more sense to students if positive numbers in the third column corresponded to observations that were greater than the observed average.

Page 143, Question 9, TR Page 245. Why do we square? Insight for teachers. One might consider taking absolute deviations, rather than squared deviations. In fact, some statisticians like to calculate the median absolute deviation from the median.

The standard deviation formula should remind teachers of how we calculate distances in n-dimensions. Another reason why we square is simply this: the function $y = x^2$ is differentiable at $x = 0$, but the function $y = \mid x \mid$ is not.

Page 143, Question 13, TR page 246. As mentioned above, the two standard deviations are *not* significantly different. (Amateurs should – and professionals do – avoid small samples.)

Page 143, Question 16 The Victorian hyperpropriety of referring to what the rest of the English-speaking world calls "screws" as "fasteners" will only draw the students' attention to the secondary meaning of the former word.

Page 144. Question 18(b), TR Page 247. Be careful with examples like this. It is a fact of life that big measurements show more variability than small measurements. It would be an interesting exercise to have students draw boxplots for heights of players on a few different NBA teams and for their own class. (Maybe restrict attention to males, and use a couple of classes.) The high school boxplot would show a shift, but we are not convinced that it would show more variability.

Page 146, 147. The discussion might be less confusing if the word "figure" were replaced by "number".

Page 150, Question 17 and TR. This is a good treatment of the topic of describing samples from a normal distribution.

Page 156 ff. Normal Distribution. The Sidebar (page 156) states "Quantities such as heights, masses and IQ scores are normally distributed." Many continuous measurements give histograms that look like the classic curves displayed on page 156. Other sets of data do *not*. Practicing statisticians acquire an instinct for when data will "look normal".

IQ scores are normal *because* they have been forced to have this characteristic, as have many standard test scores. Heights, widths and lengths often give classic bell-shaped histograms, so long as the sample comes from a single population. For example, a histogram of heights for all students in a large class could show a variety of patterns, depending on the ages of the students and the relative numbers of boys and girls. A histogram of heights for all 17-year-old males in the school would probably look much like the curves on page 156.

However, if heights look normal, then masses will not generally look normal. Some attempts to demonstrate this last statement may "fail". The reason is that the function $y = x^3$ is well approximated by a straight line over fairly large intervals of the x-axis. Thus, for example, if all apples in a bag have girth between 40 cm and 55 cm, then a plot of weight versus girth will look linear, and boxplots of both girth and weight will look much the same. (For the girth data, the coefficient of variation – standard deviation divided by average, expressed as a percentage – would be about 8%, indicating that the spread of girths is small compared to the size of the numbers recorded. Over the interval of values $40 \leq x \leq 55$, the function $y = x^3$ looks a lot like a straight line.)

On the other hand, if students went to a U-pick and picked *every* apple on (say) two selected limbs of one tree, then there would probably be more variation in apple sizes than one would see in a few bags of apples (of the same variety) bought at a store: the coefficient of variation for girth would be well above 10%; the plot of weight versus girth would look more curved, and the boxplot of weights would be more positively skewed than the boxplot of girths.

Scientists see this relationship (between a one-dimensional measurement and a measure of volume or mass) in all sorts of situations. In a stand of 50-year old trees, heights and diameters of trees may well follow a normal distribution, but volumes will not. Similar results are often observed for lengths and weights of fish. Physiologists sometimes measure the strength of athletes' legs. Though leg sizes may be normal (measured as lengths or girths, perhaps), strengths are usually not normally distributed, since strength is more highly correlated with either cross-sectional area (diameter of muscle) or mass of the leg.

Sports physiotherapists see a censored population from the high tail of the strength distribution – another, different, common cause of skewed data. Physiotherapists in general practice see a mixed distribution in which a smaller group (athletes) with a higher mean strength is mixed with a larger group having a smaller mean – yet another cause of skewness.

In a class that has no peanut allergies, have students measure lengths and weights of peanuts in the shell, bought at a bulk-food store (where there is usually not so much quality control). This experiment requires a scale calibrated for small weights. Have students construct histograms or boxplots of lengths and weights, and compare.

Ask students to shut their eyes and draw a line they think is 10 cm long. Have them measure the lengths of their lines and also have them calculate the area of a square with that side length, and the volume of a cube with that edge length. Obtain boxplots or histograms of the values for length, area, volume, and compare.

The moral is simple: some data yield histograms close to a classic bell-shaped "normal" distribution; other data do not.

Count data (such as Question 20, page 150) are usually positively skewed. (There is a lower bound, zero, but no upper bound.) With reference to Question 20, every now and then – nice day, no school – a lot of people come to the park.

As mentioned earlier, income data (for all sorts of populations) tend to be positively skewed, as do survival-time data (including situations such as that described on page 154 – "survival distance").

Looking back to the graphs on pages 237, 238 of the TR, we see symmetry when p is near one half, positive skewness when p is small, negative skewness when p is large.

TR page 157, Questions 12,13. The TR should perhaps mention that there are other correct answers (for instance, [0,1.29] for question 12a) though it is very unlikely that most students will discover them.

Page 157, Question 14 and TR. The answers given are inconsistent with the implicit normal model: the plant in (i) is further from the mean height than the plant in (v), yet it is described as more likely to come from the patch. Students with experience growing plants may not find the existence of a few stunted plants such as (iv) unlikely either.

Page 159, Example 2. Is Paula checking the first sample or not?

Page 160, Question 19. for "mathematician" read "statistician".

Page 160, Questions 23, 24; TR page 270. Distribution of an average. The answers given in the TR are incorrect.

Consider problem 23. We expect about 95% of *individuals* to listen to music for something between 141 - 32 and 141 + 32 hours. However, the standard error of *average* listening time for n individuals is $\frac{16}{\sqrt{n}}$, not 16. If the sample were indeed large, then an average of 175 would indeed indicate that rock fans listen to more music than do members of the population at large. This property of averages – larger samples give less-variable averages – was demonstrated earlier in the chapter.

Similarly, problem 24 cannot be tackled without knowing how many students are on the hockey team. But, since n is probably more than 10, we can quickly decide that hockey players are heavier than the population at large.

Since the text book does not give the required background information, problems 23 and 24 should be omitted. (See comments about page 162.)

Page 161, Paired data. Data recorded here are *matched pairs*: a before and after measurement for each student. Each student should calculate his/her change in pulse (After - Before, say), and the class should obtain a histogram or boxplot of these differences.

Page 162, Confidence interval. We repeat our comments from the *Mathematical Modeling* texts. Any range of values either contains the population parameter or does not. A 95% confidence interval is constructed *using a procedure known to be accurate 19 times out of 20.* To put it another way: 95% of 95% confidence intervals do, indeed, cover the true (unknown) value of the population parameter, while 5% of 95% confidence intervals are grossly misleading.

We do not understand the sentence "If you choose a single ... mean" in the second paragraph.

Example 3 is incorrect. The calculation gives an approximate 95% *prediction* interval for a *single observation.* A 95% confidence interval for the *population mean* is given by $15.25 \pm 2\frac{2.00}{\sqrt{40}}$.

Omit problems 28(c), 29(c), 30(c), 30(b), 31; CS4(c) and CS5(a)(iii) on page 167; TR discussion of CS5(b) on page 276; CS6(b)(iii) on page 168; Questions 21, 22 on page 178.

Page 164, Case Study 2, and TR. Question (a)(i) should say: "What are the proportions of colours of marbles in the *sample*?"

The TR answer to part (c) is misleading; the results after 5 draws are more spread out than they were with only 1.

While parts (d) and (e) are rather vague, they appear to be encouraging the creation of a prediction interval for samples of size 10. The correct solution is to pool all the data and create a confidence interval for p based on $\hat{p} = 235/400$. Then estimate the percentage of red marbles in the bag to be 59% ± 5%, using a process known to be accurate 19 times out of 20 (using the formula on page 32 of this document).

The histogram on page 274 of the TR is confusing. Intervals should be *centered* at integer values (see earlier comments).

Page 166-7, Case Study 4. If each member of the class phoned 20 people, each reported percentage would end in 5 or 0.

Part (c): The answer in the TR is wrong. The authors have confused prediction intervals with confidence intervals. The 95% *confidence* interval would be approximately $[56\%, 68\%]$.

Page 167, Case Study 5. Again, the authors have confused prediction intervals with confidence intervals. With a sample size of at least 2000 (pooling all surveys), the *confidence* interval would be approximately $[58\%, 62\%]$.

