Integration by Undetermined Coefficients

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A (moderately) fast, (often) useful, and (largely) ignored alternative to integration by parts is *integration by undetermined coefficients*. This involves making an educated guess as to the form that the solution will take, and then "putting out integral bait" in the form

$$A_1 f_1(x) + A_2 f_2(x) + \cdots + A_m f_m(x).$$

This expression (or "*Ansatz*") is then differentiated and set equal to the integrand. If the *Ansatz* is correctly chosen, the integrand can be expressed as a linear combination of the terms that appear in the derivative. Assuming these terms to be linearly independent, the equation is equivalent to a system of linear equations which may be solved for the coefficients (A_i).

This is, of course, merely a simple case of a well-known technique for solving differential equations; and it is closely related to a method widely used for deriving partial fractions expansions. The technique was published in 1971 by Grinstein [2]; and it seems unlikely that much that is actually new remains to be said about it. However, its simplicity in certain situations, and the connections to material elsewhere in the standard undergraduate curriculum, argue for its inclusion in first-year calculus courses. The present note, making no claim whatsoever of novelty, is a attempt to keep the technique alive. It may be reprinted, distributed to classes, etc. freely.

The method is particularly useful for integrals of the form $\int P(x)e^{ax}dx$:

Example Find $\int(x^3+7)e^{2x}dx$.

Based on what we have learned from integration by parts, we expect the integral to have the form

$$ax^3e^{2x}+bx^2e^{2x}+cxe^{2x}+de^{2x} \quad .$$

Differentiating this and setting it equal to the integrand:

$$\begin{aligned} x^3e^{2x}+7e^{2x} &= \frac{d}{dx}(ax^3e^{2x}+bx^2e^{2x}+cxe^{2x}+de^{2x}) \\ &= 2ax^3e^{2x}+3ax^2e^{2x}+2bx^2e^{2x}+2bxe^{2x}+2cxe^{2x}+ce^{2x}+2de^{2x} \end{aligned}$$

Breaking this up:

$$\begin{array}{rclcrcl} 2ax^3e^{2x} &=& x^3e^{2x} & \text{so} & 2a &=& 1 \\ 3ax^2e^{2x}+2bx^2e^{2x} &=& 0 & \text{so} & 3a+2b &=& 0 \\ 2bxe^{2x}+2cxe^{2x} &=& 0 & \text{so} & 2b+2c &=& 0 \\ ce^{2x}+2de^{2x} &=& 7e^{2x} & \text{so} & c+2d &=& 7 \end{array} \tag{1}$$

Solving these in order, we get $a=\frac{1}{2}, b=-\frac{3}{4}, c=\frac{3}{4}$, and $d=\frac{25}{8}$ and the solution is

$$(\frac{1}{2}x^3-\frac{3}{4}x^2+\frac{3}{4}x+\frac{25}{8})e^{2x}+C \ .$$

The fact that the system (1) is upper triangular is not, of course, a coincidence; and it should not be too hard for students to see this. This gives a conceptually useful way of looking at an integral of the form $\int x^n e^{ax} dx$: we first find a high-order term that will give the integrand as one term of its product-rule derivative, then we iteratively find lower-order terms that cancel out any unwanted by-products. This is certainly implicit in an integration by parts as well, but seems more obvious here, where everything is visible at the same time.

The method can be used for other integrals often done by other methods: for instance:

Example: Find $\int \sin(2x)\cos(3x)dx$. (Note that this can also be done using either trigonometric identities or parts.)

Ansatz: Brief consideration will show that differentiation converts "mixed" terms into terms of the form $\sin(2x)\sin(3x)$ and $\cos(2x)\cos(3x)$, and *vice versa*. We can thus keep the form simple:

$$F(x) = a\sin(2x)\sin(3x) + b\cos(2x)\cos(3x) \; .$$

Hence

$$F'(x) = 2a\cos(2x)\sin(3x) + 3a\sin(2x)\cos(3x) - 2b\sin(2x)\cos(3x) - 3b\cos(2x)\sin(3x)$$

which, broken up, yields

$$\begin{array}{rcll} 3a - 2b & = & 1 & \qquad [\sin(2x)\cos(3x)] \\ 2a - 3b & = & 0 & \qquad [\cos(2x)\sin(3x)] \end{array}$$

The system, while not upper triangular, solves immediately to $a = \frac{3}{5}$, $b = \frac{2}{5}$, and

$$F(x) = \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C \; .$$

If we had accidentally included "mixed" terms $\cos(2x)\sin(3x)$ and $\sin(2x)\cos(3x)$ in the *Ansatz*, the other two resulting equations would have been uncoupled from the two above and could have been solved to $c = 0, d = 0$ by casual inspection.

Our last example is an integral somewhat harder than most found in undergraduate calculus textbooks:

Example: Find $\int x^2 \sin(x) e^x dx$.

