ADJOINING ADJOINTS

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ABSTRACT. We construct the 2-category obtained from a category by freely adjoining a right adjoint for each morphism and isolate its universal property. Some others basic properties are also studied. Some examples in which the category is freely generated by a graph are discussed in detail. For these categories, the 2-cells are given a geometric interpretation and shown to be similar to certain diagrams which have appeared in the literature on C^*-algebras.

Introduction

The usefulness of Span in bicategory theory is well documented. For a category $\mathcal{C}$ with pullbacks, the bicategory $\mathrm{Span}(\mathcal{C})$ is characterized by the property that it is the result of freely adding right adjoints to the arrows in $\mathcal{C}$ subject to the Beck condition (cf. [4]). In this paper we describe a 2-category $\Pi_2\mathcal{C}$, which is the result of freely adding right adjoints to all arrows in $\mathcal{C}$, for any category $\mathcal{C}$. This Π_2-construction can be viewed as an extension of Span in the sense that one doesn't need the presence of pullbacks in $\mathcal{C}$ to apply it. For categories with pullbacks the two constructions will yield different results, but there are interesting relations, which are spelled out in [4].

One example of a 2-category of the form $\Pi_2\mathcal{C}$ is found in the paper [11], where the category $\mathbf{Adj} = \Pi_2\mathbb{2}$ is described in full detail (although the primary goal of the authors was to study monads).

The 2-category $\Pi_2\mathcal{C}$ is a localization of the category $\mathcal{C}$, and in many ways it resembles the hammock localization as defined by Dwyer and Kan [5], in the sense that the 2-cells in $\Pi_2\mathcal{C}$ correspond to connected components in the 2-cell category of the hammock localization. The details of the relationship between $\Pi_2\mathcal{C}$ and some of the other localizations of a category will be the subject of another paper.

The 2-cells of $\Pi_2\mathcal{C}$ are defined as equivalence classes of a type of diagram which we call fences. It is not unusual for there to be some kind of equivalence relation involved in the definition of a localization, especially for the top dimensional cells involved; see for example the category of fractions as defined in [6] or the bicategory of fractions as defined in [9]. However, the equivalence relations in those localizations are all decidable if there is enough information about the original category. This is not the case for $\Pi_2\mathcal{C}$ as we have shown in [2] and [3].

We call the result of our construction $\Pi_2\mathcal{C}$, the fundamental 2-category of $\mathcal{C}$, because the morphisms are *paths* - zig-zags of arrows - in the original category, and the 2-cells resemble homotopy classes of homotopies between these paths. The full details about the

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correspondence between this category and the homotopy group $\Pi_2(B\mathcal{C})$ of the classifying space will also be the topic of another paper.

The extension of the Π_2-construction to 2-categories and more generally, bicategories will be spelled out in a forthcoming paper. If $\mathcal{C}$ is a 2-category, the resulting $\Pi_2\mathcal{C}$ will again be a 2-category. When $\mathcal{C}$ is only a bicategory, the result will of course also be a bicategory.

We end this paper by discussing the result of applying this Π_2-construction to a category which has been freely generated by a graph. In this case the equivalence relation on the 2-cells is decidable. Moreover, they can be represented graphically by diagrams, related to the Feynman diagrams of particle physics and to the Kaufmann diagrams used in operator theory. In future research the authors plan to investigate the relationship between categories, Feynman diagrams, Kauffman diagrams, and C^*- algebras more closely.

1. The Free Adjoint Construction

1.1. Motivation. Recall that for a pair of arrows $(A \xrightarrow{f} B, B \xrightarrow{g} A)$ in a 2-category $\mathcal{C}$, the arrow f is *left adjoint* to g (and g is called *right adjoint* to f) when there are 2-cells $\eta\colon I_A \Rightarrow gf$ and $\varepsilon\colon fg \Rightarrow I_B$ satisfying the triangle identities, $\varepsilon f \cdot f\eta = i_f$ and $g\varepsilon \cdot \eta g = i_g$. It follows from these identities that (right) adjoints are unique up to canonical 2-isomorphism. So if one wants to add right adjoints to the arrows of a (2-)category in a free way, it is sufficient to add a morphism in the opposite direction for each arrow $A \xrightarrow{f} B$ in $\mathcal{C}$. We will denote this morphism by a backward arrow with the same label:

$$B \xleftarrow{f} A. \tag{1}$$

Thus, one obtains zig-zags of arrows of $\mathcal{C}$, as in any localization of a category. The difference from other localizations will be in the way we define our 2-cells. Before we explain how to construct the 2-cells, we want to consider the shape of the zig-zags more carefully; there are two issues.

First, we want the embedding of the category $\mathcal{C}$ into the 2-category $\Pi_2\mathcal{C}$ to be a 2-functor rather than only a bifunctor. In particular, we want it to preserve identity arrows. This could be done in two ways: one can take isomorphism classes of zig-zags, and require that the arrow $A \xrightarrow{I_A} A$ is equivalent to the empty zig-zag, or one can restrict the shape of the zig-zags. Moreover, for composable arrows $A \xrightarrow{f_1} B \xrightarrow{f_2} C$ and chosen adjoints f_1^*, f_2^*, and $(f_2 f_1)^*$, the arrows $f_1^* f_2^*$ and $(f_2 f_1)^*$ are isomorphic, but not necessarily equal. However, the notation of (1) suggests composability.

We can resolve both of these issues by choosing the morphisms of the 2-category $\Pi_2\mathcal{C}$ to be zig-zags of the form

$$X \xrightarrow{f_0} B_0 \xleftarrow{g_1} A_1 \cdots A_n \xrightarrow{f_n} Y,$$

i.e., beginning and ending with a forward arrow. Such zig-zags are composed by concatenation together with composition in $\mathcal{C}$. For example: $X \xrightarrow{f_0} A \xleftarrow{g} B \xrightarrow{f_1} Y$ composed with

$Y \xrightarrow{p_0} C \xleftarrow{q} D \xrightarrow{p_1} Z$ is $X \xrightarrow{f_0} A \xleftarrow{g} B \xrightarrow{p_0 f_1} C \xleftarrow{q} D \xrightarrow{p_1} Z$. The right adjoint f^* of an arrow $A \xrightarrow{f} B$ then becomes

$$B \xrightarrow{I_B} B \xleftarrow{f} A \xrightarrow{I_A} A.$$

The composition of the adjoints $C \xrightarrow{I_C} C \xleftarrow{f_2} B \xrightarrow{I_B} B$ and $B \xrightarrow{I_B} B \xleftarrow{f_1} A \xrightarrow{I_A} A$ is

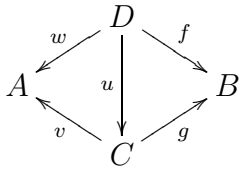
$$C \xrightarrow{I_C} C \xleftarrow{f_2} B \xrightarrow{I_B} B \xleftarrow{f_1} A \xrightarrow{I_A} A, \tag{2}$$

which is not the same as

$$C \xrightarrow{I_C} C \xleftarrow{f_2 f_1} A \xrightarrow{I_A} A. \tag{3}$$

Note that one could make different choices, *e.g.*, take equivalence classes of zig-zags, or let the embedding be only a bifunctor, and the result would be an equivalent 2-category or bicategory $\Pi_2\mathcal{C}$, although the amount of bookkeeping needed to prove this may vary.

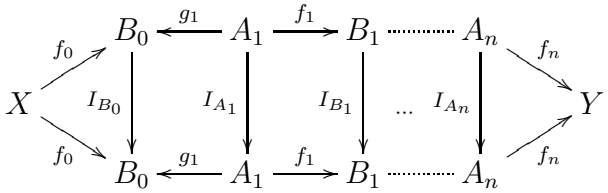
We will now discuss what the 2-cells in the category $\Pi_2\mathcal{C}$ ought to be. Recall that in the calculus of fractions as defined by Gabriel and Zisman [6], two morphisms $A \xleftarrow{w} C \xrightarrow{f} B$ and $A \xleftarrow{v} D \xrightarrow{g} B$ are equivalent if they are part of a diagram



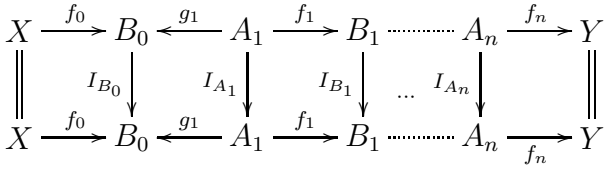
with $u \in W$ (the class of arrows to be inverted). Inverses are a special type of adjoints, with unit and counit cells that are identities; and the 2-cells of our construction will be similar to these diagrams. For example, the identity 2-cell for a zig-zag

$$X \xrightarrow{f_0} B_0 \xleftarrow{g_1} A_1 \cdots A_n \xrightarrow{f_n} Y$$

is of the form



as in Dwyer and Kan's *hammock localization* (*cf.* [5]). However, we prefer to denote this 2-cell by a diagram where we repeat the domain X and the codomain Y on the top and bottom row,



because the tops and bottoms of our (non-identity) hammocks don't need to have the same length, and this notation gives us more space. (In the hammock localization one adds identities as needed to make the top and bottom the same length, because (for instance) (2) and (3) are equivalent arrows. In our construction we have 'forgotten' the identity structure of the original category $\mathcal{C}$, so (2) and (3) are not equivalent.) We will use the $\|$-notation only for the identities at the beginning and end of the fence, and use (*e.g.*) I_A everywhere else. We will call such a diagram a *fence*.

To further motivate our definition for 2-cells, we consider the unit and counit for an adjunction. We need 2-cells from $A \xrightarrow{I_A} A$ to $A \xrightarrow{f} A \xleftarrow{f} B \xrightarrow{I_B} B$ and from $B \xrightarrow{I_B} B \xleftarrow{f} A \xrightarrow{f} B$ to $B \xrightarrow{I_B} B$. These will have the following shapes:

$$\eta = \quad \begin{array}{ccccccc} A & & \xrightarrow{I_A} & & A & & \\ \| & & & \searrow^{I_A} & & \| & \\ A & \xrightarrow[f]{} & B & \xleftarrow[f]{} & A & \xrightarrow[I_A]{} & A \end{array} \qquad \varepsilon = \quad \begin{array}{ccccccc} B & \xrightarrow{I_B} & B & \xleftarrow{f} & A & \xrightarrow{f} & B \\ \| & & & \searrow^{I_B} & & & \| \\ B & & \xrightarrow[I_B]{} & & B & & \end{array} \tag{4}$$

When one performs horizontal composition, one composes the domain and codomain of the fence as before; thereafter some of the vertical arrows in the fence may also need to be composed with arrows in the domains and codomains, so that they don't begin or end in the 'middle' of an arrow in the domain or codomain zig-zag. For example, horizontal composition of the fence for ε in (4) with an arrow $(g)\colon B \to C$ gives:

$$g\varepsilon = \quad \begin{array}{ccccccccc} B & \xrightarrow{I_B} & B & \xleftarrow{f} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \| & & & \searrow^{I_B} & & \swarrow_{I_B} & & & \| \\ B & & \xrightarrow[I_B]{} & & B & \xrightarrow[g]{} & C & & \end{array} \quad = \quad \begin{array}{ccccccc} B & \xrightarrow{I_B} & B & \xleftarrow{f} & A & \xrightarrow{gf} & C \\ \| & & & \searrow^{g} & & & \| \\ B & & \xrightarrow[g]{} & & C & & \end{array}$$

We further illustrate our definition of horizontal and vertical composition for fences by showing that these units and counits satisfy one of the triangle identities, $\varepsilon f \cdot f\eta = i_f$ (the other triangle identity is proved below in Lemma 1). Horizontal composition of the fences for η and ε with f produces the following new fences:

$$\varepsilon f = \quad \begin{array}{ccccccc} A & \xrightarrow{f} & B & \xleftarrow{f} & A & \xrightarrow{f} & B \\ \| & & & \searrow^{I_B} & & & \| \\ A & & \xrightarrow[f]{} & & B & & \end{array} \qquad \text{and } f\eta = \quad \begin{array}{ccccccc} A & & \xrightarrow{f} & & B & & \\ \| & & & \searrow^{I_A} & & & \| \\ A & \xrightarrow[f]{} & B & \xleftarrow[f]{} & A & \xrightarrow[f]{} & B \end{array}$$

For the vertical composition, we write one fence below the other and compose the vertical

and diagonal arrows that are composable:

$$\varepsilon f \cdot f\eta = \quad \begin{array}{ccccccc} A & \xrightarrow{f} & B & & A & \xrightarrow{f} & B \\ \| & \searrow^{I_A} & & & & & \| \\ A & \xrightarrow[f]{} & B & \xleftarrow[f]{} & A & \xrightarrow[f]{} & B \\ \| & & & \searrow_{I_B} & & & \| \\ A & & \xrightarrow[f]{} & & B & & \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & & \| \\ A & \xrightarrow[f]{} & B \end{array} \quad = i_f. \tag{5}$$

We will call the structure consisting of objects, zig-zags, and fences, with horizontal and vertical composition as given, $\Gamma\mathcal{C}$. As the observant reader may have noticed, this does not always satisfy the middle-four interchange law (and hence is not a 2-category); see, for instance, Example 1 below. This should not come as a surprise, given that the hammock localization has non-trivial 3-cells (cf. [5]). The objects and arrows of $\Gamma\mathcal{C}$ are the correct ones; we shall see that the 2-cells of $\Pi_2\mathcal{C}$ are obtained as equivalence classes of those of $\Gamma\mathcal{C}$.

