A variational formulation for PDEs

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Contents

In these notes we give a brief and (hopefully) clear review of what it means to give a variational formulation of a differential equation.

We will mostly focus on giving some motivations and intuitions, and we will claim fundamental results without giving proofs. For the hungry and enthusiastic readers we can refer to the classical books by Brezis [1] and Evans [2].

1 Preliminaries

Consider the following **boundary value problem**: given a function $f \in C^0([a,b])$, find a function $u : [a,b] \to \mathbb{R}$ such that it solves

$$\begin{cases} -u'' + u = f & \text{in } [a,b] \\ u(a) = u(b) = 0. \end{cases} \tag{1}$$

Definition 1. A classic solution (sometimes called **strong solution**) to the problem is a function $u \in C^2([a,b])$ such that it vanishes at the endpoints and pointwise it satisfies the equation $-u''(x) + u(x) = f(x)$.

Let's change approach. Suppose that we formally multiply the differential equation by an arbitrary function $\phi \in C^1([a,b])$ with $\phi(a) = \phi(b) = 0$ and we integrate over the interval $[a,b]$:

$$-\int_a^b u''(x)\phi(x)\mathrm{d}x + \int_a^b u(x)\phi(x)\mathrm{d}x = \int_a^b f(x)\phi(x)\mathrm{d}x. \tag{2}$$

Integrating by parts the first term on the left hand side of the equality, we get (remember that both $\phi(x)$ and $u(x)$ vanish at the endpoints)

$$\int_a^b u'(x)\phi'(x)\mathrm{d}x + \int_a^b u(x)\phi(x)\mathrm{d}x = \int_a^b f(x)\phi(x)\mathrm{d}x \qquad \forall\ \phi \in C^1([a,b]),\ \phi(a) = \phi(b) = 0. \quad (3)$$

We can notice now that for such expression to be well-defined it's enough to require $u \in C^1([a,b])$ (plus the additional vanishing condition at the boundary).

Definition 2. A function $u \in C^1([a,b])$ with $u(a) = u(b) = 0$ is a **weak solution** to the problem (1) if it satisfies (3).

Moreover, the equality (3) makes sense also if we simply require $u, u' \in L^1(a,b)$. The idea is that by expanding the functional space where we're looking for solutions we might get lucky and find the solution we are looking for.

On the other hand, some questions now arise:

1) What is the relationship of u and u' in $L^1(a,b)$? What does it mean to take the "derivative" of a function that belongs to $L^1(a,b)$?

2) What does is mean to evaluate an L^1-function on a point and impose that $u(a) = u(b) = 0$? (a point has measure zero with respect to the Lebesgue measure on $\mathbb{R}$ and L^1-functions are defined almost everywhere)

In general, if we consider a boundary- or initial-value problem for (partial) differential equations, it's really difficult to work in a classical setting and find a classical solution (i.e. functions of regularity $C^k(\overline{\Omega})$, where Ω is the domain of definition). It becomes necessary to weaken some conditions and work in bigger functional spaces where we have a weak (or **variational**) version of the form (3) of the original problem (1).

Eventually, we might even hope that the weak solution that we found could be more regular and we might be able to recover (via powerful theorems) some "classical" properties like continuity, pointwise validity (a.e.), etc.

In the following sections we will try to give a consistent answer to the two questions above, namely

1) the notion of derivatives for L^p-functions and

2) restriction of an L^p-function over a set of measure zero.

2 Distributions

The "Preliminaries" section above gives the motivation for the introduction of new functional spaces called **Sobolev spaces**.

Sobolev spaces can be thought as similar to $C^k(\overline{\Omega})$, i.e. spaces of functions where we impose conditions on their "derivatives" such that if $f \in L^p(\Omega)$, then also $\frac{\partial f}{\partial x_j} \in L^p(\Omega)$

But we still need to define what a derivative means here. To do so, we need to give some results about distributions and we will then be able to define the derivative of an L^p-function in a distributional sense.