Page 168, Case Study 6. Not only does the wording of question (b)(iii) encourage the idea of statistician-as-liar, but the question involves a fallacy akin to the once-common (but wrong) idea of "accepting" a null hypothesis. The story makes it clear that it must be shown that the average number of swims is at least 20 (actually, the story says that it is required that *every* student will go at least 20 times, but that's obviously silly.) Part (iii) implies that there is *not* enough evidence to show that the average number of visits is *not* 20. This is a logically different proposition. (Statistical inference is a difficult topic.)

Page 171, Boxplots. It is more efficient, *and* appropriate comparison is encouraged, if there is just one horizontal axis, with the two boxplots clearly labeled, as in problem 3, page 136.

Page 172, Pie chart. It is inappropriate to use a pie chart for ordinal data.

Page 175, Example 10. Incorrect calculation. One would need to know the sample size n for the new motor oil, and divide 20000 by $\sqrt{n}$.

Note that it was essential that the first sample be "large" for this crude method to be applicable. Comparison of means from two small samples is much more difficult than the text implies.

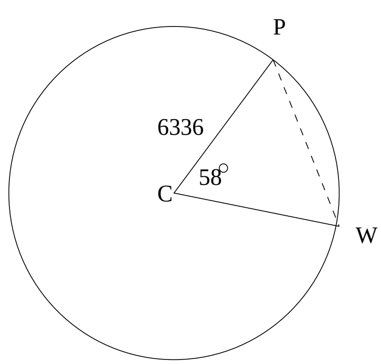
TR page 278. It is standard practice to place the axis *below* all boxplots.

TR page 279. Both the bar graph and the pie chart *are* appropriate. Note the convention for drawing bar graphs for discrete data with no obvious ordering: bars are ordered from largest to smallest (as are pie-wedges).

4.5 Chapter 5. Trigonometry

This chapter covers relatively straight forward material, with few errors. Focus B on page 188 is a botched attempt at a "real world" relevance which should be fixed.

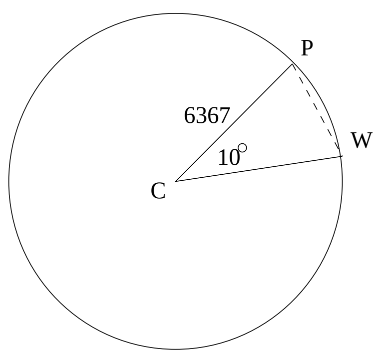
The point of Focus B is simple: in the diagram below, the distance from P to W, as measured around the circumference (6414), is larger than the distance as measured along the chord (6143). The problem is introduced in a story which seems to imply that, if the world were flat, then Columbus would have sailed through the crust of the earth, rather than along the earth's surface. Surely many students will spot the inherent contradiction in an argument that says, essentially: Let's assume the world is flat, but calculate as if the world is round.



Nevertheless, there is an important message here that needs to be addressed before going on to problems such as Question 6 on the next page. The story should be changed so as to make the following point. When we travel large distances on earth, we have to keep track of distance as traveled on the surface; approximations which amount to taking a short cut through the earth's crust will be too crude. In fact, investigations with a globe and string may help students to discover "great circles". All the little maps that appear in subsequent problems are obtained from projections which distort the surface of the earth onto the plane. Distances calculated in those problems will be slightly inaccurate. Focus B should be reorganized to help students understand that, for the distances used in subsequent problems, such approximations to the distance *on the surface* are good enough.

Before giving an example, we point out that, depending on location on the earth's surface, surveyors approximate the shape of the earth as a sphere with radius in the range 6360 km to 6400 km. The value 6367 km gives good approximations in many latitudes. In particular, the value 6336 km used in Focus B is too small.

The diagram below describes an example that could be used to illustrate the point which *should* have been made in Focus B (drawing not to scale). The distance from P to W along the chord is 1109.8; distance around the circumference is 1111.25. To three significant figures, both answers are 1110. So, in the problems that follow, we may assume that the earth is flat.



Page 180, Sidebar. "constant values" Sloppy wording. The authors have been too terse.

Of course, the authors are trying to say that the size of angle θ is all that matters; that, no matter what their side lengths, *all* right triangles with angle θ will yield the same value for $\tan\theta$. Wording of definitions must be clear and precise.

Page 181, Did you know? Greek letters are often used to represent *the measure of* angles. Angles themselves are usually represented by capital Roman letters or as (e.g.) $\angle ABC$

Page 181, Question 5; TR page 290. The TR states that 38m "is too long a ramp for an underground parking lot." Many lots surely have straight ramps this long. Other parking lots use spiral ramps.

Page 183, Question 11. Clarity. Find the *volume* of gypsum ... .

Page 183, Reflections. Formula for volume of a cone should clearly state that h denotes *vertical* height.

Page 184, Diagram. The elbow *angle* is $90°$.

Page 195, First diagram. Side length a is missing.

Page 196. Challenge yourself. Poorly worded story. The phrase "before turning" should be changed to "before turning towards her base camp."

Page 197, (b) The phrase "centre of gravity" belongs with "satellite".

Page 200, Question 5. Students who have had experience in the trade will ask about overlap of metal at the seams. It is also unlikely that exactly the metal needed can be purchased; it might be better to ask about the weight.

Page 210, Question 2. Some students will want to allow for the observer's height, which happens (for any normal adult height) to make the difference between hitting and missing the house!

Page 211, Question 6 The rope of a parasail has a very significant curve, due to its own weight. This reduces the angle that it makes with the water significantly. At the least, the assumption should be stated ("Assume that the cable is a straight line"). Better, students should be asked as a rider "Is the assumption realistic? Might this change the final answer?"

5 Constructing Mathematics 3

5.1 Chapter 1. Patterns

A sequence is a *function* from (a subset of) the non-negative integers to the real numbers. In particular, the domain is discrete. For the most part, the authors are aware of this fact. But they occasionally slip up, as on page 14. Part F should say something like this: "Graph the number of points ... sections. You should notice that all points lie on a straight line. Find both the slope and the y-intercept of the line." See also Question 5, page 15; Question 13, page 18.

Some of the more difficult problems in the chapter are not discussed in the TR.

Though the Fibonacci sequence is indeed beautiful, the primary focus of this chapter should be arithmetic and geometric sequences. (The Fibonacci sequence can be written as the sum of two geometric sequences, which would make a nice Sidebar note near page 17.)

Page 5. Sidebar. When were these students formally introduced to exponents, and rules for their manipulation?

Page 6 ff. The TR should stress that this is a difficult problem, and that closed form expression for the nth term is complicated: $\sum_{j=0}^{[n/2]} \binom{n-j}{j}$, where $[a]$ denotes the largest integer that does not exceed a.

Page 7, Questions 13, 14. These would take a *long* time!

Page 10, Question 22. Different flowers have different numbers of petals. Some varieties of clematis have 4 petals, while others have 7. This list (as subset of a larger list) is *not a sequence.*

The problem should be dropped, or reworded. See comments below about Question 27.

Page 11 and TR page 17. Golden ratio. The TR should show teachers how to construct a rectangle whose side lengths satisfy the golden ratio.

Page 26(c) and TR. The TR should note that the limiting value is the reciprocal of that found in problem 24(c).

Page 11, Question 27; TR page 18. The TR solution would "fall apart" if measurements were taken in inches. Surely students should consider *ratios*:

$$\overline{AB}\,/\,\overline{FA}, \quad \overline{FA}\,/\,\overline{FJ}, \quad \overline{FJ}\,/\,\overline{FI}.$$

The TR comments: "Some students might notice that the number of holes in the sand dollar is also a Fibonacci number."

While it is true that Fibonacci numbers arise in many biological situations and that, in many cases, scientists understand *why* such numbers arise, these text books generally err on the side of encouraging students to *see* too many patterns – refer to our numerous comments on the overuse of least-squares models.

A sand dollar has five holes (unless, as is the case in the photograph, it has been damaged). As noted in Question 22, many flowers tend to have petal-counts that are Fibonacci numbers. Sand dollars, common starfish, and sea urchins all have five-sided symmetry, and spiders have eight legs. On the other hand, insects and some starfish have six legs, and humans have four limbs.

The moral is simple: err on the side of caution; do not encourage students to "discover" patterns that do not generalize.

Page 13. We note that the authors do *not* ask students to bring real plants into school. A valuable opportunity has been lost: real data show variability about underlying patterns.

Page 14 ff. Section 1.2 needs a greater variety of examples. It considers, almost exclusively, rods in fences.

Page 18. Table. This table is confusing. A more appropriate label for row 1 is "Loan repaid after this many months:". Similarly, row three should be labeled: "Total amount paid back."

Page 19, "Think about"; TR page 34. Jason's sister charges 96% p.a. interest. The TR writes: "Some students might rightly conclude that the rate is fair because it encourages Jason to pay his sister back quickly." Jason's sister is a loan shark.