1.2. Definitions. Let $\mathcal{C}$ be any category.

1. The *objects* of $\Gamma\mathcal{C}$ (and of $\Pi_2\mathcal{C}$) are the same as the objects of $\mathcal{C}$.

2. The *arrows* of $\Gamma\mathcal{C}$ (and of $\Pi_2\mathcal{C}$) are zig-zags in $\mathcal{C}$:

$$A = A_0 \xrightarrow{f_0} B_0 \xleftarrow{g_1} A_1 \xrightarrow{f_2} B_1 \longleftarrow \cdots \longrightarrow B_{n-1} \xleftarrow{g_n} A_n \xrightarrow{f_n} B_n = B \tag{6}$$

   We denote such an arrow by

$$(g_1, \cdots, g_n; f_0, \cdots, f_n),$$

   except when $n = 0$, in which case we will write (f_0). Composition of arrows is defined by concatenating the zig-zags and composing the two composable morphisms:

$$\begin{aligned}(g_1, \cdots, g_n; f_0, \cdots, f_n) \circ (h_1, \cdots, h_m; k_0, \cdots, k_n) = \\ (h_1, \cdots, h_m, g_1, \cdots, g_n; k_0, \cdots, k_{n-1}, f_0 k_n, f_1, \cdots, f_n)\end{aligned}$$

3. A *2-cell* $(g_1, \cdots, g_n; f_0, \cdots, f_n) \Rightarrow (g'_1, \cdots, g'_m; f'_0, \cdots, f'_m)$ of $\Gamma\mathcal{C}$ is a particular kind of diagram, called a *fence*. An example of a fence is

$$\begin{array}{ccccccccccccc} A = A_0 & \xrightarrow{f_0} & B_0 & \xleftarrow{g_1} & A_1 & \xrightarrow{f_1} & B_1 & \xleftarrow{g_2} & A_2 & \xrightarrow{f_2} & B_2 = B \\ \| & \searrow^{h_1} & & \searrow^{k_0} & & & \downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle h_2} & \searrow^{h_3} & & \searrow \\ A = A'_1 & \xrightarrow[f'_0]{} & B'_0 & \xleftarrow[g'_1]{} & A'_1 & \xrightarrow[f'_1]{} & B'_1 & \xleftarrow[g'_2]{} & A'_2 & \xrightarrow[f'_2]{} & B'_2 & \xleftarrow[g'_3]{} & A'_3 & \xrightarrow[f'_3]{} & B'_3 = B \end{array}$$

Specifically, a fence is determined by a pair of order-preserving index functions

$$\phi\colon \{0,\cdots,m\} \to \{0,\cdots,n\} \text{ and } \psi\colon \{0,\cdots,n\} \to \{0,\cdots,m\}$$

together with families of arrows (which we will call *vertical*, even when they are slanted in the diagram)

$$k_j\colon B_j \to B'_{\psi(j)} \text{ and } h_i\colon A_{\phi(i)} \to A'_i.$$

These data are subject to the following conditions:

(a) $\varphi \dashv \psi$ (*i.e.*, $\psi\phi(j) \geq j$ and $\phi\psi(i) \leq i$) (note that this means that the vertical arrows don't cross, so that the resulting diagram is planar);

(b) all resulting squares commute, *i.e.*

$$f'_j h_j = \begin{cases} g'_{j+1} h_{j+1} & \text{if } \phi(j) = \phi(j+1) \\ k_{\phi(j)} f_{\phi(j)} & \text{if } \phi(j) \neq \phi(j+1) \end{cases}$$

for $j = 1, \ldots, m$;

$$k_i g_{i+1} = \begin{cases} k_{i+1} f_{i+1} & \text{if } \psi(i) = \psi(i+1) \\ g'_{\psi(i)+1} h_{\psi(i)+1} & \text{if } \psi(i) \neq \psi(i+1) \end{cases}$$

for $i = 1, \ldots, n$;

(c) $h_0 = \mathrm{id}_A$ and $k_n = \mathrm{id}_B$.

We denote such a fence by

$$(\phi; \psi; k_0, \ldots, k_n; h_0, \ldots, h_m) \text{ or } (\phi; \psi; (k_i); (h_j)). \tag{7}$$

Note that in the example above, $\psi(0) = 1$, $\psi(1) = 1$ and $\psi(2) = 3$, and $\phi(0) = 0$, $\phi(1) = 0$, $\phi(2) = 2$, and $\phi(3) = 2$.

We will now describe horizontal and vertical composition of fences.

1.3. Vertical Composition of Fences. When we want to construct the vertical composition of two fences

$$\alpha = (\phi; \psi; (k_i); (h_j))\colon (g_1, \ldots, g_m; f_0, \ldots, f_m) \Rightarrow (g'_1, \ldots, g'_n; f'_0, \ldots, f'_n)$$

and

$$\beta = (\phi'; \psi'; (k'_i); (h'_j))\colon (g'_1, \ldots, g'_n; f'_0, \ldots, f'_n) \Rightarrow (g''_1, \ldots, g''_p; f''_0, \ldots, f''_p),$$

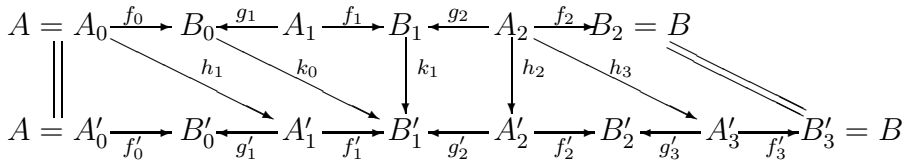
the resulting fence needs to contain a vertical arrow starting at each source B_i. The fence α provides a vertical arrow $k_i\colon B_i \to B'_{\psi(i)}$. Then the fence β contains an arrow $k'_{\psi(i)}\colon B_{\psi(i)} \to B''_{\psi'\psi(i)}$, so the required vertical arrow in the vertical composition of the fences can be obtained as

$$k'_{\psi(i)} k_i\colon B_i \to B''_{\psi'\psi(i)}.$$

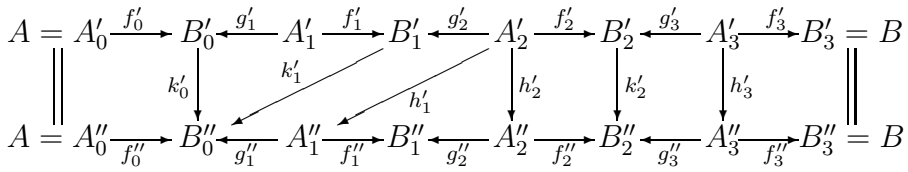
Analogously we can construct the vertical arrows into the A''_j by backtracking in two steps

$$h'_j h_{\phi'(j)} \colon A_{\phi\phi'(j)} \to A_j.$$

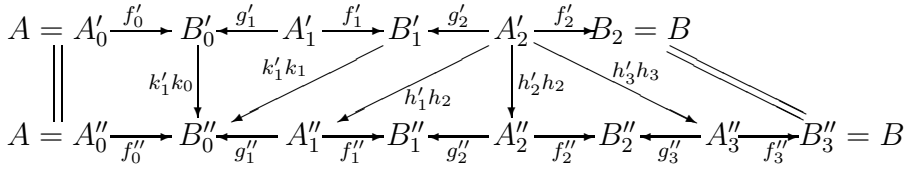
For example, composing



with



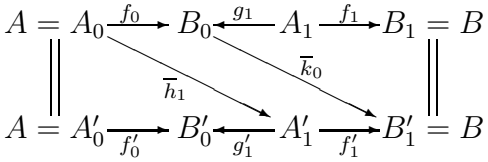
gives



In general, the index functions of the new 2-cell are the compositions of the index functions of the original 2-cells, and the vertical arrows in the fences are the compositions of the corresponding vertical arrows:

$$(\phi'; \psi'; (k'_i); (h'_j)) \cdot (\phi; \psi; (k_i); (h_j)) = (\phi\phi'; \psi'\psi; (k_{\psi(i)} k_i); (h_j h'_{\phi'(j)})).$$

1.4. Horizontal Composition of Fences. This composition is defined as an extension of the composition of zig-zags, *i.e.*, by concatenation, together with some compositions in the category $\mathcal{C}$. This is a tensor-like construction over $\mathcal{C}$. For example, when the first fence ends with the object B_n and there is a vertical arrow $k_i \colon B_i \to B_n$, where $i \neq n$, this arrow needs to be composed with the first forward arrow of the codomain zig-zag of the second fence, as in the horizontal composition of



with

$$\begin{array}{ccccccccccc} B & = & E_0 & \xrightarrow{p_0} & F_0 & \xleftarrow{q_1} & E_1 & \xrightarrow{p_1} & F_1 & = & C \\ \| & & & \searrow^{\tilde{h}_1} & & & & \swarrow_{\tilde{k}_0} & & & \| \\ B & = & E'_0 & \xrightarrow[p'_0]{} & F'_0 & \xleftarrow[q'_1]{} & E'_1 & \xrightarrow[p'_1]{} & F'_1 & = & C \end{array}$$

gives

$$\begin{array}{ccccccccccccc} A & = & A_0 & \xrightarrow{f_0} & B_0 & \xleftarrow{g_1} & A_1 & \xrightarrow{p_0 f_1} & F_0 & \xleftarrow{q_1} & E_1 & \xrightarrow{p_1} & F_1 & = & C \\ \| & & & \searrow^{\overline{h}_1} & & \searrow^{p'_0\overline{k}_0} & & \searrow^{\tilde{h}_1 f_1} & & \searrow^{\tilde{k}_0} & & & & & \| \\ A & = & A'_0 & \xrightarrow[f'_0]{} & B'_0 & \xleftarrow[g'_1]{} & A'_1 & \xrightarrow[p'_0 f'_1]{} & F'_0 & \xleftarrow[q'_1]{} & E'_1 & \xrightarrow[p'_1]{} & F'_1 & = & C. \end{array}$$

Unfortunately, the bookkeeping for the formal description of this process is a bit complicated. (This is the place where we have to pay for our choices of morphisms in $\Pi_2\mathcal{C}$.) In general, if

$$\begin{aligned} &[\overline{\phi}; \overline{\psi}; (\overline{k}_i); (\overline{h}_j)]: \\ &\quad (g_1, \ldots, g_{n_1}; f_0, \ldots, f_{n_1}) \Rightarrow (g'_1, \ldots, g'_{m_1}; f'_0, \ldots, f'_{m_1}) \end{aligned}$$

and

$$\begin{aligned} &[\tilde{\phi}; \tilde{\psi}; (\tilde{k}_i); (\tilde{h}_j)]: \\ &\quad (q_1, \ldots, q_{n_2}; p_0, \ldots, p_{n_2}) \Rightarrow (q'_1, \ldots, q'_{m_2}; p'_1, \ldots, p'_{m_2}) \end{aligned}$$

are 2-cells such that $\mathrm{cod}(f_{n_1}) = \mathrm{dom}(p_0)$ (*i.e.*, that are composable), then the horizontal composition is defined as $[\tilde{\phi}; \tilde{\psi}; (\tilde{k}_i); (\tilde{h}_i)][\overline{\phi}; \overline{\psi}; (\overline{k}_i); (\overline{h}_i)] = [\phi; \psi; (k_i); (h_j)]$, where the index functions $\psi\colon \{0, \ldots, n_1 + n_2\} \rightleftarrows \{0, \ldots, m_1 + m_2\} : \phi$ are defined by

$$\begin{aligned} \phi(j) &= \begin{cases} \overline{\phi}(j) & \text{for } \ j \le m_1 \\ \tilde{\phi}(j - m_1) + n_1 & \text{for } \ j > m_1 \end{cases} \\ \psi(i) &= \begin{cases} \overline{\psi}(i) & \text{for } \ i < n_1 \\ \tilde{\psi}(i - n_1) + m_1 & \text{for } \ j \ge n_1, \end{cases} \end{aligned}$$

and the vertical arrows are defined by

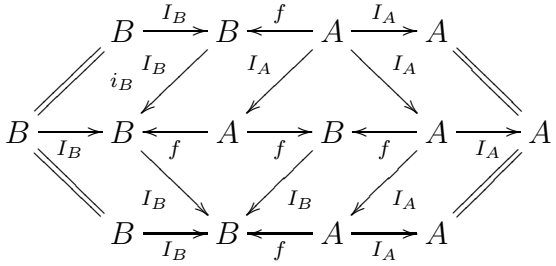
$$\begin{aligned} k_i &= \begin{cases} \overline{k}_i & \text{if } \ i < n_1 \text{ and } \overline{\psi}(i) < m_1 \\ v'_1 \overline{k}_i & \text{if } \ i < n_1 \text{ and } \overline{\psi}(i) = m_1 \\ \tilde{k}_{i - n_1} & \text{if } \ i \ge n_1 \end{cases} \\ h_j &= \begin{cases} \overline{h}_j & \text{if } \ j \le m_1 \\ \tilde{h}_{j - m_1} f_{n_1} & \text{if } \ j > m_1 \text{ and } \tilde{\phi}(j - m_1) = 0 \\ \tilde{h}_{j - m_1} & \text{if } \ j > m_1 \text{ and } \tilde{\phi}(j - m_1) > 0. \end{cases} \end{aligned}$$

Although $\Gamma\mathcal{C}$ is not a 2-category, it should be noted that the triangle identities $\varepsilon f^* \cdot f^* \eta = i_{f^*}$ and $f\varepsilon \cdot \eta f = i_f$ are unambiguous, even in the absence of the middle-four interchange law. Therefore, we can define adjunctions in the usual way (except that adjoints may not be unique).

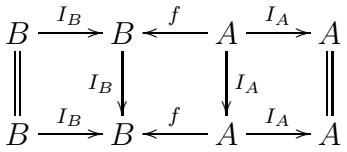
Lemma 1 *In $\Gamma\mathcal{C}$, every arrow of the form $(f)\colon A \to B$ is left adjoint to $f^* = (I_B, I_A; f)\colon B \to A$.*

Proof. The adjunction 2-cells ε and η are given in (4). It was shown in (5) that $f\varepsilon\cdot\eta f = i_f$. Moreover,

$$f^*\varepsilon\cdot\eta f^* \quad = \quad$$



$$= \quad$$



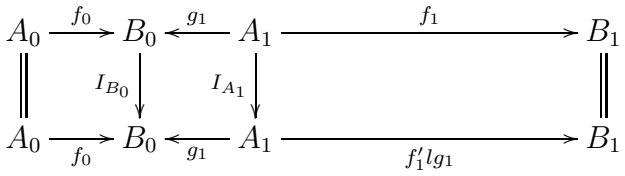
$$= i_{f^*}.$$

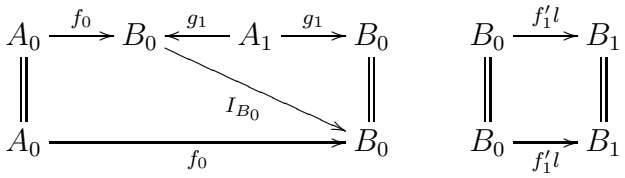
■

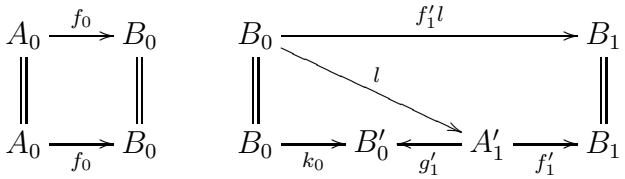
1.5. The Equivalence Relation. The structure $\Gamma\mathcal{C}$ does not satisfy the middle-four interchange law for 2-cells:

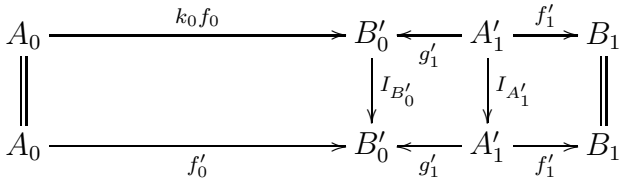
Example 1

Consider the following fences:

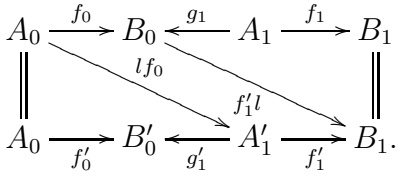






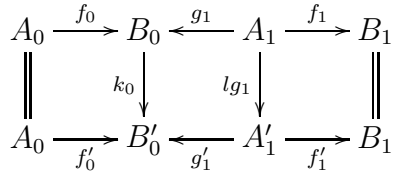


When one first takes all the horizontal compositions, and then the vertical compositions, the result is



(8)

When one first takes the vertical compositions of the four 2-cells in the center, and then the horizontal compositions of the resulting 2-cells and finally the vertical compositions with the top and bottom 2-cells, the result is:



(9)

This example shows that middle-four interchange law is not satisfied, but it motivates the equivalence relation defined below, in which we make two fences equivalent if they are related as (8) and (9) are. In the next two sections we will show that this is functorial, and that it is the smallest equivalence relation that turns $\Gamma\mathcal{C}$ into a 2-category.