From now on we will consider a set $\Omega \subseteq \mathbb{R}^n$ that is open and connected (we call it **domain**). We recall the space of functions

$$C_c^\infty(\Omega) = \{\text{the space of infinitely-differentiable functions with compact support}\} \tag{4}$$

(.i.e. for any $u \in C_c^\infty(\Omega)$ its support $\operatorname{supp} u := \{x \in \mathbb{R}^n \mid u(x) \neq 0\}$ is equal to a compact set $K \subset\subset \Omega$) and we introduce a natural topology on such space.

Definition 3. Let $\{\phi_k\}$ be a sequence of functions in $C_c^\infty(\Omega)$ and $\phi \in C_c^\infty(\Omega)$. We say that $\{\phi_k\}$ converges to ϕ in $C_c^\infty(\Omega)$

$$\phi_k \to \phi \qquad \text{in } C_c^\infty(\Omega)$$

if

1) $\exists\, K \subset\subset \Omega$ (a compact subset $K \subseteq \Omega$) which contains all the supports of ϕ_k $\forall\, k$

2) any derivative converges: $D^\alpha \phi_k \to D^\alpha \phi$ uniformly in Ω for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ (recall that $D^\alpha u := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u$)

Note 4. *We denote the space $C_c^\infty(\Omega)$ with the topology induced by the convergency definition above as $\mathcal{D}(\Omega)$.*

Remark 5. *By definition of convergency, it follows that if $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$, then $D^\alpha \phi_k \to D^\alpha \phi$ in $\mathcal{D}(\Omega)$ for any multi-index α.*

With this topology in mind, we can define the notion of continuous (linear) operator on such space.

Definition 6. An operator $\mathcal{O} : \mathcal{D}(\Omega) \to \mathbb{R}$ is **continuous** if

$$\mathcal{O}(\phi_k) \to \mathcal{O}(\phi) \qquad \text{for any sequence } \{\phi_k\} \text{ converging to } \phi \text{ in } \mathcal{D}(\Omega). \tag{5}$$

Definition 7. A **distribution** is a linear continuous operator on $\mathcal{D}(\Omega)$: $L : \mathcal{D}(\Omega) \to \mathbb{R}$. The space of all distributions is called the **dual space of** $\mathcal{D}(\Omega)$ and it's indicated as $\mathcal{D}'(\Omega)$.

The dual space is again a topological space where we can define the notion of convergency in a natural way.

Definition 8. Let $\{L_k\}$ be a sequence of distributions in $\mathcal{D}'(\Omega)$ and $L \in \mathcal{D}'(\Omega)$. We say that $\{L_k\}$ converges to L in $\mathcal{D}'(\Omega)$

$$L_k \to L \qquad \text{in } \mathcal{D}'(\Omega)$$

if for any $\phi \in \mathcal{D}(\Omega)$ (a "test function")

$$L_k(\phi) = {}_{\mathcal{D}'}\langle L_k, \phi \rangle_{\mathcal{D}} \to {}_{\mathcal{D}'}\langle L, \phi \rangle_{\mathcal{D}} = L(\phi) \tag{6}$$

(the notation $\langle \cdot, \cdot \rangle$ indicates the duality pairing).

Example 1. Consider a function $f \in L^1_{loc}(\Omega)$ (i.e. $f \in L^1(K)$ for any $K \subset\subset \Omega$) and define the functional $Gf : \mathcal{D}(\Omega) \to \mathbb{R}$ as

$$\langle Gf, \phi \rangle := \int_\Omega f(x)\phi(x)\mathrm{d}x \qquad \forall\ \phi \in \mathcal{D}(\Omega). \tag{7}$$

Then Gf is clearly linear and it's a continuous operator; indeed, given a convergent sequence $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$, we have

$$\langle Gf, \phi_k - \phi \rangle = \int_\Omega f(x)\,[\phi_k(x) - \phi(x)]\,\mathrm{d}x \leq \sup_K |\phi_k(x) - \phi(x)|\,\|f\|_{L^1(K)} \to 0 \tag{8}$$

where K is a compact subset of Ω such that $K \supseteq \{\bigcup_k \operatorname{supp}\phi_k \cup \operatorname{supp}\phi\}$.