Page 20, Chapter project. We found a web site which described bamboo growth of 60 mm per day as rapid. Sixty centimetres per day is difficult to believe. Perhaps the authors should check their sources.

Page 22, Question 7. Scatter plot. The term "scatter plot" is usually referred for situations where data (x, y) are empirical data (with inevitable random variation). The graph shown here is the graph of a well-defined function. There is nothing random about it. Simply say: "The graph shows the first seven terms of a sequence."

Page 23 ff. Both the text and the TR should give more insight into some of these examples. In Questions 9a, 10, 12 and 13, the sequences are (crude) estimates of areas of flat shapes. The fact that all sequences are quadratic illustrates the basic rule: "area is a constant times the square of some measure of width".

Similarly, the sequences in Investigation 6 and Questions 17, 19, 20, 21 are crude estimates of volume. The fact that all sequences are cubic illustrates the basic rule: "volume is a constant times the cube of some measure of height or width".

The TR alludes to this issue later, with reference to the Chapter Project (text, page 32, TR page 60).

Page 27, Question 21. There is no need to state that the balloons are "spherical" – especially so since those pictured are only approximately spherical.

Page 27, Challenge yourself. This is a nice problem.

Page 28, Question 28. Students need to be encouraged to think about how to approach this problem systematically. A suggestion: at each point on the grid, pencil in how many squares have bottom right corner at that point.

Page 33, (a). Clarity. The answer in the TR assumes that the problem had read: "Make a sequence with 10 terms, representing the total fuel cost per total distance traveled, for distances (in kilometres) 1, 2, 3, ..., 10."

Either the text or the TR should admit that this problem has been simplified: though fuel consumption may *average* 12 L/km, the actual rate of consumption is higher early in the flight (when the plane is full of fuel), and lower in the later stages of the flight.

Page 43, Question 10. Clarity. I could choose to "name" the sequence Fred. Better wording: "Which of the following could be an arithmetic sequence?"

Note: since we can see only the first few terms of the (infinite) sequences, we can't be *sure* of what the next term will be. Such technicalities will not worry most students, but teachers are obliged to set a good example. Increased clarity generally leads to less confusion.

5.2 Chapter 2. Quadratics

Factoring (Book 1, Chapter 3) is not mentioned as "assumed prior knowledge" in the TR (page 102). Consequently, this chapter fails to make the connection between factoring and the quadratic formula. Nor does it make the obvious connection with parts of Chapter 1 of Book 3: straight line functions (such as reaction times in Focus A) and arithmetic sequences; quadratic functions and sequences of squares.

In several places, students are required to use least squares on a graphing calculator to deduce the equation of a quadratic function, given a table of values. We have spoken, at length, about circumstances under which such a powerful tool would and would not be appropriate (see Book 1, Chapter 4, and our document on the *Mathematical Modeling* series). Students at this level, who have been taught how to factor quadratic equations, can readily learn how to derive formulae for many of the functions used in Section 2.1 (especially those with one x-intercept at $x = 0$).

The text leads students to believe that the problems in this chapter can be solved *only* with a TI calculator. Students should be taught that other methods can be used to solve these problems. (In fact, this is a stated goal of the curriculum.) By focusing exclusively on the calculator solution, this chapter fails to convey an important message: that mathematics is a powerful tool and many problems can be solved by manipulation of symbols (i.e. by using algebra, the language of mathematics).

Page 47, A. The problem suggests that plots of points displayed on a graphing calculator are more "accurate" than plots on graph paper.

Once again, the text suggests that students "use a sledge hammer to push a thumb-tack into a cork board": using least squares regression to find the equation of a quadratic function. Since the tables of values *exactly* fit a quadratic model, there is nothing random about the "scatter plot".

A more relevant problem would simply ask students to identify the correct equation from a list of candidate formulae, and explain how they made the choice.

Page 47, E. Extrapolation. A cautionary comment must be made whenever extrapolation is encouraged.

Page 48, Question 6. Wording. Clumsy wording, "at the same horizontal distance". Expand the problem: (a) sketches, (b) which is closest to the ground when x is 0? 10? etc.

Page 48, Question 7(c). Under the old curriculum, students were taught how to derive formulae such as this one. The "sledgehammer" approach is disappointing.

Similar comments apply to Questions 41 and 42, page 62.

Page 51, Question 13(a). "Explain why parabolas have a line of symmetry." This question is premature. It belongs with work on completing the square.

Page 52, Question 15 and Note 2. Grammar – tenses.

Page 52, Question 16. It would be more accurate to say: "The path of his jump can be described by the equation ... ".

Page 53, Focus A. The heading for Focus A should be at the top of the page.

It would be more accurate to say that the table "models" (rather than "contains") reaction distances.

The table at the top of page 54 contains supposed empirical data. The problem would be more convincing if

1. more details were provided about the experimental situation under which such data were gathered, and
2. the data exhibited more variability.

Mention should also be made that theoretical arguments (physics) justify use of the quadratic model.

Page 54, Question 17. "Explain how you know" is too strong a wording. In applied problems, many students will be all too aware that they do not know very much, and are not confident of their answers. Suggestion: "Explain why you decided ... ".

Page 55, Question 19. Inappropriate problem. The given table has (supposed) empirical data. Students are being asked to "complete the table" by extrapolating from a fitted model.

The authors are very much aware that a TI calculator can fit a polynomial of specified degree to any set of points (x, y). They are *not* aware of when the least squares curve is an attempt to approximate some underlying model, observed with error, and when the least squares curve is nothing more than a lazy way to solve an appropriate set of equations.

Page 55, Question 22(a) and TR page 86. Error. Doubling the speed doubles the reaction distance, it does not "increase it by 2"

Page 56, Question 24 or 25, and TR. Valuable exercise: using graph paper and pencil, graph all three tables (reaction time, braking distance, total stopping distance) and help students to see the "vertical" addition of two functions to obtain a third.

Page 56, Question 26(c). Crash damage index. We suspect that crash damage index has a "ceiling", in which case such extrapolation would give a silly number. (In fact, the only documentation we could find on-line used a categorical system, with numeric codes 1 to 5 for survivable accidents, and letter codes for more severe accidents.)

Page 57, Investigation 2. Realism. Cost increasing linearly with diameter is not realistic. Even a quadratic model is hard to believe: very few goats would produce hides large enough for a big drum.

Page 57, D. "Prove your prediction" using a TI calculator. The TI calculator will spit out the equation for a curve or line, whether or not such model makes much sense.

Page 58, Challenge yourself. This is a good problem. Teachers should make connections with previous problems (Chapter 1) that displayed linear, quadratic, cubic patterns.

Page 59, Question 33. Realism. (This must be a bulk-food store, since there is no base-line price for packaging. In fact, saffron is so expensive, that it is never put out in bulk food bins.)

More importantly, this problem asks for gross extrapolation, from price per fraction of a gram to price per kilogram, which would surely be a discount rate for wholesale purchase.

Page 59–60, Question 35, and TR page 93. This is a very nice problem. Students will see that some relationship are *approximately* linear, others are *approximately* quadratic, and some are neither.

However, part (c) should be omitted, since when viewed over a small interval, empirical data from an underlying quadratic or cubic model could well look like a straight line. (See our comments on Book 2, Chapter 4, page 156 ff.) On page 93 (near the bottom) the TR states: "You can find the approximate price of a drumskin using either a linear or quadratic function, although the R^2 of a quadratic function is slightly greater than the R^2 for a linear function." The R^2 is *always* higher for a more complicated model. Students should *never* be encouraged to "overfit" a model.

Note: technically, a "linear function" has the form $y = bx$. That is to say, to graph passes through the origin. One may say "linear relationship" or "function whose graph is a straight line".

Page 60, Investigation 3. Realism. The data supposedly come from a survey, but the perfect fit clearly indicates that this is not the case. Either change the data (a much more difficult problem), or make the story realistic, by (for example) saying that the numbers were obtained from a model a consultant had devised when solving the problem.

Similar comments apply to Questions 36–38. Question 39 is fine, since it clearly states that a model is being used.

Page 64 ff. Section 2.2. In order to make connections, students should use graph paper for some problems, and factoring for some problems.

Page 64, A. Clarity. With the present wording, some students might think that the equation $y = 0.05x^2 - x$ can be deduced from the given information. It would be better to say: "*Suppose* the falcon flew ...'.

Page 65, 2(b) and TR. Big idea. The TR mentions that "the x-coordinate of the vertex is found by averaging the x-intercepts," with no further comment. This is a big idea, which should be elaborated. For example, the same fact can be described geometrically by saying: "The axis of symmetry is half way between the two x-intercepts."