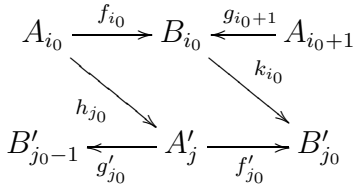
Definition 1 *Two fences*

$$\theta = (\psi; \phi; k_0, \ldots, k_n; h_0, \ldots, h_m)$$

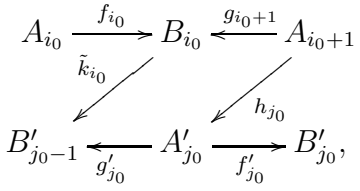
and

$$\omega = (\tilde{\psi}; \tilde{\phi}; \tilde{k}_0, \ldots, \tilde{k}_n; \tilde{h}_0, \ldots, \tilde{h}_m)$$

are directly equivalent *if there exist indices $i_0 \in \{0, \ldots, m\}$ and $j_0 \in \{0, \ldots, n\}$ such that $\psi(i_0) = j_0$, $\tilde{\psi}(i_0) = j_0 - 1$ and $\tilde{\psi}(i) = \psi(i)$ for all other values of i, and $\tilde{\phi}(j_0) = i_0 + 1$ and $\tilde{\phi}(j) = \phi(j)$ for all other values of j. Furthermore, $k_i = \tilde{k}_i$ whenever $i \neq i_0$, $h_j = \tilde{h}_j$ whenever $j \neq j_0$. This means that the fences θ and ω are the same except the cell θ contains*



where the cell ω contains



and there exists an arrow $l\colon B_{i_0} \to A'_{j_0}$ which factors both diagrams.

Note that $lf_{i_0} = h_{j_0}$, $f'_{j_0} l = k_{i_0}$, $g'_{j_0} l = \tilde{k}_{i_0}$, and $lg_{i_0+1} = h_{j_0}$.
Also, the 'direct equivalence' relation is not transitive (or reflexive).

Definition 2 *The relation '$\sim$' is the equivalence relation generated by the 'direct equivalence' relation. The* 2-cells *of* $\Pi_2\mathcal{C}$ *are equivalence classes of fences under this relation.*

It will be shown below that horizontal and vertical composition are well-defined with respect to this equivalence relation. However, as was shown in [2] and [3], for some categories $\mathcal{C}$, the problem of determining whether two fences are equivalent is undecidable. In some cases – for instance, when $\mathcal{C}$ is locally finite, or obeys certain cancellation conditions – the problem is decidable; however, there are many familiar categories, such as **Set**, **Top**, and **Grp**, for which it is undecidable.

2. Composition of 2-Cells

In this section we will show that the horizontal and vertical composition of fences is well-defined on equivalence classes of fences as defined in Definitions 1 and 2. We will also show that the class of 2-cells for $\Pi_2\mathcal{C}$ is minimal in the sense that it is generated by the units, counits, and identities.

Lemma 2 *Horizontal composition of 2-cells is well-defined on equivalence classes.*

Proof. If two 2-cells α and β are directly equivalent via a factorization l, this same arrow l will also factorize the appropriate squares after horizontal composition with a 2-cell γ, making the resulting composites directly equivalent. ∎

Lemma 3 *Vertical composition of 2-cells is well-defined on equivalence classes.*

Proof. Let

$$\omega = (\psi^\omega; \phi^\omega; (k_i^\omega); (h_j^\omega))$$

and

$$\theta = (\psi^\theta; \phi^\theta; (k_i^\theta); (h_j^\theta))$$

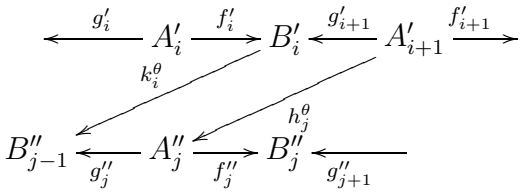
be two representatives for 2-cells with a common domain and codomain.

To prove that the vertical composition is well-defined on 2-cells, it is sufficient to prove that, if ω is directly equivalent to θ as in Definition 1, then, for any two fences γ_1 and γ_2, $\omega\gamma_1 \sim \theta\gamma_1$ and $\gamma_2\omega \sim \gamma_2\beta$ whenever these vertical composites are defined.

Suppose that ω contains

$$\begin{array}{ccccccccc}
 & \xleftarrow{g'_i} & A'_i & \xrightarrow{f'_i} & B'_i & \xleftarrow{g'_{i+1}} & A'_{i+1} & \xrightarrow{f'_{i+1}} & \\
 & & \downarrow{\scriptstyle h_j^\omega} & & \downarrow{\scriptstyle k_i^\omega} & & & & \\
B''_{j-1} & \xleftarrow[g''_j]{} & A''_j & \xrightarrow[f''_j]{} & B''_j & \xleftarrow[g''_{j+1}]{} & & &
\end{array}$$

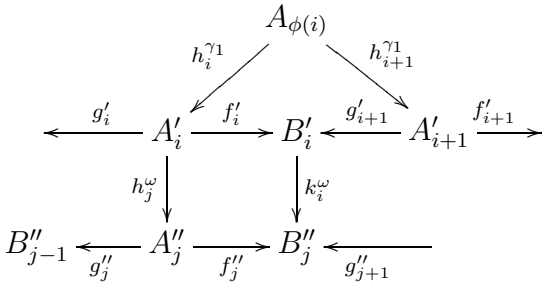
where θ contains



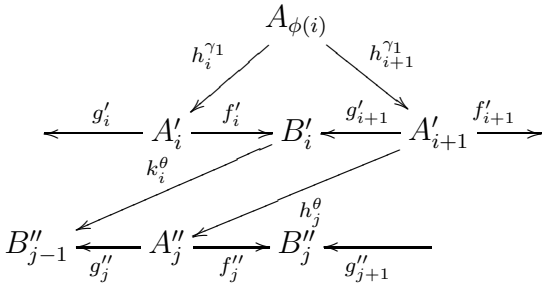
and there is a factorization $l\colon B'_i \to A''_j$ such that $lf'_i = h^\omega_j$, $k^\omega_i = f''_j l$, $h^\theta_j = lg'_{i+1}$, and $g''_j l = k^\theta_i$ as in Section 1.5.

Let $\gamma_1 = (\psi; \phi; (k^\gamma); (h^\gamma))$ and let n be such that $\phi(i+1) = \phi(i) + n$.

If $n = 0$, the relevant parts of the composing fences are:

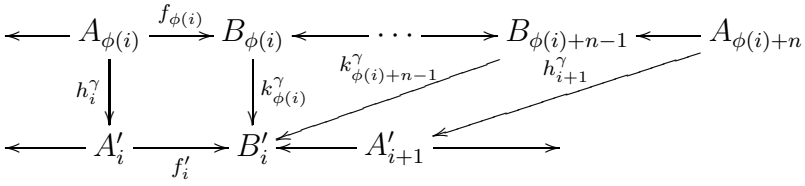


and



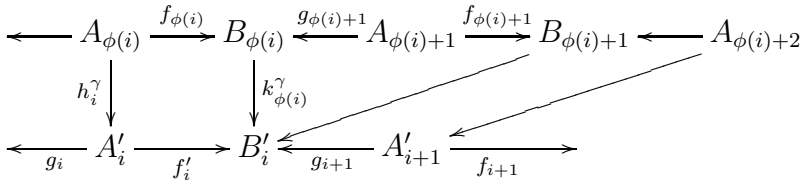
So the resulting indexing functions for the composites $\gamma_1\omega$ and $\gamma_1\beta$ will be the same and the only arrows in the families of indexed arrows for which we need to check equality are $h^{\gamma_1\omega}_j$ and $h^{\gamma_1\theta}_j$. We find that $h^{\gamma_1\theta}_j = h^\theta_j h^{\gamma_1}_{i+1} = lg'_{i+1}h^{\gamma_1}_{i+1} = lf'_i h^{\gamma_1}_i = h^\omega_j h^{\gamma_1}_i = h^{\gamma_1\omega}_j$.

For $n \geq 1$, the part of γ_1 that composes with these parts is of the following form:

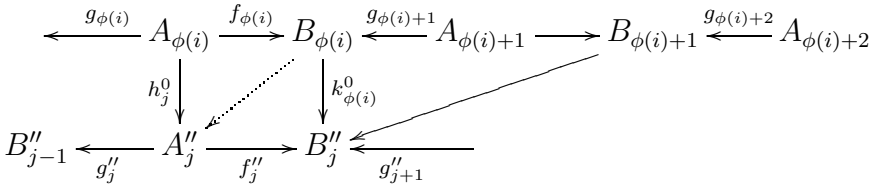


In order to keep the diagrams manageable, we will work out the case where $n = 2$; the general case is a straightforward generalization. For $n = 2$, the part of γ_1 we are working

with is of the form



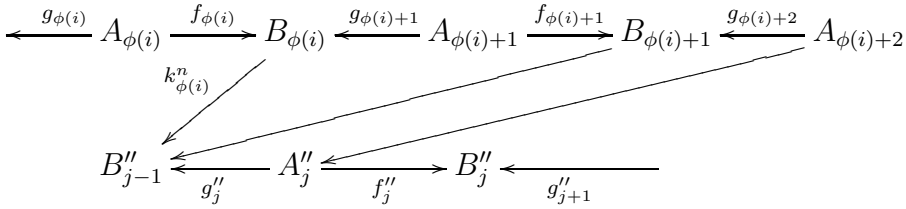
The relevant part of $\omega\gamma_1$ is



(10)

where $h^0_j = h^\omega_j h^\gamma_i$, and $k^0_{\phi(i)} = k^\omega_i k^\gamma_{\phi(i)}$.

The relevant part of $\theta\gamma_1$ is



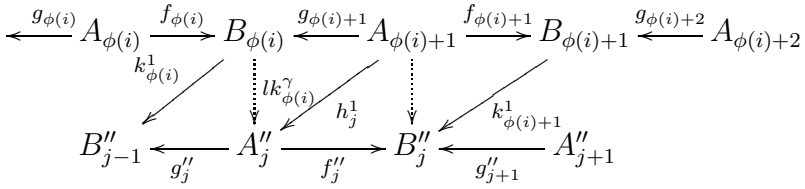
(11)

where $k^2_{\phi(i)} = k^\theta_i k^\gamma_{\phi(i)}$, $k^n_{\phi(i)+1} = k^\theta_i k^\gamma_{\phi(i)+1}$, and $h^n_j = h^\theta_j h^\gamma_{i+1}$.

In general, it will take n direct equivalences to show that $\omega\gamma_1$ and $\theta\gamma_1$ are equivalent. For $n = 2$, first notice that the arrow

$$lk^\gamma_{\phi(i)} \colon B_{\phi(i)} \to A''_j$$

factors the square in $\omega\gamma_1$. This makes $\omega\gamma_1$ directly equivalent to the fence where part (10) is replaced with



(12)

and

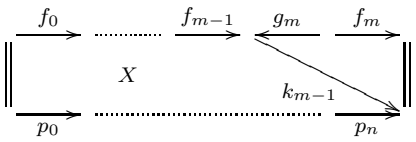
$$\begin{aligned} k^1_{\phi(i)} &= k^\theta_i k^\gamma_{\phi(i)}, \\ k^1_{\phi(i)+1} &= k^\omega_i k^\gamma_{\phi(i)+1}, \\ h^1_j &= lk^\gamma_{\phi(i)} g_{\phi(i)+1}. \end{aligned}$$

Now the arrow $lk_{\phi(i)+1}\colon B_{\phi(i)+1} \to A''_j$ factors both the second square in diagram (12) and the square in $\theta\gamma_1$, so $\theta\gamma_1$ is directly equivalent to this last fence, and therefore equivalent to $\omega\gamma_1$.