Ansatz: We suppose, based on experience with easier integrals done by parts, that we will get terms of order 2 or less in x, multiplied by sinusoidal functions and exponentials. Taking all possible terms of this type, we get:

$$F(x) = px^2 \sin(x)e^x + qx^2 \cos(x)e^x + rx \sin(x)e^x + sx \cos(x)e^x + t \sin(x)e^x + u \cos(x)e^x$$

Then

$$\begin{aligned} F'(x) \quad = \quad & px^2 \cos(x)e^x + 2px \sin(x)e^x + px^2 \sin(x)e^x - qx^2 \sin(x)e^x + 2qx \cos(x)e^x + qx^2 \cos(x)e^x \\ & +rx \cos(x)e^x + r \sin(x)e^x + rx \sin(x)e^x + s \cos(x)e^x - sx \sin(x)e^x + sx \cos(x)e^x \\ & +t \sin(x)e^x + t \cos(x)e^x + u \cos(x)e^x - u \sin(x)e^x \end{aligned}$$

Setting this equal to the integrand, term by term, we obtain

$$\begin{array}{rcll} p-q & = & 1 & [x^2\sin(x)e^x] \\ p+q & = & 0 & [x^2\cos(x)e^x] \\ 2p+r-s & = & 0 & [x\sin(x)e^x] \\ 2q+r+s & = & 0 & [x\cos(x)e^x] \\ r+t-u & = & 0 & [\sin(x)e^x] \\ s+t+u & = & 0 & [\cos(x)e^x] \end{array}$$

Solving these in pairs, in the order shown, we get $p=\frac{1}{2}$, $q=-\frac{1}{2}$; $r=0$, $s=1$; and $t=u=-\frac{1}{2}$, and the solution

$$F(x)=\frac{1}{2}\left((x^2-1)\sin(x)+(-x^2+2x-1)\cos(x)\right)e^x+C \quad .$$

We note that the unnecessary term $rx\sin(x)e^x$ of the *Ansatz* was eliminated without trouble. The system of equations was block-upper-triangular, never requiring more than two unknowns to be dealt with at once.

Discussion

It should be noted that the undetermined coefficient method is not the same as "tabular integration by parts", a streamlined way of generating the various terms without writing down the intermediate integrals. The tabular method is described in [1] and [3]; the only current textbooks that I have seen which teach it are various editions of *Thomas' Calculus* [4]) The tabular method is optimized for speed, in which regard it is probably unsurpassed; but as with other "stripped-down" algorithms (such as synthetic division of polynomials) a considerable penalty is paid in transparency. The undetermined-coefficient method, in contrast, manages a moderate improvement in speed in many problems involving repetitive or circular integration by parts, while improving the clarity of the integration.

The method does have limitations. It does not give much help in dealing with a completely new family of integrals for which the basic pattern is unknown, and certainly cannot replace integration by parts as a general technique. Also, in some problems, such as integrals of the form $\int x^n \sin(ax)dx$, it is not as fast as integration by parts unless the initial *Ansatz* is extremely selective.

However, it has the advantage of combining a minimum of calculus with familiar algebra to get a nontrivial result. As the calculus used is entirely differentiation, it illustrates the fundamental theorem of calculus nicely. It also encourages students to find and think about patterns in integrals, and to see connections between different parts of the undergraduate mathematical curriculum.

Normally this method would be introduced immediately after integration by parts. I have done this in a first-year calculus class, with some success. Students were aware that this method was not in the textbook; but they were given a handout, upon which this note is loosely based. Some students continued to prefer integration by parts; others quickly came to prefer the method of undetermined coefficients. Students who were taking the course for a second time commented spontaneously on the similarity of this method to the use of undetermined coefficients in finding partial fraction expansions. It seemed that this early exposure to undetermined coefficients made the partial fraction method easier to learn.

In a fairly radical "reform" course, in which the instructor's input is kept to a minimum, integration by undetermined coefficients could play an even more important part. As it is based solidly on differentiation, it could be introduced immediately after the fundamental theorem of calculus, and used to find a number of nontrivial integrals. Moreover, it could probably be used to discover the rule for integrating $f(ax)$ where the antiderivative of $f(x)$ is known, and perhaps even integration by parts; however, I have not tested it in such a context.

References

[1] L. Gillman, More on Tabular Integration by Parts *College Math Journal* **22** (1991) 407-410

[2] L. S. Grinstein, Integration by Undetermined Coefficients *College Math Journal* **2** (1971) 98-101

[3] D. Horowitz, Tabular Integration by Parts, *College Math Journal* **21** (1990) 307-311

[4] D. W. Weir *et al.*, *Thomas' Calculus (Early Transcendentals)*, 11th edition, 2006, Pearson Education Inc., Boston