Page, 66, Question 5. Clarity. In order to answer this question as posed, students would need to know university-level calculus. The question should ask "How far away did the ball land?" or "When the ball hit the ground, how far was it from the point at which it was kicked?"

In fact, using a graph drawn on paper, a piece of string, and a ruler, it would be a good exercise to have students compare horizontal (and vertical) distance traveled with actual distance traveled.

See also Question 21, page 72.

Page 69, Note. The note provides an algorithm for solving the equation $x^2+5x+1004 = 998$. The chapter seems to place high emphasis on graphs, yet neither the text not the TR demonstrates the connection between this problem and interpretation of the points where the horizontal line $y = 998$ crosses the curve $y = x^2 + 5x + 1004$.

See also the note at the bottom of page 72.

Page 71, Investigation. Introduction to quadratic formula. The quadratic formula is introduced as a "black box", with no motivation. Students learned how to find roots by factoring way back in Book 1, but no connection is made. At the very least, students should be asked to work a few problems both ways, to see that the new formula seems to work.

Page 71 and TR page 112, Think about. According to the TR, the $\pm$ is there "to remind us that two solutions are possible." Derivation of the quadratic formula (possibly for a specific function) would clearly demonstrate the source of the $\pm$ sign.

Page 73, Question 26. Interpretation. Some students will want explanation of the x-intercepts. The intercept $x = 10$ means this: when hamburgers are priced at $10 each, nobody will buy them. If hamburgers cost $0, then no money comes in (the intercept $x = 0$).

It would be a nice exercise (and a valuable "connection") to have students graph number of hamburgers sold versus cost ($y = 6000 - 600x$), and interpret both slope and y-intercept.

Page 74 - 75, Focus C. Students are asked to:

1. obtain a table of values,
2. use least squares to find an equation,
3. use the quadratic formula to find the roots.

Neither the TR nor the text note that some (many?) students may be happier to derive the equation algebraically, then factor to find the roots.

These comments are relevant to the rest of the problems in the section.

5.3 Chapter 3. Exponential Growth

Any revisions of both the *Constructing Mathematics* and *Mathematical Modeling* series should involve a more coherent introduction to exponents and laws for their manipulation. There should be:

1. Careful construction of a few "key" graphs, using graph paper: $y = x^2$, $y = x^3$, etc., using hand and calculator calculations to check that interpolation seems to be appropriate.

2. Discussion of how square-roots, (cube-roots, etc.) can be found from the graph of $y = x^2$ ($y = x^3$, etc.).

3. Introduction of notation such as $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, etc.

4. Careful construction of more "key" graphs, using graph paper: $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$, etc., using hand and calculator calculations to check that interpolation seems to be appropriate.

5. Use of the graphs obtained in Steps 1 and 4 to demonstrate that, for example,

$$(x^3)^{\frac{1}{2}} = (x^{\frac{1}{2}})^3 = x^{\frac{3}{2}}.$$

   (The class could work in groups, with some students investigating $y = x^{\frac{2}{5}}$, etc.)

6. Use of a table of values obtained in Step 5 to graph points for the function $y = x^{\frac{3}{2}}$ ($y = x^{\frac{2}{5}}$, etc.), using calculator and hand calculations to check that interpolation seems to be appropriate.

7. (Eventually), introduction to graphs of functions such as $y = 2^x$, $y = \frac{1}{3}^x$, $y = \left(\frac{2}{3}\right)^x$.

8. Comparison of the graphs of $y = 3^x$ and $y = \frac{1}{3}^x$, with discussion of negative exponents.

9. Many problems requiring interpretation.

10. "Bare hands" checking that rules for exponents do, indeed, work (referring to the graphs obtained in Step 7).

11. Practice using the rules for exponents *until* the rules become part of each student's "mathematical tool-box".

12. Practice using the rules for exponents in applied problems.

13. Many students have difficulty grasping the huge difference in the role played by x in these two functions: $y = 2^x$ and $y = x^2 + 1$. *for non-negative values of* x. In fact, it would be a good exercise to have some students graph these two functions using the same set of coordinate axes (graph paper, $x > 0$). Another group could graph the functions $y = 3^x$ and $y = x^3 + 1$, another group could graph $y = (\frac{1}{2})^x$ and $y = x^{\frac{1}{2}} + 1$, etc.

We believe that the authors had intended a graphical introduction to exponents, similar to that outlined above. We suspect that the illogical treatment of exponents (in *both* series of texts) stems from a lack of communication between teams working on different chapters.

Some of the material described above should be introduced before the tenth grade. (For example, x^n for x positive and n an integer.) We have not looked at mathematics texts that precede this series.

Several problems in this chapter are poorly worded, with mixed tenses.

The chapter starts well, making connections with the geometric and quadratic sequences of Chapter 1. Regrettably, the term “arithmetic sequence” is not used in this chapter, only “linear sequence”. To make a better connection with Chapter 1, teachers should include a few cubic sequences.

The authors should consider moving the material from Section 3.3 (Finance) back into Chapter 3 of Book 2 (which needs major revisions). Then financial examples could provide a familiar lead-in to exponential functions.

Page 98 ff. and TR. Investigation 1. An interesting introduction would be to offer a moderate prize (say $5) to any student who can fold a (supplied) sheet of newspaper accurately in half 10 times. (Ten should be sufficient even for a broadsheet; if it were cut and stacked that many times the pile would be less than 3cm square and about 10cm thick!)

The investigation is a good idea, but there are several problems with it in its present form.

Firstly, the text readily admits that the folding process creates problems. Thus, the tables for area and thickness describe a “model”, they do *not* describe “data” as stated in the TR.

It would be a valuable exercise to have students compare real data with the theoretic model: a nice example of predictable bias away from a theoretic model.

In order to avoid the confounding effects of folding, students could carefully *cut*, rather than fold the paper.

Problems 1 – 6 all deal with paper folding. For variety, teachers could bring in a wooden rod, cut in half, then one half cut in half, one of those pieces cut in half, etc. Students could graph (Weight of a sub-rod) versus (Cut-number at which this sub-rod was obtained). For added impact, and happy students, bring in a (long) log-shaped cake.

Page 100, Question 8. To avoid frustrating students (repeating calculations), ask (b) before (a).

Page 101, Question 12. Magnification. This is a nice problem.

In order to avoid confusion, the second bullet should read: "Jacqueline enlarges the *dimensions of* the image by 10%..." (The point of the problem is that the overall size increases by more than 10%.)

Students should be encouraged to pause and think about *why*, when calculating areas "The common ratio is the product of the common ratios in parts (a)and (b)." (TR page 137.) We suggest that students use graph paper: draw a rectangle with specified dimensions and count the number of enclosed squares (i.e. area). Then increase each dimension by 10%, carefully counting the number of *extra* enclosed squares – which should be added to the previous total. Students should obtain a diagram that reminds them of algebra-tile problems, leading to a review of the expansion of $(a+b)^2$.

Problems 12 – 14 are all too similar. Perhaps one could count pixels, and another could describe a problem in which Derek is copying and enlarging (or scaling down) a paper map, by hand. Teachers may also consider asking students to investigate what happens to the volume of a cube when edge size increases by 10%.

Some students may ask why exponential functions are obtained in these problems, but sequences of squares were obtained in the problems on page 23. Have students compare the pattern obtained in Question 12 with the pattern that would be obtained if Jacqueline simply added 2 cm to the length and 1 cm to the width at each step. This would be a good time to review the different roles of x in functions such as $y = ax^2 + bx + c$ and $y = (1.1)^x$.

Page 103, Note. Axes should be clearly labeled. On page 107, the labels for the table are explained in the margin, but there is plenty of room for more detailed labels right on the table. (Remember: a text book is a model, which students imitate.)

Page 105, Note. The correct way to discover the value of 2^0 is via the formula $a^b a^c = a^{b+c}$, *not* as an artifact of the TI calculator. We note that this important formula is not introduced in the text, despite its elementary nature and the fact that it, in turn, is easily discovered (at least in simpler cases).

An investigation might start out by asking students:

Compute some values of 2^n. Make a table for $n = 1, 2, 3, \ldots, 10$ showing n, 2^n as a product, and 2^n as a number. The first few rows will look like

n	2^n	2^n
1	2	2
2	2×2	4
3	$2 \times 2 \times 2$	8
.	.	.

Use your table to solve: $2^2 \times 2^3 = 2^x$. (Several questions here, all "multiplicative".)

Describe the pattern.

What equation with exponents is illustrated by

$(2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) = (2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2)$?
(Answer: $2^3 \times 2^4 = 2^7$.)