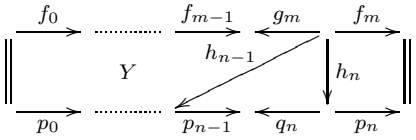
The proof that $\gamma_2\omega \sim \gamma_2\theta$ is similar to this one, but the focus is on the B's, rather than the A's. ∎

Proposition 1 *The 2-cells in $\Pi_2\mathcal{C}$ are generated, through horizontal and vertical composition, by the identity 2-cells i_f and i_{f^*}, the units η_f , and the counits ε_f, for $f \in \mathcal{C}_1$ (as in (4)).*

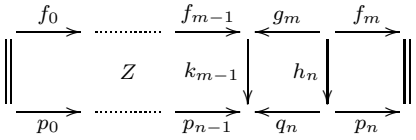
Proof. We will prove this by induction on the length of the fence. The shortest fences are the ones with one right arrow on the top and the bottom. These are identity 2-cells and hence satisfy the requirements. All other fences are of one of the following forms:



(13)

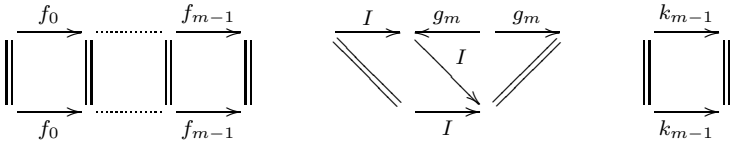


(14)

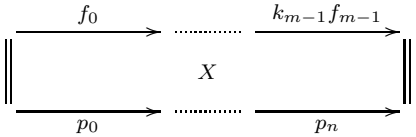


(15)

A diagram of the form (13) can be written as the horizontal composition of the following cells

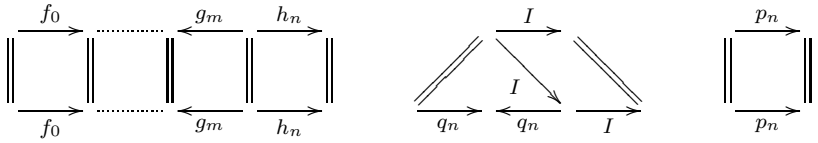


composed vertically with



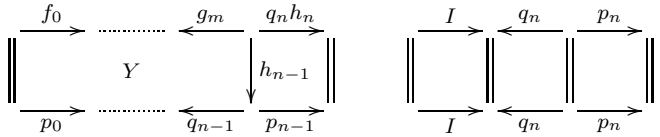
This last fence is shorter than the one in (13), so it can be decomposed as required by the induction hypothesis. A fence of the form (14) can be written as the horizontal

composition of the following cells

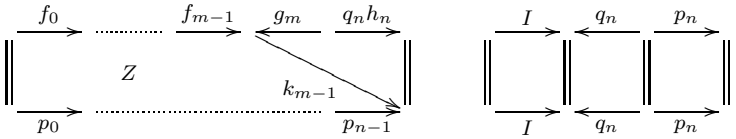


(16)

composed vertically with the horizontal composition of the following cells



Note that the left-hand fence here is again shorter than the original one, so we may apply the induction hypothesis. A fence of the form (15) can be written as the horizontal composition of the fences in (16), composed vertically with the horizontal composition of the following fences:



Once more, the induction hypothesis can be applied to the left-hand fence. ∎

Note that the proof of this proposition effectively constitutes an algorithm for factoring 2-cells into units, counits, and identities. Equivalent fences will give rise to the same sets of units and counits, but the identity cells may differ.

3. The Middle-Four Interchange Law

In this section we finish our proof that $\Pi_2\mathcal{C}$ is a 2-category by showing that it satisfies the interchange law for horizontal and vertical composition. We will also show that the middle-four interchange law forces the equivalence relation $\sim$ on fences.

Theorem 1 *The horizontal and vertical composition for equivalence classes of fences in $\Pi_2\mathcal{C}$, as defined on representatives in Sections 1 and 2, satisfy the middle four interchange law.*

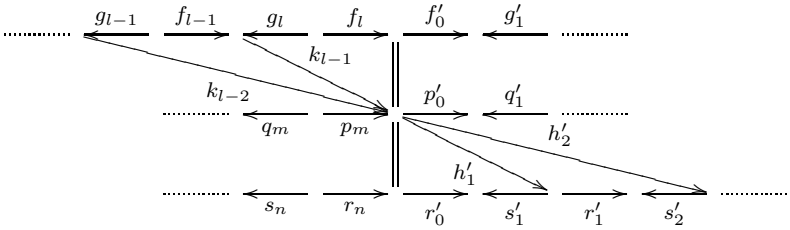
Proof. Suppose that

$$\begin{aligned}
\alpha = [\phi;\psi;(k_i);(h_j)] &: (g_1,\ldots,g_l;f_0,\ldots,f_l) \Rightarrow (q_1,\ldots,q_m;p_0,\ldots,p_m),\\
\beta = [\phi^\dagger;\psi^\dagger;(k_i^\dagger);(h_j'^\dagger)] &: (g_1^\dagger,\ldots,g_L^\dagger;f_0^\dagger,\ldots,f_L^\dagger) \Rightarrow (q_1',\ldots,q_M';p_0',\ldots,p_M'),\\
\gamma = [\phi^\ddagger;\psi^\ddagger;(k_i^\ddagger);(h_j^\ddagger)])] &: (q_1,\ldots,q_m;p_0,\ldots,p_m) \Rightarrow (s_1^\ddagger,\ldots,s_n^\ddagger;r_0^\ddagger,\ldots,r_n^\ddagger),\\
\delta = [\phi';\psi';(k_i');(h_j')])] &: (q_1',\ldots,q_M';p_0',\ldots,p_M') \Rightarrow (s_1',\ldots,s_N';r_0',\ldots,r_N'),
\end{aligned}$$

and $\mathrm{cod}(p_m) = \mathrm{dom}(p'_0)$, so that $(\delta \circ \gamma) \cdot (\beta \circ \alpha)$ and $(\delta \cdot \beta) \circ (\gamma \cdot \alpha)$ are well-defined in $\Pi_2\mathcal{C}$. The index function ψ must map l to m; and ϕ' must map 0 to 0 (*cf.* Section 1.2). The proof of the following lemma, which we will need below, is straightforward.

Lemma 4 *If $\psi^{-1}(m)$ or $\phi'^{-1}(0)$ contain no other elements, then the index functions of $(\delta \circ \gamma) \cdot (\beta \circ \alpha)$ and $(\delta \cdot \beta) \circ (\gamma \cdot \alpha)$ are the same and middle four interchange holds for the fences themselves.*

If the conditions of Lemma 4 are not satisfied, $\psi(l-1) = m$ and $\phi'(1) = 0$, and the relevant part of the pasting diagram is as follows.

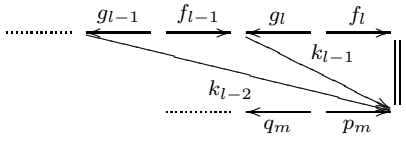


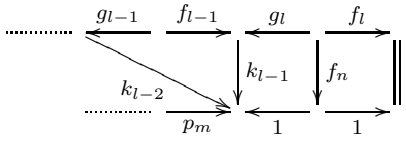
(17)

We can factorize α as

$$\alpha = \tilde{\varepsilon} \cdot a \tag{18}$$

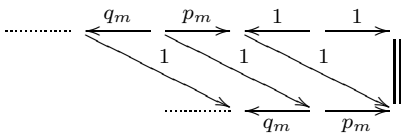
where a is obtained from α by the substitution

$\alpha =$ 

$\mapsto$ 

$= a$

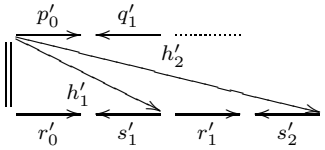
while $\tilde{\varepsilon} = \varepsilon_1 \circ i_{(q_1,\ldots,q_m;p_0,\ldots,p_m)}$, *i.e.*,

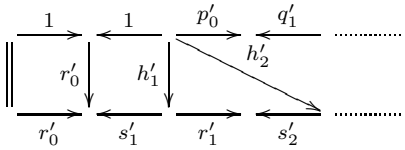
$\tilde{\varepsilon} =$ 

Analogously, we can factorize δ as

$$\delta = d \cdot \tilde{\eta} \tag{19}$$

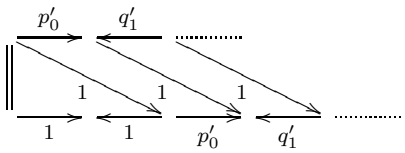
where d is obtained from δ by the substitution

$\delta =$ 

$\mapsto$ 

$= d$

and

$\tilde{\eta} =$ 

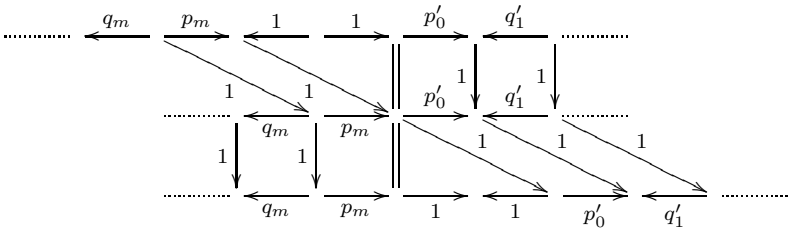
In the rest of this proof we use the following notation for horizontal and vertical composition of 2-cells:

$$\frac{\alpha \quad \beta}{\gamma \quad \delta} = (\delta \circ \gamma) \cdot (\beta \circ \alpha) \quad \text{and} \quad \left.\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}\right| = (\delta \cdot \beta) \circ (\gamma \cdot \alpha)$$

(blocks of 2-cells separated by lines should be composed first, as if parenthesized). We first prove that the following instance of middle-four interchange holds.

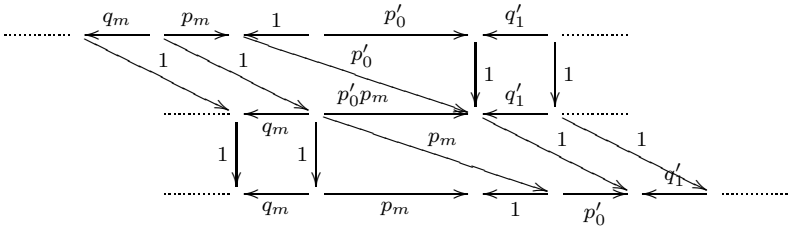
$$\left.\begin{matrix} \tilde{\varepsilon} & I \\ I & \tilde{\eta} \end{matrix}\right| \quad = \quad \frac{\tilde{\varepsilon} \quad I}{I \quad \tilde{\eta}} \tag{20}$$

This is the pasting of the following fences:

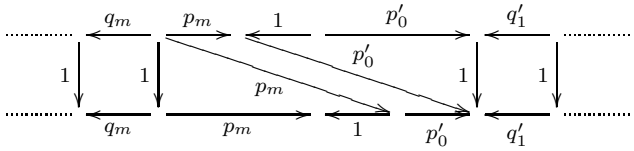


(21)

Performing the horizontal compositions first gives:

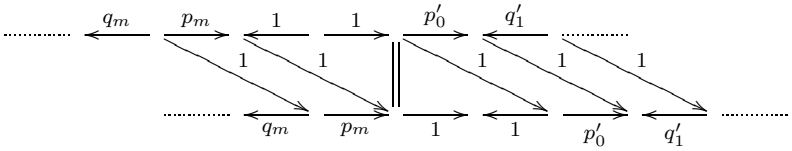


and then by vertical composition, we obtain

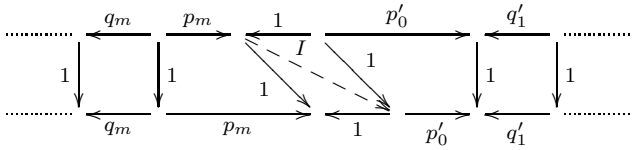


(22)

Composing the fences in diagram (21) in the vertical direction first gives:



which after horizontal composition yields:



(23)

The fences (22) and (23) are directly equivalent via the identity arrow labelled I in (23). We are now able to prove the general case of the middle-four interchange law:

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array} = \begin{array}{|c|c|}\hline a & \beta \\ \hline \tilde{\varepsilon} & \mathrm{id} \\ \hline\hline \mathrm{id} & \tilde{\eta} \\ \hline \gamma & d \\ \hline \end{array} \qquad \text{by (18) and (19);}$$

$$= \begin{array}{|c|c|}\hline a & \beta \\ \hline \tilde{\varepsilon} & \mathrm{id} \\ \hline\hline \mathrm{id} & \eta \\ \hline \gamma & d \\ \hline \end{array} \qquad \text{by Lemma 4;}$$

$$= \begin{array}{|c|c|}\hline a & \beta \\ \hline\hline \tilde{\varepsilon} & \mathrm{id} \\ \hline \mathrm{id} & \tilde{\eta} \\ \hline\hline \gamma & d \\ \hline \end{array} \qquad \text{by (20);}$$

$$= \begin{array}{|c|c|}\hline a & \beta \\ \hline \tilde{\varepsilon} & \mathrm{id} \\ \hline \mathrm{id} & \tilde{\eta} \\ \hline\hline \gamma & d \\ \hline \end{array}$$

$$= \begin{array}{|c|c|}\hline a & \beta \\ \hline \tilde{\varepsilon} & \mathrm{id} \\ \hline \mathrm{id} & \tilde{\eta} \\ \hline \gamma & d \\ \hline \end{array} \qquad \text{both by Lemma 4;}$$

$$= \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array} \qquad \text{by (18) and (19),}$$

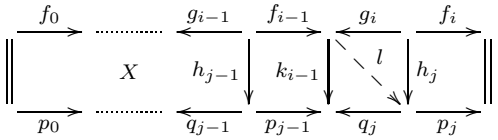
completing the proof of the theorem. ∎

The rather brief appearance of the equivalence relation in the previous proof might lead us to wonder whether its full strength is needed; the next theorem shows that it is.