Therefore, Gf is a distribution and with abuse of notation we can identify the distribution Gf simply with the function f itself.

Proposition 9. *For any $f \in L^1_{loc}(\Omega)$, $f \in \mathcal{D}'(\Omega)$.*

Remark 10. *Given that $L^p(\Omega) \hookrightarrow L^1_{loc}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ for $1 \leq p < +\infty$ (the symbol $\hookrightarrow$ means continuous embedding: $A \hookrightarrow B \Leftrightarrow A \subseteq B$ and $\forall\ v \in A$: $\|v\|_B \leq c\|v\|_A$, for some constant $c \in \mathbb{R}_+$), any function $f \in L^p(\Omega)$ is a distribution.*

Example 2. Suppose that $0 \in \Omega \subseteq \mathbb{R}^n$ and define the functional $\delta_0 : \mathcal{D}(\Omega) \to \mathbb{R}$ as

$$\langle \delta_0, \phi \rangle := \phi(0) \qquad \forall\ \phi \in \mathcal{D}(\Omega). \tag{9}$$

Then δ_0 is linear and continuous (if $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$, then $\phi_k(0) \to \phi(0)$ in $\mathbb{R}$).

We call the distribution δ_0 as Dirac Delta distribution.

Remark 11. *Such distribution cannot be identifiable with any function in any L^p-space.*

2.1 Distributional derivative

Consider as before a domain $\Omega \subseteq \mathbb{R}^n$ (i.e. open, connected set) and assume its boundary $\partial\Omega$ to be sufficiently regular.

If $u \in C^1(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$ (test function), then by Gauss-Green formula we have

$$\int_\Omega \frac{\partial u}{\partial x_j}\phi(x)\mathrm{d}x = -\int_\Omega u(x)\frac{\partial \phi}{\partial x_j}\mathrm{d}x \tag{10}$$

(the boundary term $\int_{\partial\Omega} u(\sigma)\nu_j\phi(\sigma)\mathrm{d}\sigma$ is zero since $\phi \in \mathcal{D}(\Omega)$). We can generalize this expression and extend it to the space of distributions in the following way.

Definition 12. Let $F \in \mathcal{D}'(\Omega)$. We define the **distributional derivative** $\frac{\partial F}{\partial x_j}$ as the distribution

$$_{\mathcal{D}'}\langle \frac{\partial F}{\partial x_j}, \phi \rangle_{\mathcal{D}} := -\ _{\mathcal{D}'}\langle F, \frac{\partial \phi}{\partial x_j} \rangle_{\mathcal{D}} \qquad \forall\ \phi \in \mathbb{D}(\Omega).$$

Remark 13.

- *If $u \in C^1(\Omega)$, then the classical derivative $\frac{\partial u}{\partial x_j} \in C^0(\Omega)$ coincides with the distributional one.*

- *A distributional derivative can always be defined because* $-\,_{\mathcal{D}'}\langle F, \frac{\partial \phi}{\partial x_j}\rangle_{\mathcal{D}}$ *is a linear operator on* $\mathcal{D}(\Omega)$ *and it is also continuous: if* $\phi_k \to \phi$ *in* $\mathcal{D}(\Omega)$*, then* $D^\alpha \phi_k \to D^\alpha \phi$ *in* $\mathcal{D}(\Omega)$ *and therefore* $-\langle F, \frac{\partial \phi_k}{\partial x_j}\rangle \to -\langle F, \frac{\partial \phi}{\partial x_j}\rangle$ *in* $\mathbb{R}$*.*

- *Every distribution admits infinitely many (distributional) derivatives and the following holds*

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial x_i}\right) \tag{11}$$

(same for higher order derivatives).