Write an equation like the one above to illustrate that $3^3 \times 3^1 = 3^x$.
What is the value of x? (Check your equation!)

Make up and check two other problems like this, one with 3 as the base, the other with 5.

Write an expression for $3^x \times 3^y$; $5^x \times 5^y$; $17^x \times 17^y$; $a^x \times a^y$.

Explain the sentence "To multiply two powers of the same number, you add the exponents".

We ask for what sort of numbers a we've tested this rule. We briefly take out the calculator to see if the calculator values for similar calculations involving bases such as 0.7, -2.35, or 3.14159 seem to obey the same rule, using BOTH the power key and multiplication. (Note that the power key may not work for negative base, but multiplication does work.)

The investigation then progresses to "division-type" problems such as $3^4 \times 3^y = 3^7$.

Then we throw in $2^x \times 2^5 = 2^5$. We ask for a solution as a power of 2. Then we ask what the numerical value of 2^0 ought to be if this is going to be true. We ask for an illustration with a product, hoping for:

$$\begin{aligned} (\ )(2 \times 2 \times 2 \times 2 \times 2) &= (2 \times 2 \times 2 \times 2 \times 2) \\ \text{or}\quad (1)(2 \times 2 \times 2 \times 2 \times 2) &= (2 \times 2 \times 2 \times 2 \times 2) \end{aligned}$$

We point out that students are outgrowing the repeated-multiplication model and will soon want to leave it behind.[7]

We try other bases, and elicit a general formula for a^0.

Negative powers, $(a^b)^c$, and fractional powers can all be discussed as extensions of this investigation.

Page 108, Question 28. "How can you decide if a set of data represents an exponential relationship?" The explanation given in the TR, "As the x-values change by the same amount, the y-values change by the same amount," is correct for a *table of values describing a model*. For real (empirical) *data*, it is actually difficult to recognize an exponential model. (A statistician would plot $\log(y)$ versus x, see whether the relationship looked like a straight line *and* ask the client whether an exponential relationship would make *practical* sense.)

Page 108, Question 29 and TR page 145. Some of the TR answers (such as that for 29 (d)) are wrong. In other cases, the common differences or ratios given in the TR may be confusing. The sidebar on page 145 of the TR mentions a "common ratio" or

[7] A point that should be made more often with "training wheel" techniques and manipulatives.

"common difference" between successive y values when "the x-values change by the same increment"; this is hinted at on page 108 of the text, as well.

Thus, in problem 29 (e), we have x-values changing by an increment of 2, and we see common differences of -4 between successive y values. What the TR actually gives in some cases are common ratios or differences betwen successive y values when the x values change *by 1*, interpolating if necessary. This explains the TR's statement that the common difference is -2. (In problem 29 (g), however, the other convention is followed. The "$\Delta x = 1$" answer, too difficult for this course, would be a common ratio of $\sqrt[3]{1.5}$)

We suggest that problems in which the x values do not assume successive integer values are too difficult for this course; and that tables such as (h) and (i) that are deliberately disordered may cause too much trouble for the slight lesson that they teach.

Page 111, Question 35, Situation 1. Some students will want to draw up a table of total interest payment set aside. In order to encourage the table shown, change the story to say that Peter may repay the loan at the end of any month, etc.

Page 112, Investigation 3. "b^x when x is an integer."

See our comments at the beginning of the chapter. The rules for exponents, including the meaning of symbols such as 5^3, 5^{-3} and 5^0 have been carefully developed over the years, so that the language of mathematics is consistent (and useful). These rules should *not* be introduced as an artifact of the TI calculator, to be "discovered" by enterprising high school students.

See comments on Note, page 105.

Page 116, Chapter project. This problem pays far more attention to interest than does the chapter on finance in Book 2. We note that the TR sheds no light on what the solutions to parts (b) - (d) may be.

Page 117 ff. Section 3.2, Regression Analysis. Investigation 4. At this level, students should graph data (preferably on graph paper) and use various methods to demonstrate to themselves that growth is *faster* than quadratic or cubic, or quartic, ... Such is the nature of "exponential growth".

The TR places great emphasis on finding a model with high R^2. Some students will obtain larger values of R^2 using a polynomial than using an exponential model. We have written at length about the dangers inherent in encouraging students to look for patterns in data without adequate training.

Page 121, Question 18 and 19d,e. Extrapolation. Students should never be encouraged to extrapolate beyond the range of observed data.

Page 123, Focus C. The measurement of bacteria population density in bacteria per square centimeter is appropriate to two-dimensional culture media (such as an agar film in a Petri dish), not to substances such as ground meat.

Page 127, Challenge yourself. Realism. The fact that the TI cannot fit an exponential model to data containing the vales $(0, 0)$ is a technicality best avoided with students at this level. A far more relevant observation is this: the table suggests that there would be *no* traffic accidents if everybody stopped drinking.

Page 129, tables. There should be more information above these tables.
Table A: compounding period is 1 year; interest rate is 6% per compounding period.
Table B: compounding period is 6 months, or half a year; interest rate is $\frac{6}{2} = 3\%$ per compounding period.
Table C: compounding period is 1 month, or $\frac{1}{12}$th year; interest rate is $\frac{6}{12} = 0.5\%$ per compounding period.

We are certainly disappointed that students have been asked to "use technology" to find the equations. *If* they have understood the preceding section, then students should be able to:

- check that the given entries are correct,
- calculate the missing entries.

It would be an interesting exercise in rules for manipulation of exponents to have students enter their data for all three schemes with time measured in *years*, then figure out *how* to derive the TI formula (time in years) from their formulae (time measured in six-month or one-month intervals).

Page 135, Investigation 7. Introduction to an annuity. It is always *dangerous* to print an incorrect solution to a problem.

If, back in Chapter 1, students had seen a formula for the sum of a geometric progression, then they would have greater appreciation for Marcus' (correct) solution. Then, when calculators are introduced (Focus F), students could check that the TI agrees with the solution obtained by Marcus.

Material on financial mathematics should cross-reference topics in the *revised* Chapter 3 of Book 2.

5.4 Chapter 4. Geometry of Design

Overall, this is an original and well-designed chapter, likely to interest students who will be involved (either on an amateur or professional basis) in crafts, graphic arts, design, etc. much more than a traditional section on geometry would do. There is some intelligent and appropriate use of ethnomathematics, which could probably be extended.

A little more explanation about the actual geometry and construction techniques would not go amiss. This chapter, like many others in this and the *Mathematical Modeling* series, is written as if sketchiness were a virtue, and would result in students working everything out for themselves. If sketchiness is taken to extremes, then students will completely miss the point.

One disappointing omission is the n-section of a line segment by parallel lines. This technique, needed in various places in the chapter, is never mentioned, and has wide applicability in carpentry and design. Another is the characterization of a cyclic quadrilateral; this would have fit in very naturally around Investigation 12/Focus C.

An annoying flaw is the introduction of mysticism on the subject of crop circles. Crop circles are not "unexplained"; plenty of people have admitted to making them (see the site www.circlemakers.org). Both Winterbourne Basset and Beckhampton, mentioned on page 170, are sites where human-produced crop circles, of known origin, have been documented. (Put either name into the local search engine on the circlemakers website for details.) We cannot understand why the authors - who surely do not believe this nonsense themselves - should indulge in this gratuitous mystification. For many years now (except among the exceptionally gullible), crop circles have not been considered as a serious attempt at hoax, but as a form of landscape art. Suggestions to the contrary may mislead a few students, and damage the authority of the textbook among the wiser ones.

Moreover, by not stressing the human creativity involved, the authors are missing some great opportunities to involve students. While the use of standing grain would probably be inadvisable, classes could easily create a design of their own, on a similar scale, in snow, sand, or other media. The main tool involved is a rope, used both as ruler and as compasses. It would be far more appropriate for the authors to encourage students to try it themselves than to propagate pseudoscience.

There is also considerable confusion about the properties of the incenter and circumcenter. The authors are apparently under the misconception that the former point minimizes the sum of the distances to the edges – and the latter, to the vertices – of the triangle. Neither of these is the case; and several of the "practical applications" in section 4.5 are thus, unfortunately, wrong.

Page 158, Question 10, and TR. The procedure shown in the TR is a shortcut, in which one arc (the first one drawn, locating the centers of the other arcs) is not shown, and the arcs used in the first stage of the construction are used again in the second stage. This is not appropriate as a first exercise.

Page 158, Question 16 and TR. The TR makes it clear (and the textbook does not) that the distance from the center to the four inner vertices is arbitrary.