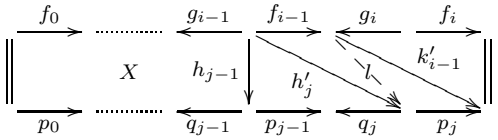
Theorem 2 *The equivalence relation given in Definitions 1 and 2 is the minimal relation such that horizontal and vertical composition of equivalence classes of fences satisfies the middle-four interchange law.*

Proof. We need to show that if any two fences are directly equivalent, there is an instance of the middle-four interchange law which implies that they should be equal. Because of

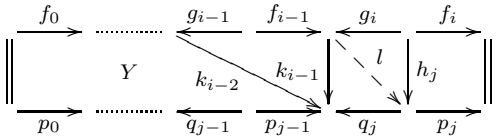
the induction argument in the proof of the previous proposition, it is sufficient to check the following two cases of the equivalence relation. In case 1, θ is of the form:



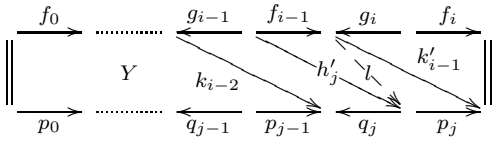
and ω is of the form



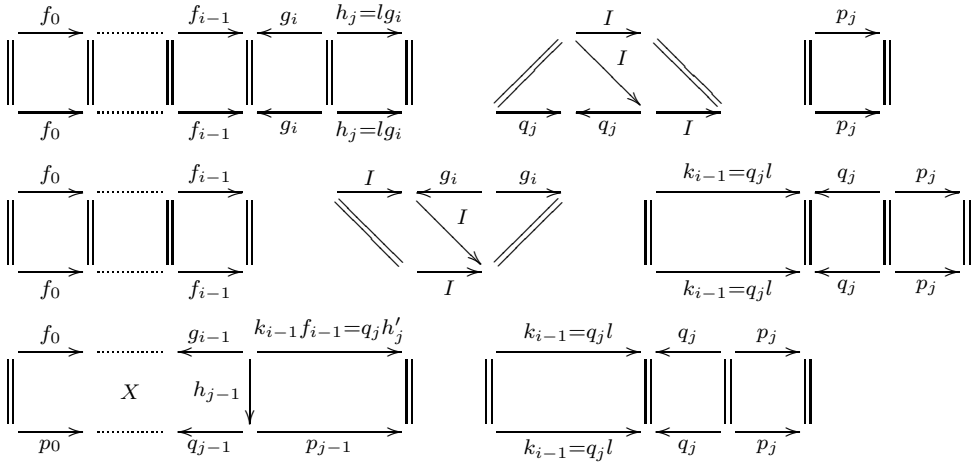
In case 2, θ is of the form



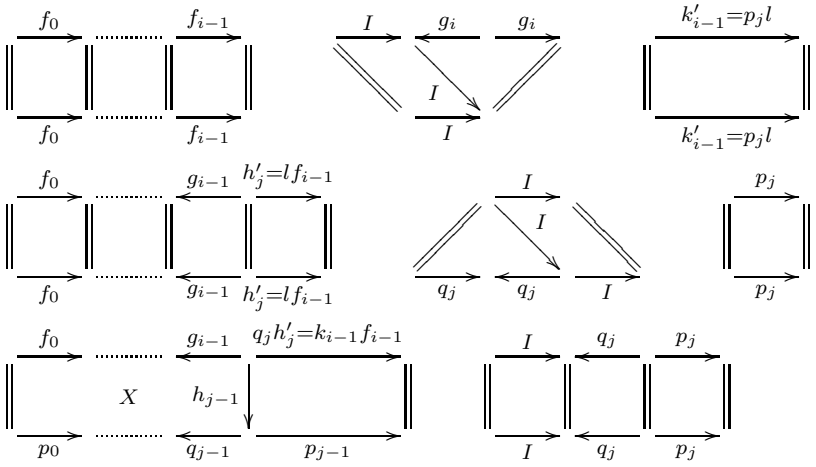
and ω is of the form



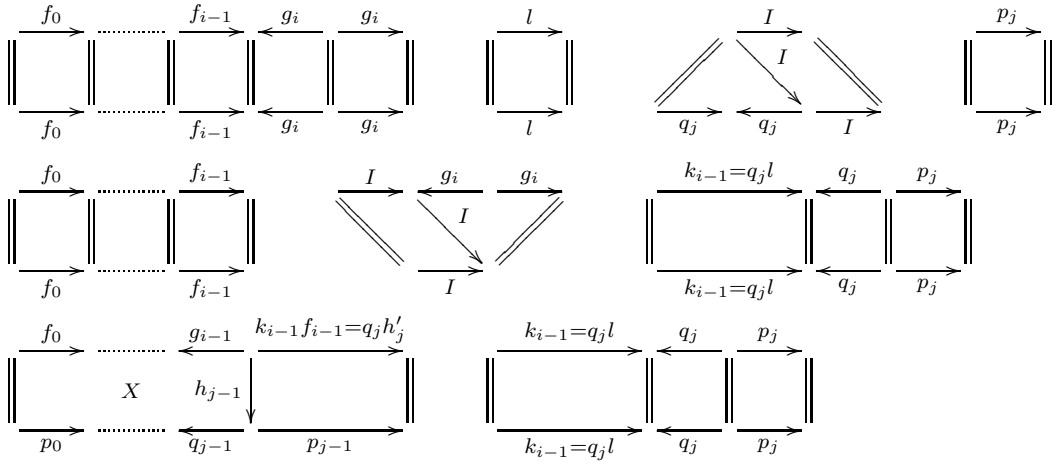
We will do case 1 and leave case 2 to the reader. The algorithm of the proof of Proposition 1 decomposes θ as a pasting of the following fences



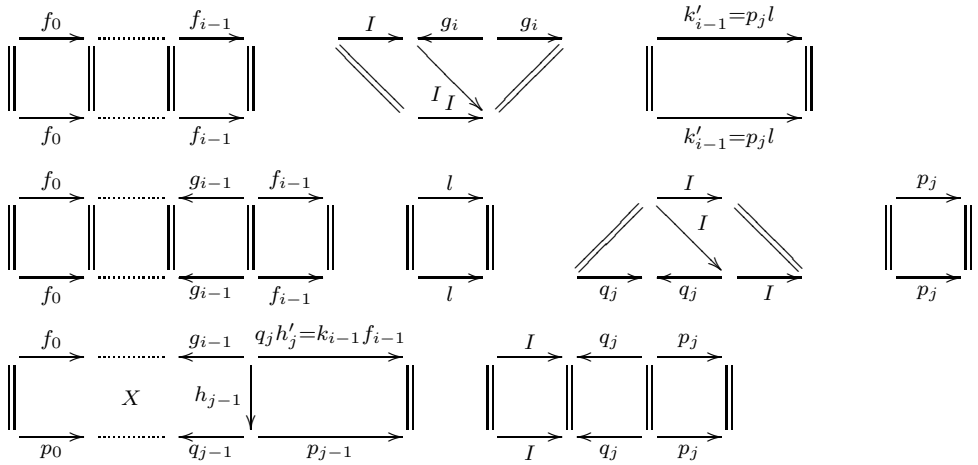
and ω can be decomposed as



Since $h_j = lg_i$ and $k_{i-1} = q_jl$, these decompositions can be rewritten in the following way. θ is the pasting of the fences



and ω is the pasting of the fences



It is now obvious that one can be obtained from the other by middle-four interchange. ∎

4. Universal Property of $\Pi_2\mathcal{C}$

There is an obvious embedding functor $(-)_*\colon \mathcal{C} \to \Pi_2\mathcal{C}$, which is the identity on objects and sends morphisms to sequences of length 1: $f_* = (f)$. It is easy to check that it has the following property:

Lemma 5 *The embedding $(-)_*\colon \mathcal{C} \to \Pi_2\mathcal{C}$ is locally fully faithful.*

It also has the property that it sends arrows in $\mathcal{C}$ to arrows which are left adjoints in $\Pi_2\mathcal{C}$. We will call any functor which send arrows to left adjoints *sinister*.

We will also need the notion of a *sinister natural transformation* between sinister functors. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be sinister functors, and let $\alpha\colon F \Rightarrow G$ be a strong transformation. Recall (cf. [1]) that this means that α is determined by a family of arrows $\alpha_A\colon FA \to GA$ in $\mathcal{D}$, for objects $A \in \mathcal{C}_0$ together with a natural family of isomorphisms $\alpha_f\colon \alpha_B Ff \Rightarrow Gf\alpha_A$ in $\mathcal{D}$, for arrows $f\colon A \to B$ in $\mathcal{C}$. For any arrow $f\colon A \to B$, choose right adjoints $(Ff)^*$ and $(Gf)^*$ for Ff and Gf respectively. Together with α_f these give rise to the following squares in $\mathcal{D}$:

$$\begin{array}{ccccc} FA & \xrightarrow{Ff} & FB & \xrightarrow{(Ff)^*} & FA \\ {\scriptstyle\alpha_A}\downarrow & \cong\alpha_f & \downarrow{\scriptstyle\alpha_B} & \Downarrow\beta_f & \downarrow{\scriptstyle\alpha_A} \\ GA & \xrightarrow[Gf]{} & GB & \xrightarrow[(Gf)^*]{} & GA, \end{array}$$

where $\beta_f\colon (Gf)^*\alpha_B \Rightarrow \alpha_A(Ff)^*$ is the mate of α_f^{-1}, *i.e.*,

$$\beta_f = ((Gf)^*\alpha_B\varepsilon_{Ff}) \cdot ((Gf)^*\alpha_f^{-1}(Ff)^*) \cdot (\eta_{Gf}\alpha_A(Ff)^*).$$

Definition 3 *A strong natural transformation $\alpha\colon F \Rightarrow G$ between two sinister functors $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ is called* sinister *if $\alpha_{I_A} = i_{\alpha_A}$ for all objects $A \in \mathcal{C}_0$. Moreover, it is called* strongly sinister *if for each arrow $f \in \mathcal{C}_1$ the mate β_f of α_f^{-1} is an isomorphism.*

Recall (cf. [1]) that in general a natural transformation α between two bifunctors $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ can be represented by a bifunctor $K_\alpha\colon \mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$. This bifunctor is defined as follows: on objects,

$$K_\alpha(A) = \begin{array}{c} FA \\ \downarrow{\scriptstyle \alpha_A} \\ GA \end{array} \quad ;$$

on arrows,

$$K_\alpha(f) = \begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \alpha_A}\downarrow & \alpha_f & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array} \quad ;$$

and on 2-cells, $K_\alpha(i_f)$ is the cylinder with identity 2-cells on the top and the bottom and α_f on the front and the back. Definition 3 is justified by the following result.

Proposition 2 *A transformation $\alpha\colon F \Rightarrow G$ between sinister 2-functors $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ is sinister if and only if its representing bifunctor $K_\alpha\colon \mathcal{C} \to Cyl(\mathcal{D})$ is a sinister 2-functor.*

Proof. Let $\alpha\colon F \Rightarrow G$ be a sinister (strong) natural transformation between sinister 2-functors, and let $K_\alpha\colon \mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$ be its representing functor. Let $f \in \mathcal{C}_1$ be any morphism. We claim that β_f, the mate of α_f^{-1} as defined above, is right adjoint to $K_\alpha f$. The unit and counit of the adjunction are the pairs (η_{Ff}, η_{Gf}) and $(\varepsilon_{Ff}, \varepsilon_{Gf})$ respectively. Moreover, K_α is a 2-functor, since $K_\alpha(I_A) = i_{\alpha_A I_{FA}}$ for all $A \in \mathcal{C}_0$ and it preserves composition. To prove the converse, suppose that $\alpha\colon F \Rightarrow G$ is a natural transformation between sinister 2-functors, such that its representing bifunctor $K_\alpha\colon \mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$ is a sinister 2-functor. Since K_α is a 2-functor, we have that $\alpha_{I_A} = i_{\alpha_A}$. To check that α is a strong transformation, let $f \in \mathcal{C}_1$ be any morphism. Since K_α is sinister, there exists an arrow $(K_\alpha f)^*$ in $\mathrm{Cyl}(\mathcal{D})$, *i.e.*, a square

$$\begin{array}{ccc} FB & \xrightarrow{(Ff)^*} & FA \\ {\scriptstyle \alpha_B}\downarrow & \Downarrow (K_\alpha f)^* & \downarrow{\scriptstyle \alpha_A} \\ GB & \xrightarrow[(Gf)^*]{} & GA \end{array}$$

where $(Ff)^*$ and $(Gf)^*$ are right adjoint to Ff and Gf respectively. The mate of $(K_\alpha f)^*$ can be shown to be the inverse of α_f. ∎

Let $\mathrm{Weak}(\mathcal{C},\mathcal{D})$ denote the category of 2-functors and weak natural transformations, let $\mathrm{Strong}(\mathcal{C},\mathcal{D})$ denote the category of 2-functors and strong natural transformations, let $\mathrm{Sin}(\mathcal{C},\mathcal{D})$ denote the category of sinister 2-functors and sinister natural transformations, and let $\mathrm{SSin}(\mathcal{C},\mathcal{D})$ denote the category of sinister 2-functors and strongly sinister natural transformations.

Theorem 3 *The functor $(-)_*\colon \mathcal{C} \to \Pi_2\mathcal{C}$ has the property that for each arrow f in $\mathcal{C}$ the arrow f_* in $\Pi_2\mathcal{C}$ has a right adjoint in $\Pi_2\mathcal{C}$ (i.e. f_* is a left adjoint). Moreover, this functor is universal with this property in the sense that composition with $(-)_*$ defines the following equivalences of categories*

$$\mathrm{Weak}(\Pi_2\mathcal{C},\mathcal{D}) \xrightarrow{\sim} \mathrm{Sin}(\mathcal{C},\mathcal{D}),$$

$$\mathrm{Strong}(\Pi_2\mathcal{C},\mathcal{D}) \xrightarrow{\sim} \mathrm{SSin}(\mathcal{C},\mathcal{D}).$$

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a sinister 2-functor. Our first goal is to construct a 2-functor $\tilde{F}\colon \Pi_2\mathcal{C} \to \mathcal{D}$ such that $\tilde{F} \circ (-)_* \cong F$. Choose, for each arrow $(f\colon A \to B) \in \mathcal{C}_1$, a specific arrow $(Ff)^* \in \mathcal{D}_1$, and 2-cells

$$\eta_{Ff}\colon I_{F(A)} \Rightarrow (Ff)^*F(f) \text{ and } \varepsilon_{Ff}\colon F(f)(Ff)^* \Rightarrow I_{F(B)}$$

such that η_{Ff} and ε_{Ff} satisfy the triangle identities, *i.e.*,

$$F(f)\varepsilon_{Ff} \cdot \eta_{Ff}F(f) = i_{F(f)} \text{ and } \varepsilon_{Ff}(Ff)^* \cdot (Ff)^*\eta_{Ff} = i_{(Ff)^*}.$$

On objects, $\tilde{F}$ is the same as F. On arrows, $\tilde{F}$ is defined by

$$\tilde{F}(g_1,\ldots,g_n;f_0,\ldots,f_n) = F(f_n)(Fg_n)^*F(f_{n-1})\cdots(Fg_1)^*F(f_0)$$

Since the 2-cells in $\Pi_2\mathcal{C}$ are generated, by identities, ε's, and η's as in (4), it sufficient to define $\tilde{F}$ for these particular 2-cells, which is done in the obvious way:

$$\tilde{F}\left(\begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & & \| \\ A & \xrightarrow[f]{} & B \end{array}\right) = F(i_f) = i_{Ff}$$

$$\tilde{F}\left(\begin{array}{ccccc} & & A & \xrightarrow{I_A} & A & & \\ & \swarrow\!\!\swarrow & & \searrow & & \swarrow\!\!\swarrow & \\ A & \xrightarrow[f]{} & B & \xleftarrow[f]{} & A & \xrightarrow[I_A]{} & A \end{array}\right) = \eta_{Ff}$$

$$\tilde{F}\left(\begin{array}{ccccc} B & \xrightarrow{I_B} & B & \xleftarrow{f} & A \xrightarrow{f} B \\ & \searrow & & \swarrow_{I_B} & \\ & & B & \xrightarrow[I_B]{} & B \end{array}\right) \;=\; \varepsilon_{Ff}$$

The functor $\tilde{F}$ is well-defined on equivalence classes of 2-cells, since this equivalence relation is equivalent to having middle-four interchange for the vertical and horizontal composition of fences (cf. Theorem 2). Finally, it is clear that $\tilde{F} \circ (-)_* = F$. This shows that the functor 'composition with $(-)_*$' is essentially surjective on objects for both equivalences described in the theorem.