Example 1. Consider the Heaviside function

$$H(x) = \begin{cases} 0 & -1 \le x \le 0 \\ 1 & 0 < x \le 1 \end{cases} \tag{12}$$

$H \in L^1_{loc}(-1,1) \hookrightarrow \mathcal{D}'(-1,1)$. If we calculate its classical derivative we have $H'(x) \equiv 0$ on $[-1,1]\backslash\{0\}$.

On the other hand, if we calculate its distributional derivative, we have by definition

$$\langle H', \phi\rangle := -\langle H, \phi'\rangle = -\int_{-1}^{1} H(x)\phi'(x)\mathrm{d}x = -\int_0^1 \phi'(x)\mathrm{d}x = \phi(0) - \phi(1) = \phi(0) \tag{13}$$

$\forall\ \phi \in \mathcal{D}(-1,1)$; therefore, $H' = \delta_0 \in \mathcal{D}'(\Omega)$ in the distributional sense.

Example 2. Consider the function $u \in C^0(0,2)$

$$u(x) = \begin{cases} x & 0 < x \le 1 \\ 1 & 1 < x < 2 \end{cases} \tag{14}$$

then its distributional derivative is

$$\langle u', \phi\rangle := -\langle u, \phi'\rangle = -\int_0^2 u(x)\phi'(x)\mathrm{d}x = -\int_0^1 x\phi'(x)\mathrm{d}x - \int_1^2 \phi'(x)$$
$$= -\left[x\phi(x)\right]_0^1 + \int_0^1 \phi(x)\mathrm{d}x - \phi(2) + \phi(1) = -\phi(1) + \int_0^1 x\phi'(x)\mathrm{d}x + \phi(1) = \int_0^1 x\phi'(x)\mathrm{d}x \tag{15}$$

$\forall\ \phi \in \mathcal{D}(0,2)$; thus $u' \in \mathcal{D}'(0,2)$ is defined as

$$u'(x) = \begin{cases} 1 & 0 < x \le 1 \\ 0 & 1 < x < 2. \end{cases} \tag{16}$$

In this case the distributional derivative is identifiable with a function in $L^1_{loc}(0,2)$.

Example 3. Consider the function $v \in \mathcal{D}'(0,2)$

$$v(x) = \begin{cases} x & 0 < x \leq 1 \\ 2 & 1 < x < 2 \end{cases} \tag{17}$$

then its distributional derivative is

$$\langle v', \phi \rangle := -\langle v, \phi' \rangle = -\int_0^2 v(x)\phi'(x)\mathrm{d}x = -\int_0^1 x\phi'(x)\mathrm{d}x - \int_1^2 2\phi'(x)$$
$$= -\left[x\phi(x)\right]_0^1 + \int_0^1 \phi(x)\mathrm{d}x - 2\phi(2) + 2\phi(1) = -\phi(1) + \int_0^1 x\phi'(x)\mathrm{d}x + 2\phi(1) = \int_0^1 x\phi'(x)\mathrm{d}x + \phi(1) \tag{18}$$

$\forall\ \phi \in \mathcal{D}(0,2)$; thus $v' \in \mathcal{D}'(0,2)$ is defined as

$$v'(x) = u' + \delta_1 \tag{19}$$

where $u' \in L^1_{loc}(0,2)$ has been defined in the previous example.

3 Sobolev spaces

We can now give a definition of Sobolev spaces. From now on we will only consider domains $\Omega \subset \mathbb{R}^n$ which are bounded (unless otherwise stated), for the sake of simplicity.

Definition 14. Let Ω be a bounded domain in $\mathbb{R}^n$. For any $p \in [1, +\infty]$ we define

$$\begin{aligned} W^{1,p} &:= \left\{ u \in L^p(\Omega) \,\middle|\, \frac{\partial u}{\partial x_j} \in L^p(\Omega),\ \forall\ j = 1, \dots, n \right\} \\ &= \{ u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega),\ \forall \text{ multi-index } |\alpha| = 1 \} \end{aligned} \tag{20}$$

The definition above can be generalized to the case of derivative of higher order.

