Math 1251: Worksheet #5 **Winter 2018**

Name: SOLUTIONS	A#:	Section:

1. Let $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$.

   (a) Find all critical points of f (i.e. points where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$).

$$\frac{\partial f}{\partial x} = 6xy - 6x \quad , \quad \frac{\partial f}{\partial y} = 3x^2 + 3y^2 - 6y$$

So $\frac{\partial f}{\partial x} = 0 \iff 0 = 6xy - 6x$
$\iff 0 = 6x(y-1)$
$\iff x = 0$ or $y = 1$

If $x = 0$ then $\frac{\partial f}{\partial y} = 0$ gives $3y^2 - 6y = 0$
$\iff 3y(y-2) = 0$
$\iff y = 0$ or $y = 2$

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So we have critical pts $(0,0)$ and $(0,2)$

If $y = 1$ then $\frac{\partial f}{\partial y} = 0$ gives $3x^2 - 3 = 0$
$\iff 3(x-1)(x+1) = 0$
$\iff x = \pm 1$.

So we get critical points $(1,1)$ and $(-1,1)$

Altogether: $\boxed{(0,0),\ (0,2),\ (1,1),\ (-1,1)}$

(b) Use the second-derivative test to classify the critical points of f as local maxima, local minima, or neither.

$$\frac{\partial^2 f}{\partial x^2} = 6y-6, \quad \frac{\partial^2 f}{\partial y^2} = 6y-6, \quad \frac{\partial^2 f}{\partial y\,\partial x} = 6x$$

So $D(x,y) = (6y-6)^2 - (6x)^2$

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- $D(0,0) = 36 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0,0) = -6 < 0 \Rightarrow \boxed{(0,0) \text{ is local MAX}}$
- $D(0,2) = 36 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0,2) = 6 > 0 \Rightarrow \boxed{(0,2) \text{ is local MIN}}$
- $D(1,1) = D(1,-1) = -36 < 0$, so $\boxed{(1,1) \text{ and } (1,-1) \text{ are saddle points (ie neither max nor min)}}$

2. Find the least-squares line for the data points $(-1,-5), (1,0), (3,4), (5,7)$.

Let $y = ax+b$ be the desired line. Then the least-squares error is

$$E(a,b) = (-a+b+5)^2 + (a+b)^2 + (3a+b-4)^2 + (5a+b-7)^2$$

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Then
$$\frac{\partial E}{\partial a} = 2(-a+b+5)(-1) + 2(a+b) + 2(3a+b-4)(3) + 2(5a+b-7)(5)$$
$$= 2[36a + 8b - 52]$$

$$\frac{\partial E}{\partial b} = 2(-a+b+5) + 2(a+b) + 2(3a+b-4) + 2(5a+b-7)$$
$$= 2[8a + 4b - 6]$$

So $\frac{\partial E}{\partial a} = \frac{\partial E}{\partial b} = 0 \Leftrightarrow$ ① $36a + 8b = 52$, ② $8a + 4b = 6$ $\Rightarrow 20a = 40$ (① $- 2\times$②)

$\Rightarrow a = 2$

$\Rightarrow b = \frac{6-8\cdot 2}{4} = -\frac{5}{2}$

The line is $\boxed{y = 2x - \frac{5}{2}}$