Page 159, Investigation 3. The concepts of "incenter" and "circumcenter" should be explained in more detail. In particular, the authors should try to make it clear that it is *surprising* when three lines meet, not something to be taken for granted in a definition placed off in a sidebar.

Page 161, chapter project, trisection. The students have (implicitly) been shown how to bisect a line segment with compass and straight-edge on page 159. They have not been shown how to trisect a segment, an operation that is required here. The TR does not show how this is done, and the method used for bisection cannot be generalized. (See also our comment about page 197, below.)

Page 167, Question 17, igloo size. Igloo A in the diagram has a radius (and height) of 6m. This is large enough to contain a small two-story wood-frame house. This would seem implausible given traditional construction techniques. (If the size is indeed as claimed, it is sufficiently unusual that it should not pass without comment, and perhaps a picture.)

Dr B. Davis of Saint Mary's University suggests that the scale of this diagram would make sense if the units were feet rather than meters; and that it may have been incorrectly adapted from one so labeled.

Page 167, Question 17b-d. The answers in the TR are based upon the unreasonable (and unstated) assumption that you would walk by way of the center of any intervening igloo.

Page 167, Question 18. The description of the Leaning Tower as "54m tall" is ambiguous; students could reasonably take this to be vertical height.

The answer given in the TR (53.8m) requires students to ignore what they were taught in Grade 10 about significant figures and measurement; the correct answer is 54m.

A minor quibble - a straight rope could not be lowered from the top of the tower, due to the step-back at the penultimate floor (see the picture in the text).

Page 170-1, crop circles. As mentioned above, the textbook should make it clear that crop circles are not of unknown origin, but a form of landscape art. The techniques are well-known and lend themselves to outdoor activities.

Page 172, paragraph 1. A triangle *is* a polygon. Betty should ask "What would happen if I used a different polygon?"

Page 175, Question 14 and TR. Students will certainly "have some difficulty locating the centers for the smaller circles" if they follow the suggestion in the TR. As can be seen by careful inspection of the figure in the textbook, the midpoint of the radii are *not* the points of tangency. Rather, the points of tangency are at distance $\sqrt{2}-1$ from the origin. This problem is probably too difficult for most students in this stream.

Page 175, Question 16, TR. The solution suggested in the TR assumes the centers of the four circles to be known; they are not given in the diagram and the procedure for constructing the center of a circle is not given until Investigation 19.

Page 175, Question 17d and TR. A problem such as this, that is too difficult for the target group, is not made acceptable by telling the teacher - as does the TR - to accept a vague answer rather than the precise one implicitly requested. ("What is the radius of the larger circle? How do you know?")

The correct answer would be more usefully given as $(3+2\sqrt{3})r/3$ than as "about 2.155 times the radius". (Giving both is probably best.)

Page 176, Question 21. The triangular partitions shown appear to be fictitious, not what "would be used" for fragile items. (We asked two Halifax-area manufacturers of packages for fragile items and the response, in each case, was that they had never heard of this design.) Such designs are *not* more appropriate for fragile items. Firstly, the stress on the item is divided among only three points instead of four; secondly, the looser packing fraction would make the objects more likely to become loose in transit; and thirdly, the wider spacing between points of contact would cause a "wedging" effect that would increase the force at points of contact. Finally, the threefold intersections would be weak, as 2/3 of each sheet would be cut away.

Page 182, Investigation 12. The result used here is sometimes called the *Star Trek Theorem* - students may enjoy this name. The proof (while not obvious) is not very difficult. It should be presented to students, so that they know that there is a reason for the relation between the angles. This would also be a good time to introduce basic results on cyclic quadrilaterals.

Page 187, note. This is somewhat confusing. Athletes do not need to know the scoring angle, merely whether they have a reasonable chance of achieving the required precision.

Page 188, Question 25. The elevation of the football goal may make this question more difficult for those students familiar with football. It certainly makes the question of scoring angle far less meaningful than it is for the soccer example.

Page 191, Question 30 b-d. This formula gives only an *upper bound* on the car's velocity, unless we assume that the car is, while on the circular path, on the verge of skidding. Note that the point D is misplaced; it should be right on the tire track.

Page 192 Question 31 and TR. The first four of Hawkins' "theorems" are (despite the praise lavished on them by crop-circle cranks) extremely trivial results. For instance, one of them states that if circle C is inscribed in square S, in turn inscribed in circle C', then the area of C is half that of C'. This would be obvious to any mathematician (and would be an appropriate exercise for this textbook).

The "fifth theorem" may be apocryphal; in the original announcements Hawkins did not reveal it. One of the standard references, to Volume 91 of the *Mathematics Teacher* refers to a somewhat incoherent paid advertisement that the editorial board were ill-advised enough to accept, and which again contains no statement of a theorem.

In part (a), students are not "using Hawkins' [fifth] theorem" (which the book, understandably in light of the above, never states) to produce the diagram. This careless use of words suggests confusion on the authors' part as to what a "theorem" is.

The suggested method of finding the circles centered on the circumcenter and tangent to the sides (TR, bottom of page 264) is invalid. The correct way is to draw a larger circle, intersecting the given side in two points; then find the point half way between those two points of intersection. This is the point at which the new circle will touch the side of the triangle.

Page 195, Question 4. The term "optimum point" is introduced (using italics and the word "called") as if it were a definition of a new term. Students may be puzzled by the apparent lack of difference between the "optimum point" and the circumcenter. The note in the sidebar of page 196, in which it is apparently redefined to be the incenter, completes the confusion.

Page 197, "Challenge yourself" and TR. It is not clear what the authors have in mind for the golf course. Is it supposed to be understood that the three holes are end-to-end on a straight strip of land? If so, the greens are *not* equal distances apart - this could only be achieved by placing them at the vertices of an equilateral triangle.

The solution in the TR is one of the most bizarre we have ever come across. The authors should surely understand that the trisection - or n-section - of a given segment by parallel lines is an easy construction, of great practical value and widely used among carpenters and other craftspeople. Its absence from this chapter is surprising and disappointing.

Instead, the even easier - and different - problem of constructing a segment of three times the length of a known segment is solved - incorrectly! (In the Old Testament (I Kings 7:23), 3 is used as an easily-constructed approximation to π; this is the first time we have ever seen the opposite done.)

Page 197, Question 13; also other questions. The optimum location for the pump is surely at any point on the triangle, with one pipe feeding the others. If this is inadmissible, the point minimizing the sum of the distances to the edges (this is the vertex with the largest angle) would be a better choice.

Page 197, Question 16-17. These questions make the dubious assumption that the circumcenter is "optimal." The most usual criterion would be minimizing the sum of the distances to the three vertices, which is achieved by the Fermat point.

Page 197, Question 19. This question recognizes that the circumcenter is not optimal, but is poorly worded: "... the distances to the chalets are equal and as small as possible." Making the distances equal uniquely specifies the point completely; no further optimization is possible.

Page 197, Question 20. We are apparently meant to infer that *each pair* of chalets is connected. What we are told is that all the chalets are connected, a different condition.

Realism: This loop main is presumably fed from somewhere; a direct connection to that point would give at least the same water pressure.

The shower is said to be "connected to the plumbing of each chalet". The TR answer makes it clear that it is connected to the plumbing *between each pair of chalets"*

As observed in the comment on Question 13, the incenter does NOT minimize the sum of the distances to the edges.

Page 197, Question 21. As observed above, the incenter does not minimize the amount of wire needed.

Page 198, Investigation 16. This is a good topic and well presented; however, one of the subsequent questions is unrealistic.

Page 199, Question 25. Depending on materials used for walls, the assumption that the angle of incidence equals the angle of reflection for the miniature golf ball of the previous question may be valid. However, as any student who shoots pool seriously will know, a pool ball does not bounce in this way, because friction with the cushion converts some transverse momentum into spin. Therefore, the ball will rebound at an angle greater than the angle of incidence..

Page 200, Question 29,30,32b. These questions, again, are invalidated by the authors' misconceptions about the properties of the incenter and circumcenter.

Page 200, Question 31. The authors apparently propose to supply electricity *to* the fence via one of the two sockets of a standard outlet at $(4, 0)$. This would require a highly dangerous (and probably illegal) male-male cable. The correct solution would be to run a cable from the fence to the power source, which would not have an electrical outlet at the fence end. Thus, the outlet would be placed as near to the tree as possible at $(12, 0)$.

Page 206, Investigation 18. This is an excellent and practical topic.

Page 208, Challenge Yourself. This is a good elementary presentation of an advanced and beautiful piece of mathematics.

Page 210, Case Study 1. This, too, is an excellent idea. It might be made more interesting by encouraging students to find some styles of arch themselves (perhaps from non-European cultures).