Definition 15. Let Ω be a domain in $\mathbb{R}^n$. For any $p \in [1, +\infty]$, $k \in \mathbb{N}$ we define

$$W^{k,p} := \{ u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega),\ \forall \text{ multi-index } |\alpha| \leq k \} \tag{21}$$

Remark 16. *Functions that belong to a Sobolev space ($\forall\ p, k$) have distributional derivatives which admits a representation as a L^1_{loc}-function (there are no delta distributions).*

The case $p = 2$ deserves special attention, as it will be clear a few lines below, and it has its own definition.

Definition 17. Let Ω be a domain in $\mathbb{R}^n$. For any $k \in \mathbb{N}$ we define

$$H^k(\Omega) = \left\{ u \in L^2(\Omega) \,\middle|\, D^\alpha u \in L^2(\Omega),\ \forall \text{ multi-index } |\alpha| \leq k \right\} \tag{22}$$

We will now state some of the main properties of the Sobolev spaces. We introduce a norm in such spaces as

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \qquad \forall\, p \in [1, +\infty) \tag{23}$$

and

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)} \qquad p = +\infty \tag{24}$$

In other words, the norm of a function $u \in W^{k.p}(\Omega)$ is the sum of the norm of u and the norms of all its derivatives in $L^p(\Omega)$.

For the particular case of $H^k(\Omega)$ we can define an inner product as

$$(u, v)_{H^k(\Omega)} := \sum_{0\leq\alpha\leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} = \sum_{0\leq\alpha\leq k} \int_\Omega D^\alpha u(x) D^\alpha v(x) \mathrm{d}x \tag{25}$$

which automatically induces a norm

$$\|u\|_{H^k(\Omega)} := \left(\sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{26}$$

The following theorem holds

Theorem 18.

- *The space $W^{k,p}(\Omega)$ with the norm defined above is a Banach space for any $p \in [1, +\infty]$.*
- *In particular, $W^{k,p}(\Omega)$ is reflexive for any $p \in (1, +\infty)$ and separable for any $p \in [1, +\infty)$.*
- *The space $H^k(\Omega)$ with the inner product defined above is a separable Hilbert space.*

At this point, one might wonder whether functions in $W^{k,p}(\Omega)$ might enjoy further regularity properties.

We may argue that, for example, if p is a high value, then the functions decay more rapidly, or if k is high, then the functions admits more "derivatives" and should be more regular.

On the other hand, we can intuitively guess that if we are in high dimension (big value of n), then there are more degrees of freedom and we might gain less regularity.

There exist indeed a relation between these parameters and the regularity of the functions in the Sobolev space $W^{k,p}(\Omega)$, when $\Omega \subseteq \mathbb{R}^n$. All the results are known as "Sobolev embedding theorems".

Theorem 19 (Sobolev embedding Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficient regularity along its boundary $\partial\Omega$.*

- *If $kp < n$, then*
$$W^{k,p}(\Omega) \subseteq L^q(\Omega) \qquad \forall\ q \in [1, p^*],\ p^* = \frac{pn}{n-kp}$$
and the inclusion is continuous for any $q \in [1, p^*]$, i.e.
$$\|u\|_{L^q(\Omega)} \leq c\, \|u\|_{W^{k,p}(\Omega)}\,, \tag{27}$$
and compact for any $q \in [1, p^*)$, i.e.
$$\text{for any bounded sequence } \{v_k\} \subseteq W^{k,p}(\Omega),\ \exists\ \left\{v_{k_j}\right\} \text{ such that } v_{k_j} \to v \text{ in } L^q(\Omega). \tag{28}$$

- *If $kp = n$, then*
$$W^{k,p}(\Omega) \subseteq L^q(\Omega) \qquad \forall\ q \in [1, +\infty)$$
and the inclusion is continuous and compact for all $q \in [1, +\infty)$.