It remains to be shown that for any two functors $F, G\colon \Pi_2\mathcal{C} \rightrightarrows \mathcal{D}$, the functor 'composition with $(-)_*$' induces an isomorphism between the appropriate sets of natural transformations, *i.e.*,

$$\mathrm{Weak}(F,G) \cong \mathrm{Sin}(F \circ (-)_*, G \circ (-)_*) \text{ and } \mathrm{Strong}(F,G) \cong \mathrm{SSin}(F \circ (-)_*, G \circ (-)_*).$$

Let $\alpha\colon F \circ (-)_* \Rightarrow G \circ (-)_*$ be a sinister natural transformation. By Proposition 2, its representing functor $K_\alpha\colon \mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$ is sinister. We need to show that there is a unique natural transformation $\tilde{\alpha}\colon F \Rightarrow G$, such that $\tilde{\alpha} \circ (-)_* = \alpha$. To obtain $K_{\tilde{\alpha}}$, take the lifting $\tilde{K}_\alpha\colon \Pi_2\mathcal{C} \to \mathrm{Cyl}(\mathcal{D})$ of K_α such that $\tilde{K}_\alpha(f)$ is the square

$$\begin{array}{ccc} FB & \xrightarrow{F(f;I_B,I_A)} & FA \\ {\scriptstyle \alpha_B}\downarrow & \Downarrow \beta_f & \downarrow{\scriptstyle \alpha_A} \\ GB & \xrightarrow[G(f;I_B,I_A)]{} & GA \end{array}$$

where β_f is the mate of α_f^{-1}. It is clear that this is the only lifting of K_α for which the top is F and the bottom is G. Moreover, α is strongly sinister if and only if $\tilde{\alpha}$ is a strong transformation. ∎

4.1. Adding Right Adjoints for a Class of Arrows. As mentioned in the introduction, the Span construction adds adjoints freely subject to the Beck condition. If instead one adds adjoints only for monomorphisms (still subject to the Beck condition), one obtains instead the closely related partial map construction. Similarly, in the calculus of fractions one usually obtains more useful results by inverting only a distinguished class of arrows (*e.g.*, the homotopy equivalences) in a category. In the same way, we may want to add right adjoints for only a class of arrows $W \subset \mathcal{C}$. To do this it is necessary to require that the g_i in every zig-zag (6) be in W. The class of fences with such domains and codomains is closed under horizontal and vertical composition. The proofs that horizontal and vertical composition are well-defined (Lemma 2 and 3) and of the middle-four interchange property (Theorem 2) do not involve the creation of new backward arrows; thus these fences form a 2-category, which we will denote by $\mathcal{C}[W^*]$. Since the proof of the universal property of $\Pi_2\mathcal{C}$ (cf. Theorem 3) doesn't involve the creation of any new

backward arrows either, we have in fact proved a universal property for $\mathcal{C}[W^*]$. Define W-sinister 2-functors and (strongly) W-sinister natural transformations in the obvious way. Let $\mathrm{Sin}_W(\mathcal{C},\mathcal{D})$ denote the category of W-sinister 2-functors and W-sinister natural transformations; and let $\mathrm{SSin}_W(\mathcal{C},\mathcal{D})$ denote the category of W-sinister 2-functors and (strongly) W-sinister natural transformations.

Theorem 4 *Let $\mathcal{C}$ be a category and $W \subset \mathcal{C}_1$ a distinguished class of arrows. The functor $(-)_*\colon \mathcal{C} \to \mathcal{C}[W^*]$ has the property that for each arrow f in W the arrow f_* in $\mathcal{C}[W^*]$ has a right adjoint in $\mathcal{C}[W^*]$ (i.e., f_* is a left adjoint). Moreover, this functor is universal with this property in the sense that composition with $(-)_*$ defines the following equivalences of categories*

$$\begin{aligned} \mathrm{Weak}(\mathcal{C}[W^*],\mathcal{D}) &\xrightarrow{\sim} \mathrm{Sin}_W(\mathcal{C},\mathcal{D}), \\ \mathrm{Strong}(\mathcal{C}[W^*],\mathcal{D}) &\xrightarrow{\sim} \mathrm{SSin}_W(\mathcal{C},\mathcal{D}). \end{aligned}$$

5. Some Basic Properties

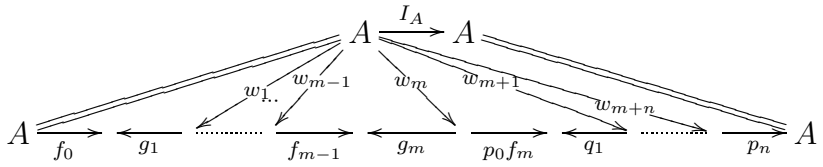
5.1. Arrows with Right Adjoints. We have shown that the arrows of the form $f_* = (f)$ in $\Pi_2\mathcal{C}$ have right adjoints. For a category $\mathcal{C}$ with pullbacks, we know that in $\mathrm{Span}\,\mathcal{C}$ the only left adjoints are the arrows in the image of $\mathcal{C}$. Although $\Pi_2\mathcal{C}$ and $\mathrm{Span}\,\mathcal{C}$ are distinct categories in general, we see that this result also holds for $\Pi_2\mathcal{C}$; we show that up to 2-isomorphism, the arrows of the form f_* are the only left adjoints in $\Pi_2\mathcal{C}$.

Proposition 3 *Let $\mathcal{C}$ be any category. If a morphism $(g_1,\ldots,g_n;f_0,\ldots,f_n)$ in $\Pi_2\mathcal{C}$ has a right adjoint, there exists an arrow $f \in \mathcal{C}_1$, with a 2-isomorphism*

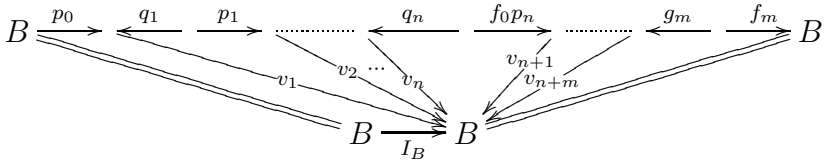
$$(g_1,\ldots,g_n;f_0,\ldots,f_n) \cong (f).$$

Proof. Suppose that the arrow $(g_1,\ldots,g_m;f_0,\ldots,f_m)\colon A \to B$ has a right adjoint in $\Pi_2\mathcal{C}$, say $(q_1,\ldots,q_n;p_0,\ldots,p_n)\colon B \to A$. Let the unit and counit of the adjunction be

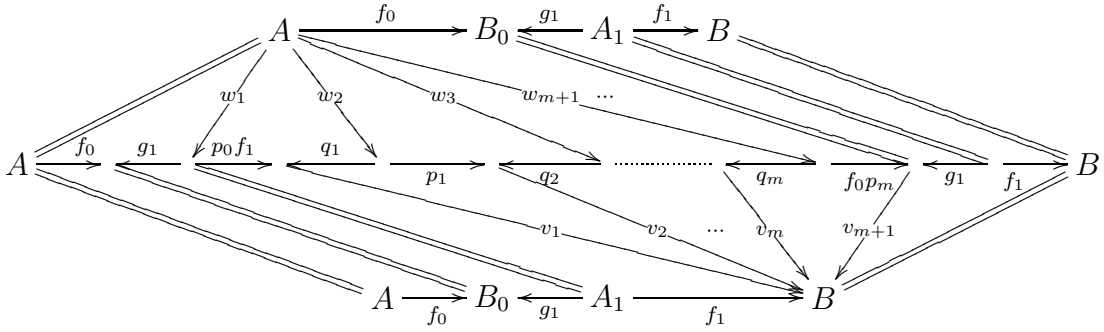
represented by the fences



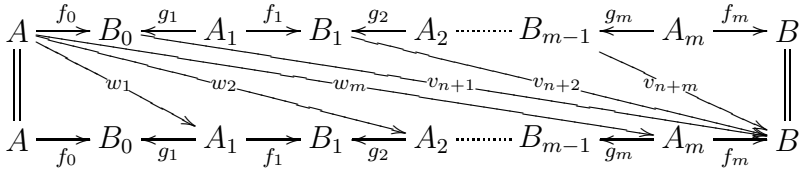
and



respectively. One of the triangle identities states that the following composition of 2-cells is the identity 2-cell for the morphism $(g_1; f_0, f_1)$. (We draw the diagram only for $m = 1$.)

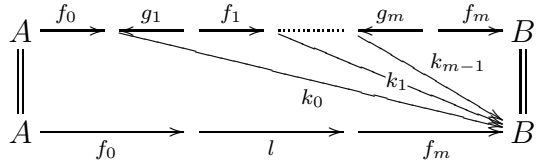


Vertical composition results in:

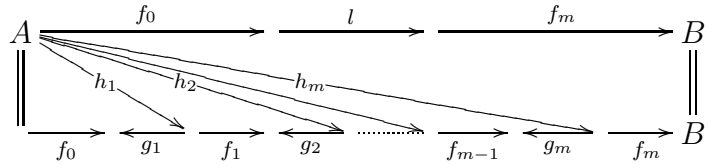


(24)

Since this 2-cell is equivalent to the identity 2-cell, there is an arrow $l\colon B_0 \to A_m$, such that $w_m = lf_0$ and $f_m l = v_{n+1}$. We claim that $(g_1, \ldots, g_m; f_0, \ldots, f_m)$ is isomorphic to $(f_m l f_0)$ in $\Pi_2\mathcal{C}$. Indeed, the required 2-cells are:

$\Theta_1 =$ 

and

$\Theta_2 =$ 

It is easy to see that $\Theta_1\Theta_2 = \mathrm{id}_{f_m l f_1}$ (after all, the original category has only identity 2-cells). The composite $\Theta_2\Theta_1$ is equal to the cell represented by fence (24), and by assumption this cell is equivalent to the identity 2-cell. ∎

5.2. Limits and Colimits in $\Pi_2\mathcal{C}$. The nonempty hom-sets (we are considering the underlying *category* of $\Pi_2(\mathcal{C})$) are always infinite; if $A \overset{f_0}{\to}\overset{g_1}{\leftarrow} \ldots \overset{f_n}{\to} B$ is a zig-zag, then so is $A \overset{f_0}{\to}\overset{g_1}{\leftarrow} \ldots \overset{f_n}{\to}\overset{f_n}{\leftarrow}\overset{f_n}{\to} B$, and they are not equal. Thus, it follows that except in vacuous cases, $\Pi_2(\mathcal{C})$ never has a terminal (or initial) object.

Similar arguments show that the underlying category of $\Pi_2(\mathcal{C})$ has only trivial products.

Proposition 4 *The product $A \times B$ exists in $\Pi_2(\mathcal{C})$ if and only if $A = B$ and no arrow except for I_A maps into A; in this case, $A \times A = A$.*

Proof. Suppose that $A \times B$ exists in $\Pi_2(\mathcal{C})$; then the projections

$$A \overset{p_0}{\longleftarrow} A \times B \overset{p_1}{\longrightarrow} B$$

are zig-zags

$$A \overset{f_m}{\leftarrow} \ldots \overset{g_1}{\to}\overset{f_0}{\leftarrow} A \times B \overset{f_0'}{\to}\overset{g_1'}{\leftarrow} \ldots \overset{f_n'}{\to} B$$

and it is evident that the zig-zags

$$A \overset{f_m}{\leftarrow}\overset{f_m}{\to}\overset{f_m}{\leftarrow} \ldots \overset{g_1}{\to}\overset{f_0}{\leftarrow} A \times B \overset{f_0'}{\to}\overset{g_1'}{\leftarrow} \ldots \overset{f_n'}{\to}\overset{f_n'}{\leftarrow}\overset{f_n'}{\to} B$$

can only factor through the projections if

$$f_0 = g_1 = f_1 = \cdots = f_m = f_0' = g_1' = f_1' = \cdots = f_n' \,. \tag{25}$$

As we always have the option of using an identity within a zig-zag, (25) is only forced under the conditions stated in the proposition. ∎

Note: The dual of this proposition also holds; the only coproducts in $\Pi_2(\mathcal{C})$ are of the form $A \coprod A$ where only I_A maps out of A.

However, there may be nontrivial equalizers and coequalizers, and there are always nontrivial pullbacks and pushouts.

Proposition 5 *A diagram* $A \underset{\underset{f_0'}{\overset{f_0}{\rightrightarrows}}}{\overset{g_1}{\leftleftarrows}} \cdots \underset{f_n'}{\overset{f_m}{\rightrightarrows}} B$ *has an equalizer in* $\Pi_2(\mathcal{C})$ *if and only if* $m = n$, $g_1 = g_1', \ldots, f_m = f_m'$, *and* $A \underset{f_0'}{\overset{f_0}{\rightrightarrows}} X$ *has an equalizer in* $\mathcal{C}$.

