Math 1251: Taylor Polynomial Practice — Winter 2018

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1. Find the 4th degree Taylor polynomials at $x = 0$ for the following functions:
   (a) $f(x) = x^3 - 3x^2 + x - 1$
   (b) $f(x) = x^5 + 2x + 1$
   (c) $f(x) = \ln(1 + 3x)$
   (d) $f(x) = \sqrt{1 - 4x}$
   (e) $f(x) = \cos 2x$
   (f) $f(x) = \frac{1}{2}(e^x + e^{-x})$

2. Find the n-th degree Taylor polynomial at $x = 0$ for the following functions:
   (a) $f(x) = \dfrac{1}{1 - 2x}$
   (b) $f(x) = \dfrac{1}{(1 - x)^2}$
   (c) $f(x) = \ln(1 - x)$

3. Use 4rd degree Taylor polynomials to approximate the following integrals, and then find the exact value of the integrals (through antidifferentiation) to compare the results.
   (a) $\displaystyle\int_0^{1/2} \ln(1 - x)\, dx$
   (b) $\displaystyle\int_0^{\pi/4} x \sin x\, dx$

4. Use a 3rd degree Taylor polynomial to approximate $\int_0^1 e^{-x^2}\, dx$. (The true value to 4 decimal places is 0.7468.)

5. Use a 2nd degree Taylor polynomial to approximate $\int_0^1 \sqrt{\cos x}\, dx$. (The true value to 4 decimal places is 0.9140.)

6. For each of the following, use the n-th degree Taylor polynomial of $f(x)$ to approximate the value of $f(a)$, and give a bound on the error of your estimate. Then use your calculator to evaluate $f(a)$ exactly and find the true error of your approximation.
   (a) $n = 4, \quad f(x) = \sin x, \quad a = \dfrac{\pi}{12}$
   (b) $n = 2, \quad f(x) = \dfrac{1}{\sqrt{1 + x}}, \quad a = \dfrac{1}{10}$
   (c) $n = 3, \quad f(x) = \ln(1 - x), \quad a = \dfrac{1}{4}$

Answers

1. (a) $p_4(x) = x^3 - 3x^2 + x - 1$
   (b) $p_4(x) = 2x + 1$
   (c) $p_4(x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4$
   (d) $p_4(x) = 1 - 2x - 2x^2 - 4x^3 - 10x^4$
   (e) $p_4(x) = 1 - 2x^2 + \frac{2}{3}x^4$
   (f) $p_4(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$

2. (a) $p_n(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots + 2^n x^n$
   (b) $p_n(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n$
   (c) $p_n(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots - \frac{1}{n}x^n$

3. (a) Approximation: $\int_0^{1/2} \ln(1-x)\,dx \approx \int_0^{1/2}(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4)\,dx \approx -.15260417$
   Real: $\int_0^{1/2} \ln(1-x)\,dx = -\int_1^{1/2} \ln u\,du = -(u \ln u - u)\Big|_1^{1/2} \approx -0.15342641$
   (b) Approximation: $\int_0^{\pi/4} x \sin x\,dx \approx \int_0^{\pi/4}(x^2 - \frac{1}{6}x^4)\,dx \approx 0.15152945$
   Real: $\int_0^{\pi/4} x \sin x\,dx = (\sin x - x \cos x)\Big|_0^{\pi/4} \approx 0.15174641.$

4. $\int_0^1 e^{-x^2}\,dx \approx \int_0^1 (1 - x^2)\,dx = \frac{2}{3} \approx 0.6667.$

5. $\int_0^1 \sqrt{\cos x}\,dx \approx \int_0^1 (1 - \frac{1}{4}x^2)\,dx = \frac{11}{12} \approx 0.9167.$

6. (a) Taylor polynomial $p_4(x) = x - \frac{1}{6}x^3$ gives approximation $p_4(\frac{\pi}{12}) = 0.25880881$.
   Since $|f^{(5)}(c)| = |\cos c|$ is at most 1 for all c between 0 and $\frac{\pi}{12}$, the error is at most $\frac{1}{5!}(\frac{\pi}{12})^5 \approx 1.025 \times 10^{-5}$.
   The true value to 8 decimal places is $\sin(\frac{\pi}{12}) = 0.25881905$, so the true error is roughly 1.024×10^{-5}.
   (b) Taylor polynomial $p_2(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2$ gives approximation $p_2(\frac{1}{10} = \frac{763}{800} = 0.95375$.
   Since $|f^{(3)}(c)| = |\frac{15}{8}(1+c)^{-7/2}|$ is at most $\frac{15}{8}$ between $x = 0$ and $x = \frac{1}{10}$, the error is at most $\frac{15/8}{3!}(\frac{1}{10})^3 \approx 3.125 \times 10^{-4}$.
   The true value to 8 decimal places is $\frac{11}{10}^{-1/2} = 0.95346259$, so the true error is roughly 2.874×10^{-4}.
   (c) Taylor polynomial $p_3(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3$ gives approximation $p_3(\frac{1}{4}) = -\frac{55}{192} = -0.28645833$.
   Since $|f^{(4)}(c)| = |6(1-c)^{-4}|$ is at most $6(1-\frac{1}{4})^{-4} = \frac{512}{27}$ between $x = 0$ and $x = \frac{1}{4}$, the error is at most $\frac{512/27}{4!}(\frac{1}{4})^4 \approx 3.086 \times 10^{-3}$.
   The true value to 8 decimal places is $\ln \frac{3}{4} = -0.28768208$, so the true error is roughly 1.224×10^{-3}.