5.5 Chapter 5. Probability

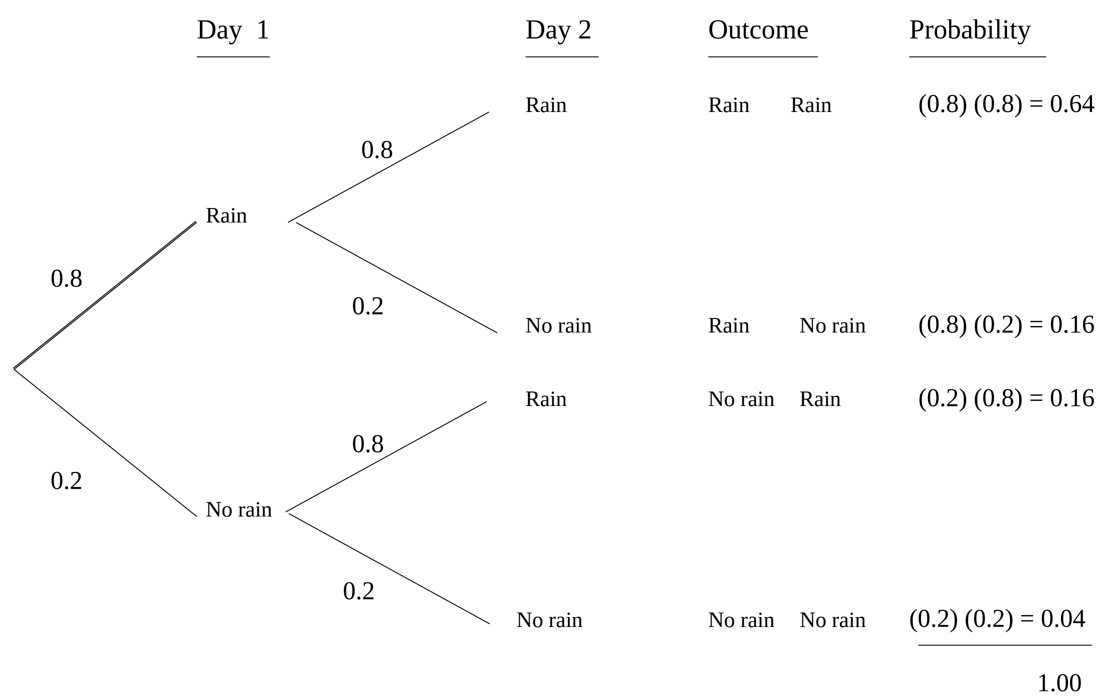
Sections 1 through 6 deal with probability in a round about, disorganized way, and could be improved with careful reorganization. Early sections deal with experiments with equally probable outcomes, though that constraint is not discussed. Section 5.1 introduces the addition rule for probabilities, and Section 5.3 introduces multiplication of probabilities with no clear guidelines as to *when* or *what* to multiply. Tree diagrams with probabilities do not appear until Section 5.6. If tree diagrams were introduced earlier, much of the confusion over when to add and when to multiply could be avoided. In particular, the addition rule could be demonstrated, though all problems could be solved without recourse to the formula.

We pause to discuss details of tree-diagram construction. With reference to the tree on page 257, we have found (over many years' experience) that the protocol described below leads to far less confusion.

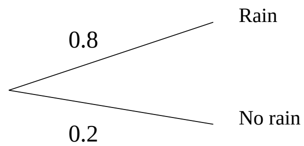
Note details of the tree diagram:

1. Outcomes are listed at the *tips* of branches.
2. Each column of outcomes is clearly labeled: outcomes for Day 1, and outcomes for Day 2. (We have found that forcing students to think of column labels helps them to formalize the problem.)
3. Probabilities are written beside branches, and *nothing else* is written beside branches.

Students intuitively "multiply probabilities along the branches". The fact that probabilities in the last column add to one convinces most students that they should "add probabilities down the extreme tips".

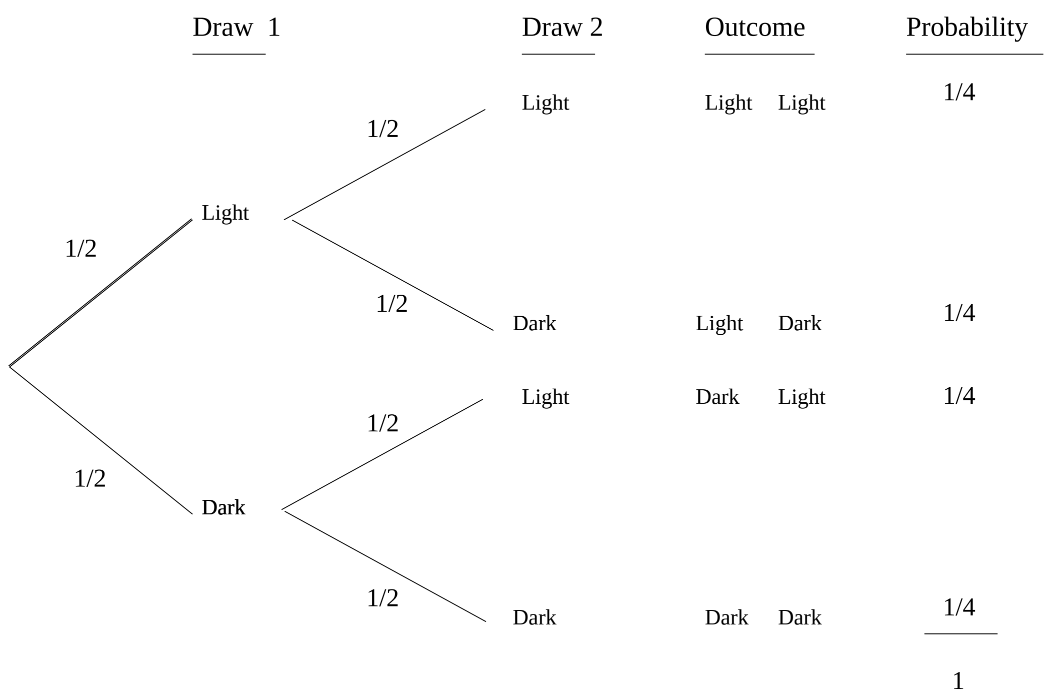


In this problem, occurrence of rain on day 2 is assumed *independent* of rain on day 1. And that is why the two subtrees leading to Day 2 outcomes are identical:



Section 5.2 introduces students to sampling with and without replacement (issues relevant to Book 2, Chapter 4, but not mentioned there). The key idea is this: when sampling *with replacement*, outcome on the second draw is independent of outcome on the first draw; when sampling *without replacement*, outcome on the second draw is *not* independent of outcome on the first draw. Question 6 demonstrates that the distinction between sampling with and without replacement matters only when the underlying population is small – once again, an issue relevant to Book 2, Chapter 4, but not mentioned there. The two tree diagrams below (with calculations) demonstrate the point of Investigation 8, page 250. In the first tree diagram (sampling with replacement), all subtrees leading to Draw 2 outcomes are identical. In the second tree diagram (sampling without replacement), subtrees leading to Draw 2 outcomes are *not* identical. Some sample calculations appear below each tree.

Tree diagram for Investigation 8, page 250. Sampling with replacement.



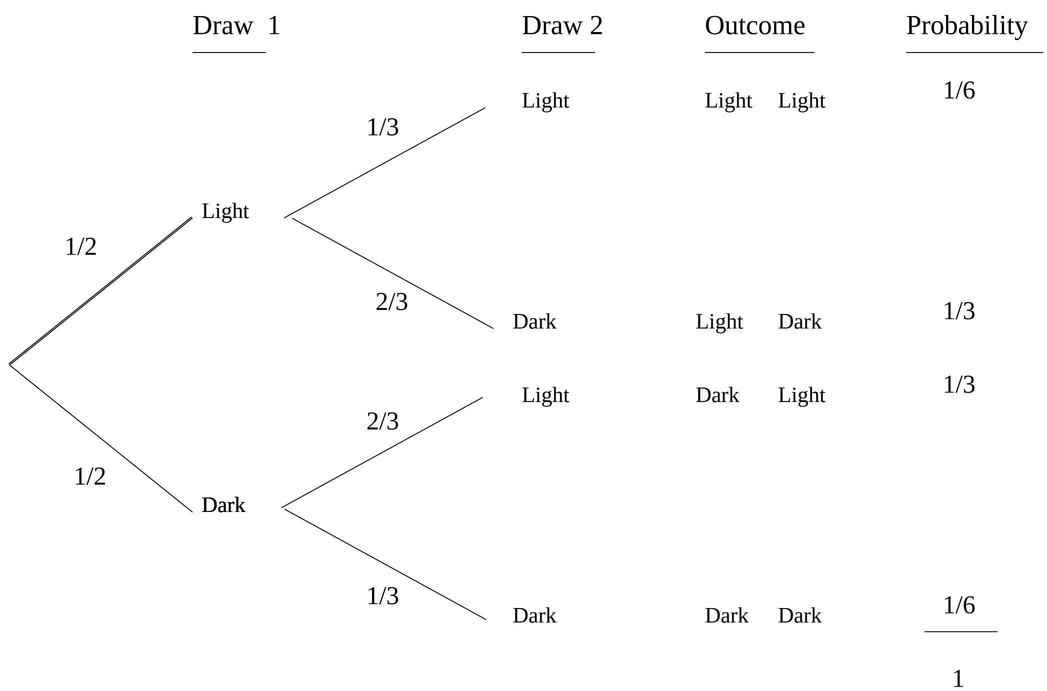
P(Two light chips removed) = P(Light, Light) = 1/4.