Proof. If the two zig-zags $A \overset{f_0}{\rightarrow} \overset{g_1}{\leftarrow} \cdots \overset{f_m}{\rightarrow} B$ and $A \overset{f_0'}{\rightarrow} \overset{g_1'}{\leftarrow} \cdots \overset{f_n'}{\rightarrow} B$ are different lengths, or if they differ anywhere except at the extreme left, we can never have

$$(A' \overset{f_0''}{\rightarrow} \overset{g_1''}{\leftarrow} \cdots \overset{f_0 f_k''}{\longrightarrow} \overset{g_1}{\leftarrow} \cdots \overset{f_m}{\rightarrow} B) = (A' \overset{f_0''}{\rightarrow} \overset{g_1''}{\leftarrow} \cdots \overset{f_0' f_k''}{\rightarrow} \overset{g_1'}{\leftarrow} \cdots \overset{f_n'}{\rightarrow} B) \; . \tag{26}$$

Conversely, if $m = n$, $g_1 = g_1', \ldots, f_m = f_m'$, the zig-zags in (26) are equal if and only if $f_0 f_k'' = f_0' f_k''$. Should this be the case, the length-1 zig-zag $A' \overset{f_k''}{\rightarrow} A$ satisfies (26) itself, and cannot factor through any longer zig-zag; so only length-1 zig-zags can be equalizers in $\Pi_2(\mathcal{C})$. Finally, if

$$E \overset{e}{\rightarrow} A \underset{f_0'}{\overset{f_0}{\rightrightarrows}} B$$

is an equalizer diagram in $\mathcal{C}$, then $f_0 e = f_0' e$, and if (26) holds, $A' \overset{f_0''}{\rightarrow} \overset{g_1''}{\leftarrow} \cdots \overset{f_k''}{\rightarrow} B$ factors uniquely up to isomorphism through e, as $A' \overset{f_0''}{\rightarrow} \overset{g_1''}{\leftarrow} \cdots \overset{q}{\rightarrow} E \overset{e}{\rightarrow} A$ where $eq = f_k''$ in $\mathcal{C}$. ∎

We thus see that the existence of nontrivial equalizers in $\Pi_2(\mathcal{C})$ depends on their existence in $\mathcal{C}$. Coequalizers are, of course, computed analogously. The next proposition, too, has two dual forms for pushouts and pullbacks; in this case, the colimit form is notationally easier.

Proposition 6 *A diagram*

$$A \overset{f_m}{\leftarrow} \ldots \overset{g_1}{\rightarrow} \overset{f_0}{\leftarrow} C \overset{f_0'}{\rightarrow} \overset{g_1'}{\leftarrow} \ldots \overset{f_n'}{\rightarrow} B \; . \tag{27}$$

has a pushout in $\Pi_2(\mathcal{C})$ *if either*

i) $f_0 = f'_0$, $f_i = f'_i$ *and* $g_i = g'_i$ *for* $i = 1, \ldots, \min(m, n)$; *or*

ii) $m = n$, $f_0 = f'_0$, $f_i = f'_i$ *and* $g_i = g'_i$ *for* $i = 1, \ldots, m-1$, $g_m = g'_m$, *and* $A \xleftarrow{f_m} X \xrightarrow{f'_m} B$ *has a pushout in* $\mathcal{C}$.

Proof. We may suppose, without loss of generality, that $m \leq n$. We consider two cases, depending upon whether $m < n$ or $m = n$. In the first case, if we can fill out the square at all, there exists

$$A \xrightarrow{\hat{f}_m} \xleftarrow{g_{m+1}} \ldots \xrightarrow{f_k} D \xleftarrow{f'_k} \ldots \xrightarrow{g'_{n+1}} \xleftarrow{\hat{f}'_n} B \tag{28}$$

such that

$$f_0 = f'_0,\ g_1 = g'_1,\ \ldots,\ \hat{f}_m f_m = f'_m,\ \ldots, f_n = \hat{f}'_n f'_n,\ \ldots,\ f_k = f'_k\ . \tag{29}$$

A necessary condition on (27) for this to happen is that

$$f_0 = f'_0,\ g_1 = g'_1,\ \ldots,\ g_m = g'_m$$

and that $\hat{f}_m \in \mathcal{C}_1$ with $\hat{f}_m f_m = f'_m$ should exist. We can fill the square out with

$$A \xrightarrow{\hat{f}_m} \xleftarrow{g_{m+1}} \ldots \xrightarrow{f_n} B \xleftarrow{I_B} B\ . \tag{30}$$

Any other commuting completion (28) factors uniquely (up to isomorphism) through $B \xrightarrow{\hat{f}_n} \xleftarrow{g_{n+1}} \ldots \xrightarrow{f_k} D$. We thus conclude that (30) is a pushout. We note that, while $\hat{f}_m$ may not be unique, all choices give isomorphic pushouts.

If, on the other hand, $m = n$, then we have, instead of (29),

$$f_0 = f'_0,\ g_1 = g'_1,\ \ldots,\ \hat{f}_m f_m = \hat{f}'_m f'_m,\ \ldots,\ f_k = f'_k \tag{31}$$

and (27) takes the form

$$A \xleftarrow{f_m} X \xrightarrow{g_m} \ldots \xrightarrow{g_1} \xleftarrow{f_0} C \xrightarrow{f_0} \xleftarrow{g_1} \ldots \xleftarrow{g_m} X \xrightarrow{f'_m} B\ . \tag{32}$$

Again, any pair $\hat{f}_m, \hat{f}'_m$ consistent with (31) allows us to close the square with

$$A \xrightarrow{\hat{f}_m} Y \xleftarrow{\hat{f}'_m} B\ . \tag{33}$$

If (33) is the pushout of

$$A \xleftarrow{f_m} X \xrightarrow{f'_m} B \tag{34}$$

in $\mathcal{C}$, any commuting completion

$$A \xrightarrow{\tilde{f}_m} \tilde{Y} \xleftarrow{g_{m+1}} \ldots \xrightarrow{f_k} D \xleftarrow{f_k} \ldots \xrightarrow{g_{m+1}} \tilde{Y} \xleftarrow{\tilde{f}'_m} B$$

of (32) can be factored (in $\Pi_2(\mathcal{C})$) through (33) by the zig-zag

$$Y \xrightarrow{h} \tilde{Y} \xleftarrow{g_{m+1}} \ldots \xrightarrow{f_k} D \ ,$$

where h is the unique arrow factoring $A \xrightarrow{\tilde{f}_m} \tilde{Y} \xleftarrow{\tilde{f}'_m} B$ through (33) in $\mathcal{C}$. Again, we conclude that (33) is a pushout in $\Pi_2(\mathcal{C})$. Conversely, if (34) does not have a pushout in $\mathcal{C}$, it does not have one in $\Pi_2(\mathcal{C})$. ∎

We have, then, obtained necessary and sufficient conditions for the existence of specific pushouts in $\Pi_2(\mathcal{C})$. These conditions are such that we can see that (on the one hand) $\Pi_2(\mathcal{C})$ never has *all* pushouts except in trivial cases; but (on the other hand) it always has *some* pushouts. Clearly, analogous conditions govern the existence of pullbacks.

The virtual nonexistence of products, and the restrictions on limits of other types, are not peculiar to $\Pi_2(\mathcal{C})$. Rather, they result from the formal nature of this category, which means that only rather predictable factorizations exist. The free category on a graph has similarly restricted limits.

It may be noted that the conditions of this section (especially the nonexistence of nontrivial products) give a quick way to show that most familiar 2-categories do not arise as $\Pi_2(\mathcal{C})$ for any $\mathcal{C}$.

6. Π_2 of a Free Category

Let $\mathbf{G}$ be a directed graph, and $F\mathbf{G}$ the free category generated by $\mathbf{G}$. All morphisms are mono and epi, so the word problem for 2-cells in $\Pi_2 F\mathbf{G}$ is solvable. In fact there is a nice geometric description of the 2-cells.

As the morphisms of $F\mathbf{G}$ are paths of edges of $\mathbf{G}$ all in the forward direction, the morphisms of $\Pi_2 F\mathbf{G}$ will be paths of edges of G going in either direction:

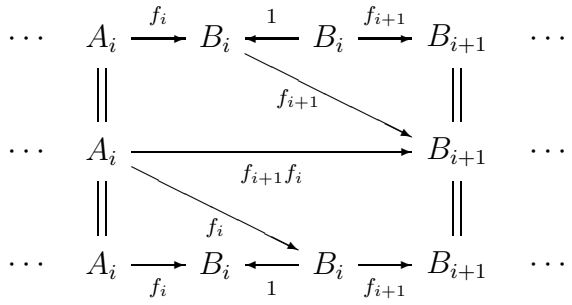
$$A_0 \overset{a_1}{\rightarrow} A_1 \overset{a_2}{\rightarrow} A_2 \overset{a_3}{\leftarrow} A_3 \overset{a_4}{\rightarrow} A_4 \overset{a_5}{\leftarrow} A_5 \overset{a_6}{\leftarrow} A_6.$$

We denote such a path by $(a_1^+, a_2^+, a_3^-, a_4^+, a_5^-, a_6^-)$, where the superscript denotes the direction of the edge.

Strictly speaking, the morphisms of $\Pi_2 F\mathbf{G}$ are a bit more complicated than this as we are allowing our paths to contain identities. This makes the general theory easier and is necessary for the two end morphisms. As identities are paths of length zero in $F\mathbf{G}$, it is impractical to put them in here. This does not affect the universal property as we get a locally equivalent 2-category with the same objects. Indeed if in the general construction a path of length k contains

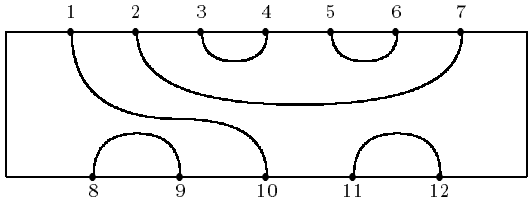
$$\cdots A_i \overset{fi_1}{\to} B_i \overset{1_{B_i}}{\leftarrow} B_i \overset{f_{i+1}}{\to} B_{i+1} \cdots$$

then the cells



are inverse isomorphisms with the path of length $k-2$ in which the identity is removed and the adjacent arrows composed.

Let $\mathbf{a} = (a_i^{\epsilon(i)})_{i \le p}$ and $\mathbf{b} = (b_j^{\eta(j)})_{j \le q}$ be two paths with common domain and codomain. In order to describe the 2-cells $\mathbf{a} \Rightarrow \mathbf{b}$, we recall the following definition from [7]. A (p, q) *diagram* is a partition P of the set $\{1, 2, \ldots, p+q\}$ into subsets of size two. We represent this pictorially as a rectangle with p marked points on the top edge numbered from 1 to p and q on the bottom numbered from $p+1$ to $p+q$. If $\{i, j\} \in P$, the point i is joined to the point j.



A (7, 5)-diagram

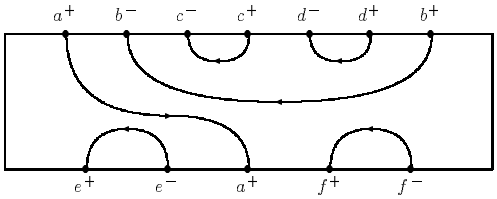
It is a *planar* (p, q)-diagram if corresponding points can be joined by non-intersecting arcs which lie inside the rectangle, as in the above example. (It would not be planar if, for example, $\{4, 6\}$ and $\{5, 7\} \in P$.) Planar (p, q)-diagrams are also called *Kauffman diagrams* in [8], at least when $p = q$.

Given paths $\mathbf{a}$ and $\mathbf{b}$ as above, define $a_i^{\epsilon(i)} = b_{i-p}^{\eta(i-p)}$ for $p < i \le p+q$. We define a *planar* $(\mathbf{a}, \mathbf{b})$*-diagram* to be a planar (p, q)-diagram such that for all $\{i, j\}$ in P we have

$$\begin{gathered} a_i = a_j \quad \text{and} \\ \epsilon(i) = - \quad \text{and} \quad \epsilon(j) = + \quad \text{if} \quad i < j \le p \\ \epsilon(i) = + \quad \text{and} \quad \epsilon(j) = - \quad \text{if} \quad p < i < j \\ \epsilon(i) = \epsilon(j) \quad \text{otherwise.} \end{gathered}$$

Thus the points that are joined are labeled by the same edge. These edges go in the same direction if one is on top and one on the bottom, and in opposite directions otherwise. If two top points are joined the edges point away from each other whereas on

the bottom they point toward each other. These conditions on the directions of joined edges can be coded into an orientation on the joining strands. They are oriented so that rotating from the forward direction of the edge to the forward direction of the string is clockwise:

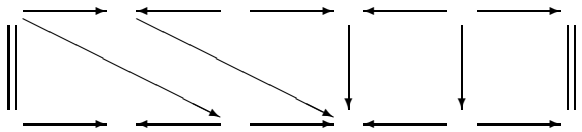


The strands joining points at the same level go from right to left.

Planar $(\mathbf{a}, \mathbf{b})$-diagrams can be pasted horizontally or, if they are compatible, vertically. Because of the orientations of the strands no loops are formed when pasting.

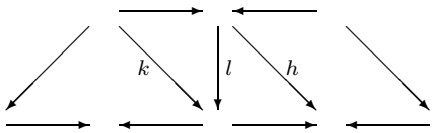
Proposition 7 *Given two morphisms* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\Pi_2 F\mathbf{G}$ *as above, there is a bijection between 2-cells* $\mathbf{a} \Rightarrow \mathbf{b}$ *and isotopy classes of planar* $(\mathbf{a}, \mathbf{b})$*-diagrams. Furthermore, horizontal (vertical) composition of cells corresponds to horizontal (resp. vertical) pasting of diagrams.*

Proof. Suppose we have a morphism in $\Pi_2 F\mathbf{G}$ represented by a diagram such as the following:

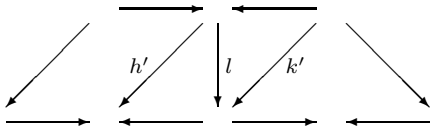


Each commutative square in this diagram is an equality of arrows in the free category $F\mathbf{G}$, so the same edges appear in each composite and in the same order. The corresponding edges may be joined by non-intersecting curves with the correct orientation. Each edge occurring in a vertical arrow borders two regions so there will be a curve entering and one leaving. Thus the end points of the pasted curves lie on the top or bottom of the diagram, giving a planar $(\mathbf{a}, \mathbf{b})$-diagram. As the vertical arrows consist of downward edges, the strands cross them from right to left.

The $(\mathbf{a}, \mathbf{b})$-diagram structure is independent of which representative of the 2-cell we choose. Let l be as in the diagram below and consider the two squares adjoining the one containing l (there are four possible combinations of shapes and they all work the same).

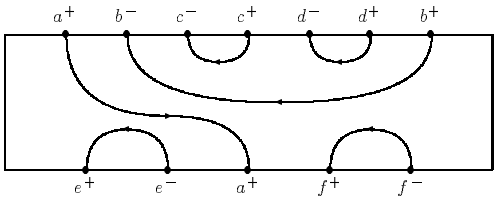


Our equivalence says that we can replace this diagram by one of the form

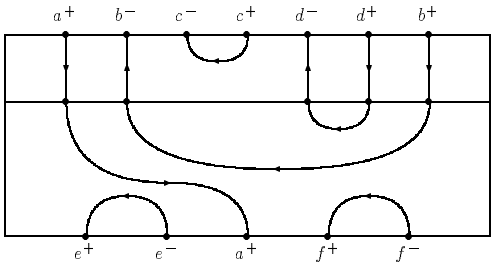


The curves joining corresponding arrows in the middle square can be drawn by first joining corresponding edges in each of the triangles made by l and then pasting those that pass through l together. These curves are now pasted to the matching ones in the squares on the right and left. But this could have been accomplished by erasing the h and k (resp. h' and k') but keeping the l and then joining the corresponding edges in the remaining two commutative pentagons. The result will be the same in both cases.

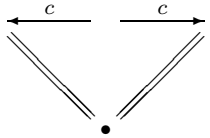
To show that every planar $(\mathbf{a}, \mathbf{b})$-diagram represents a 2-cell, note first of all that if every top edge is paired with a bottom edge and conversely, then the diagram represents the identity 2-cell. Otherwise, there will be two top (or bottom) edges paired together and so two adjacent ones paired together. Remove these two and we get a smaller $(\mathbf{a}, \mathbf{b})$-diagram which, by induction, represents a 2-cell. The original diagram is obtained by pasting this diagram with one having just two adjacent edges joined and augmented with identities on either side. For example



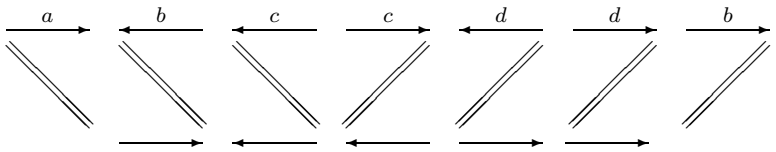
can be written as



The top diagram represents the 2-cell



with the appropriate identities pasted on either side, i.e.

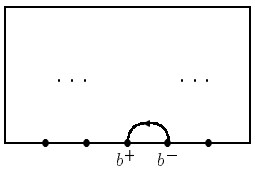


■

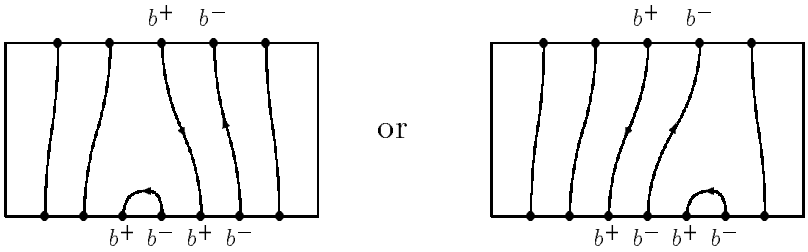
The hom categories in $\Pi_2 F\mathbf{G}$ have a lot of nice structure. For example they have epi-mono factorizations.

Proposition 8 *An* $(\mathbf{a}, \mathbf{b})$*-diagram is an epimorphism (monomorphism) if and only if there are no bottom (resp. top) hoops.*

Proof. Consider an $(\mathbf{a}, \mathbf{b})$-diagram α with a hoop on the bottom as follows

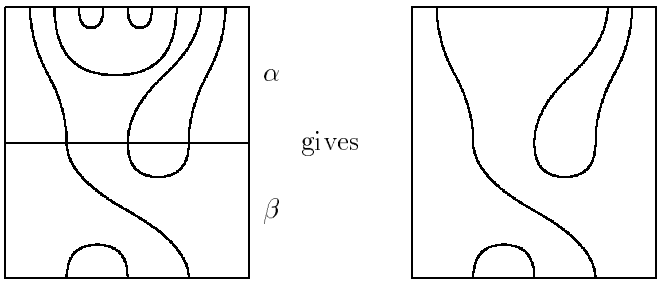


If we compose with



we get the same result, namely a diagram like α but with two hoops instead of one. So α is not epi.

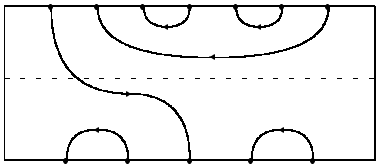
Conversely, if no bottom vertex is joined to another bottom vertex, then each bottom vertex is joined to a top one with the same label and the same sign. If we paste α with a cell β, we can recover β from $\alpha \cdot \beta$ by picking out those through strands of α together with the strands they are joined to in β. For example



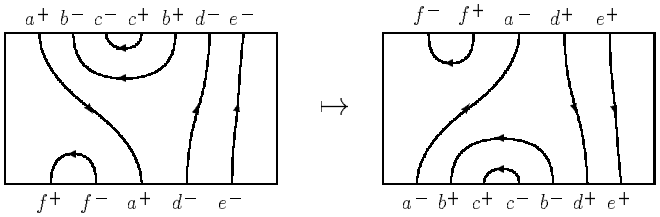
which is isotopic to β. Thus α is epi.

The corresponding statement for monos is the same. ■

Now given an $(\mathbf{a}, \mathbf{b})$-diagram , it can be factored into an epi followed by a mono as follows. Deform the strands isotopically so that no hoop is more than one third the height of the diagram. Then draw a line across the middle of the diagram and it is factored.

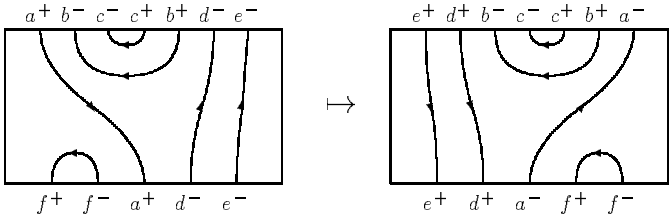


The description of monos also follows by the duality $\Pi_2 F(\mathbf{G})^{co} \cong \Pi_2 F(\mathbf{G}^{op})$ obtained by taking an $(\mathbf{a}, \mathbf{b})$-diagram, changing the sign of each node and then reflecting in a horizontal line. For example,



in $\Pi_2 F(\mathbf{G})$ in $\Pi_2 F(\mathbf{G}^{op})$

There is another duality, namely $\Pi_2 f(\mathbf{G})^{op} \cong \Pi_2 f(\mathbf{G})$, obtained by reflection in a vertical line and changing the signs of the nodes:



in $\Pi_2 F(\mathbf{G})$ in $\Pi_2 F(\mathbf{G})$

Let us clarify what is happening with dualities. For an arbitrary category $\mathbf{A}$ we always have $\Pi_2(\mathbf{A}^{op}) \cong \Pi_2(\mathbf{A})^{coop}$ gotten by reversing everything in our diagrams for 2-cells. But there is also another duality which holds for an arbitrary category, namely $(\ )^* : \Pi_2(\mathbf{A})^{op} \simeq \Pi_2(\mathbf{A})$ which is the identity on objects. An arrow, which is a zig-zag path of morphisms from $\mathbf{A}$, is sent to the same zig-zag in the reverse direction with identities added at each end to give it the right shape and the same is done to 2-cells. $(\ )^{**}$ is not exactly the identity. It is on objects but only induces an equivalence on the hom categories. We might call it a *pseudo-equivalence*. This is not a serious problem and arises only because of our choice of shape of zig-zag (starting and ending with forward arrows). If we combine these two equivalences we get $\Pi_2(\mathbf{A}^{op}) \simeq \Pi_2(\mathbf{A})^{co}$. As we are ignoring identities in the case of free categories, this gives an actual equivalence $\Pi_2 F(\mathbf{G})^{co} \cong \Pi_2 F(\mathbf{G}^{op})$.

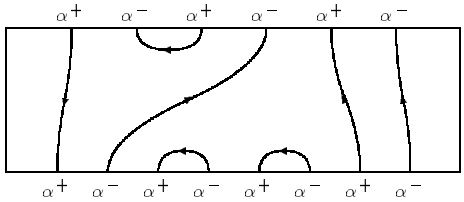
We now consider some special cases.

Example 2

The simplest non-trivial example is for the graph $\mathbf{2} = 0 \to 1$. A path from 0 to 0 must look like

$$0 \to 1 \leftarrow 0 \to 1 \leftarrow 0 \to 1 \leftarrow 0$$

i.e. a number $n \in \mathbf{N}$ of $0 \to 1 \leftarrow 0$. An ($\mathbf{a}$, $\mathbf{b}$)-diagram from $(\alpha^-\alpha^+)^n$ to $(\alpha^-\alpha^+)^m$ must look like:

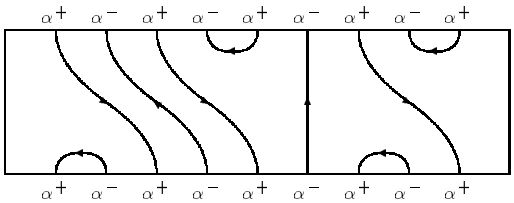


i.e. there are no nested hoops. Furthermore, the region between a down strand and the up strand immediately to its right must involve an α^+ on the bottom and the α^- immediately to its right otherwise the α^- to its right could be joined to nothing. Thus to each α^+, α^- on the top there is associated a unique such pair on the bottom and planarity says that this is an order preserving map $\mathbf{n} \to \mathbf{m}$. Conversely, any order preserving map $\mathbf{n} \to \mathbf{m}$ is realized this way. Indeed if $\varphi : \mathbf{n} \to \mathbf{m}$ is order preserving, then we join the $i^{\text{th}}\,\alpha^+$ on the top to the $\varphi(i)^{\text{th}}\,\alpha^+$ on the bottom and the $\varphi(i)^{\text{th}}\,\alpha^-$ on the bottom to the $j^{\text{th}}\,\alpha^-$ on the top where j is the largest number such that $\varphi(j) = \varphi(i)$. Those not thus joined are joined to the appropriate neighbour. Thus we see that ($\mathbf{a}$, $\mathbf{b}$)-diagrams from $(\alpha^-\alpha^+)^n$ to $(\alpha^-\alpha^+)^n$ are in bijection with order preserving maps $\mathbf{n} \to \mathbf{m}$. So the hom category $\Pi_2 F(\mathbf{2})(0,0)$ is $\mathbf{\Delta}$, the category of finite ordinals.

Paths $0 \to 1$ look like

$$0 \to 1 \leftarrow 0 \to 1 \leftarrow 0 \to 1 \leftarrow 0 \to 1$$

i.e. $\alpha^+(\alpha^-\alpha^+)^n$. An ($\mathbf{a}$, $\mathbf{b}$)-diagram from $\alpha^+(\alpha^-\alpha^+)^n$ to $\alpha^+(\alpha^-\alpha^+)^m$ might look like:



But now the last α^+ on the bottom cannot be paired with anything else on the bottom so must be paired with an α^+ on the top. So with the same argument as above applied to the α^+s, we see that such a diagram corresponds to an order preserving map $\mathbf{n+1} \to \mathbf{m+1}$ which preserves the top element. Thus the hom category $\Pi_2 F(\mathbf{2})(0,1)$ is $\mathbf{\Delta}_*$, the category of non-zero finite ordinals and top preserving morphismes.

By the duality $\Pi_2 F(\mathbf{G})^{co} \cong \Pi_2 F(\mathbf{G}^{op})$ mentioned above, we see that the hom category $\Pi_2 F(\mathbf{2})(1,1)$ is $\mathbf{\Delta^{op}}$ and $\Pi_2 F(\mathbf{2})(1,0)$ is $\mathbf{\Delta_*^{op}}$. Hence we recover the Schanuel-Street description of the free adjoint [11].

Example 3

A related example is for the graph $\mathbf{G}$ with 2 vertices, 0 and 1, and a set C of edges $0 \longrightarrow 1$ (colours). Paths from 0 to 0 will be as in Example 2 but now each node is assigned a colour. The 2-cells are $(\mathbf{a}, \mathbf{b})$-diagrams but now the threads are also coloured, joining nodes of the same colour.

7. Further Research

In the previous section we have seen some examples of the surprisingly rich structure of 2-categories which arise as Π_2 of a free category. Further results in this area and more examples will be discussed in a forthcoming paper.

The $(\mathbf{a}, \mathbf{b})$-diagrams discussed above resemble Kauffman diagrams, but have different properties due to their directedness. If instead one freely adds arrows to a free category which are simultaneously left and right adjoint to the existing arrows, the 2-cells can be described as 'honest' Kauffman diagrams without directions. This construction together with some applications will also be discussed in a forthcoming paper. There we will also consider presentation for 2-cells of Π_2 of an arbitrary category in terms of equivalence classes of diagrams. In the general case, the arcs join and branch (representing composition and factorization), and thus resemble Feynman diagrams rather than Kauffman diagrams.

References

[1] J. Bénabou, Introduction to bicategories, in *Reports of the Midwest Category Seminar*, LNM 40, Springer Verlag, New York, 1967, p. 1-77.

[2] R. J. MacG. Dawson, R. Paré, D. A. Pronk, Undecidability and free adjoints, in *Proceedings of the World Multiconference on Systemics, Cybernetics and Informatics 2001*, Volume XIV, N. Callaos, F. G. Tinetti, J. M. Champarnaud, J. K. Lee (Eds), International Institute of Informatics and Systemics, Orlando, 2001, pp. 156-161

[3] R. J. MacG. Dawson, R. Paré, D. A. Pronk, Undecidability of the Free Adjoint Construction, preprint.

[4] R. J. MacG. Dawson, R. Paré, D. A. Pronk, Spans and free adjoints, preprint.

[5] W. G. Dwyer, D. M. Kan, Calculating Simplicial Localizations, *J.P.A.A.* 18 (1980), p. 17-35.

[6] P. Gabriel, M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer Verlag, New York, 1967.

[7] V. F. R. Jones, A quotient of the affine Hecke algebra in the Brauer algebra, *L'enseignement Mathématique*, 40, 313-334 (1994).

[8] Z. A. Landau, Fuss-Catalan Algebras and Chains of Intermediate Subfactors, *Pacific Journal of Mathematics*, Vol. 197, No. 2, 2001.

[9] D. A. Pronk, Etendues and stacks as bicategories of fractions, *Comp. Math.* 102 (1996), p. 243-303.

[10] D. Quillen, *Homotopical Algebra*, LNM 43, Springer Verlag, New York, 1967.

[11] S. Schanuel, R. Street, The free adjunction, *Cahiers Topologie Géom. Differentielle Catégoriques* 27 (1986), p. 81-83.

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