P(At least one light chip removed) =
P(Light, Light) + P(Light, Dark) + P(Dark, Light) = 1/4 + 1/4 + 1/4 = 3/4.

The addition rule gives:

P(At least one light chip removed) =
P(Light on first draw) + P(Light on second draw) - P(Light on both draws) =
{ P(Light, Light) + P(Light, Dark) } + { P(Light, Light) + P(Dark, Light)}
- P(Light, Light) = (1/4 + 1/4) + (1/4 + 1/4) - (1/4) = 3/4.

Tree diagram for Investigation 8, Sampling without replacement.



P(Two light chips removed) = P(Light, Light) = 1/6.

P(At least one light chip removed) =
P(Light, Light) + P(Light, Dark) + P(Dark, Light) = 1/6 + 1/3 + 1/3 = 5/6.

The addition rule gives:
P(At least one light chip removed) =
P(Light on first draw) + P(Light on second draw) - P(Light on both draws) =
{ P(Light, Light) + P(Light, Dark) } + { P(Light, Light) + P(Dark, Light) }
- P(Light, Light) = (1/6 + 1/3) + (1/3 + 1/6) - (1/6) = 5/6.

Page 226, Note 2. The definitions of probability given in the sidebars of pages 226, 228, and 234 assume *equally probable outcomes*. This assumption must be clearly stated. (Some card tricksters rely on certain cards – e.g. those with a small nick in them – being selected more often than others.)

It is easy to construct examples where outcomes are not equally likely. For example, place balls or rods of varying sizes (colour-coded for size, for quick identification) in a bag, and ask students to reach in and select one item. If students are forced to select quickly, then bigger items will be selected more often than smaller items.

Page 235 ff. Odds. This definition of "odds" works well enough when there are a finite number of equally likely outcomes. Students actually need an extension of the definition to answer some of the questions that follow. Strictly speaking,

$$\text{Odds of an event } = \frac{\text{Probability of the event}}{1 - \text{Probability of the event}}.$$

Thus, in Question 33 on the same page,

$$\text{Odds of winning} = \frac{\text{Probability of winning}}{\text{Probability of losing}} = \frac{0.4}{0.6} = \frac{2}{3}.$$

Using the correct definition of odds ratio provides valuable practice with arithmetic using fractions.

Page 227, Experimental probability. Mathematicians and statisticians do not use this phrase. In order to avoid confusion, it would be better to say (every time) "experimental estimate of probability". In particular, the opening paragraph on page 237 should state:
"This is different from the *experimental estimate of probability*, or *estimated probability*, based on experimental data."

In problem 9 on page 229, it makes sense to ask "What is the probability that ...": there is an obvious model to use when calculating probabilities. But, in problems such as 33 and 34 on page 235, students can provide only *estimates* of probabilities and odds ratios, based on previous data. Similarly, problems H and 3 on page 238 should say "estimate" rather than "calculate" or "find". There are numerous other places where these comments apply.

Neither the text nor the TR pay sufficient attention to the important distinction between theoretical and estimated probabilities.

Page 227. Note 2. "P(win) = 2/5 means there are two favourable outcomes and five possible outcomes." The wording is too sloppy. For example, if I select a ball at random from a bag containing 4 red and 6 blue balls, then P(red) = 2/5.

Page 228. Note 1, Question 8. Technically, mathematicians describe probabilities as numbers between 0 and 1, but we often convert to percentages if the audience prefers. In problem 8, one might say "The probability that the team will win the next game is 0.7," or "The team has a 70% chance of winning the next game."

Probability is a difficult topic. Careful attention to language helps to avoid confusion. See a similar comment about pages 116, 117 of Book 2.

Page 232. Stereotyping. The hockey coach is not too good at math.

Page 232. Mathematical "or". Neither the text nor the TR provides the following insight: when a mathematician says "A or B", he or she means: "A or B or both".

Page 232 ff. Addition rule. We suggest that the addition rule be delayed until students are more comfortable with probabilities.

Page233, Question 24. Realism. "There is no chance of both rain and snow flurries." Given non-zero probabilities for each of rain and flurries, students will spot this assumption as ridiculous. The problem would be more relevant to the section if there were, indeed non-zero probability of both. (Make sure that the probability of both is less than the probability of rain *and* less than the probability of flurries. Then draw a Venn diagram, with probabilities marked.)

Page 235, Challenge Yourself. We need a little more information about this lottery. The TR solution assumes: there are 14 million distinct tickets; exactly one of them wins; all tickets are equally likely to be selected.

These are reasonable assumptions, but they must be clearly stated.

Page 236, Question 37. Both the text and the TR fall short of explaining *why* "In a horse race, odds are not written in the standard way": because odds describe winnings from a specified size of bet *if* the horse wins. In fact, one hears "paying fifty to one", etc. (In particular, the odds quoted in horse races are not quite the same as the odds of losing, since bookies skim off a profit.)

Page 240. Multiplying probabilities. The spinner model suggested in the TR tacitly assumes that performance on the second hockey game is independent of performance on the first. Avid hockey players will dispute such an assumption. *If* tree diagrams had been introduced prior to this Investigation, then students would have all necessary tools to model a more interesting (and realistic) problem.

Page 241, Question 3; TR page 330. The TR provides a beautiful demonstration of sampling with and without replacement (but with no connection made to Book 2, Chapter 4).

Page 241. Chapter Project. The point has been made many times. Students can tackle this problem *without* resorting to simulation.

Page 248 ff. Independent events. The TR provides solutions in which probabilities are multiplied, but provides no insight as to why such multiplication is justified. Definitions of dependent and independent events appear later, on page 252, but no "connection" is made in either the text or the TR. Once again, tree diagrams would help here, allowing students to model both independence and dependence in "free throw" problems such as Question 32 on page 249.

Page 256, Question 7. To avoid confusion, break this into two problems, one for each given square.

Page 260 ff. Tree diagrams can be used effectively to obtain the ordered lists relevant to permutations. (For Investigation 10, column headings would be: First player, Second player, Third player chosen.)

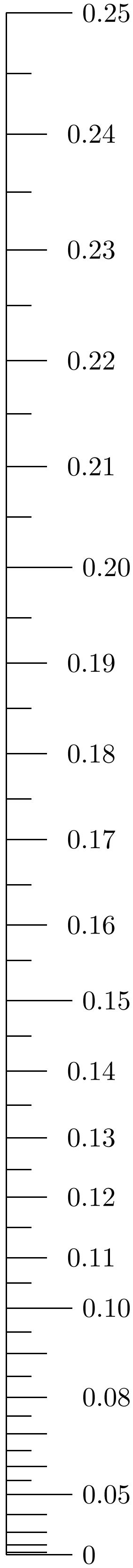
Page 262, Question 6(a). Students would find this problem easier to think about if the word "identical" were omitted.

TR Page 364, Sidebar, A point of interest for teachers: $n! = \Gamma(n+1)$, where the continuous function Γ is indeed defined for all positive real numbers. In fact,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$

Page 264, Focus H. While both notations are valid, the $\binom{n}{r}$ notation is more widely used by mathematicians and statisticians.

Section 5.7, dealing with combinations, fails to make the obvious connection with the "handshake" type problems on pages 7 and 8.



REACTION TIME METER MASTER

Photocopy this onto **legal** paper at a magnification of 140%. (This step is very important!) Cut it out, and stick it to a ruler or strip of wood about 30cm long. The experimenter holds it by the top; the subject holds his or her fingers apart on either side, parallel with the 0 mark. The experimenter drops it without warning; the subject pinches it to catch it. The reaction time in seconds is read off the ruler.