- *If $kp > n$, then*
$$W^{k,p}(\Omega) \subseteq C^m(\overline{\Omega}) \qquad \text{with } m = \left[k - \frac{n}{p}\right]$$
(the symbol $[\cdot]$ denotes here the integer part) and the inclusion is continuous and compact.

Remark 20. *The third bullet can be re-stated in the following way: if $u \in W^{k,p}(\Omega)$ (with $kp > n$), then there exists a function $\tilde{u} \in C^m(\overline{\Omega})$ such that $u = \tilde{u}$ almost everywhere.*

Example: the Hilbert space $H^1(\Omega)$. We recall that
$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \,\middle|\, \frac{\partial u}{\partial x_j} \in L^2(\Omega),\ \forall\ j = 1, \ldots, k \right\} \tag{29}$$

1) If $\Omega \subset \mathbb{R}$ (an interval), then $H^1(\Omega) = W^{1,2}(\Omega) \subseteq C^0(\overline{\Omega})$. Therefore, any function in $H^1(\Omega)$ admits a continuous representative, but we have no information about the derivatives.

2) If $\Omega \subset \mathbb{R}^2$ (a subset of the 2D-plane), then $H^1(\Omega) = W^{1,2}(\Omega) \subseteq L^p(\Omega)$ for all $p \in [1, +\infty)$, but $u \in H^1(\Omega)$ might not be bounded.

   For example, consider the unit disk $\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\right\}$ and the function $u(x) = \left[-\log(x^2 + y^2)\right]^{\frac{1}{4}}$ defined on it. It can be proven that $u \in H^1(\Omega)$, but u is clearly not bounded in a neighbourhood of zero.

3) If $\Omega \subset \mathbb{R}^3$ (a portion of the 3D-space), then $H^1(\Omega) = W^{1,2}(\Omega) \subseteq L^p(\Omega)$ where the range of p is more limited: $p \in [1, p^*]$, with $p^* = \frac{pn}{n-kp} = 6$.

Remark 21. *More generally, it follows from Sobolev embedding Theorem that $H^1(\Omega) \subseteq L^2(\Omega)$ (with continuous embedding) for any dimension of the space $\mathbb{R}^n$.*

Indeed, we saw already that this is true for the cases $n = 1, 2, 3$; for $n \geq 4$ we have that $H^1(\Omega) \subseteq L^p(\Omega)$ with $p \in [1, p^*]$, but $p^* = \frac{2n}{n-2} > 2$ for any n, therefore (Ω is bounded) $H^1(\Omega) \subseteq L^{p^*}(\Omega) \subseteq L^2(\Omega)$.

4 Density and Trace

With the previous section we gave an answer to the first question stated in the Preliminaries. We need to tackle now a way to define consistently what does it mean to restrict a function (that is defined almost everywhere) on a set of measure zero.

We will first start with giving some density results. We recall that

Proposition 22. *$C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $p \in [1, +\infty)$.*

Is such space also dense in a Sobolev space $W^{k,p}(\Omega)$ for some k, p? The short answer is that the density of $C_c^\infty(\Omega)$ depends on the domain Ω.

The motivation for studying density properties of the space $C_c^\infty(\Omega)$ into Sobolev spaces comes from the fact that if indeed $\overline{(C_c^\infty(\Omega))}^{\|\cdot\|_{k,p}} = W^{k,p}(\Omega)$, then every function $u \in W^{k,p}(\Omega)$ can be approximated by a sequence of infinitely-differentiable functions for which the operation of restriction over the boundary of Ω (or over any set of $\mathbb{R}^n$-dimension equal zero) is well defined. By a limit process, we can then extend the notion of restriction to any function in $W^{k,p}(\Omega)$.

Definition 23. Given a bounded domain Ω (with sufficient boundary regularity), for any function $u : \Omega \to \mathbb{R}$ the operation of restriction over the boundary of Ω (denoted as $u|_{\partial\Omega} : \partial\Omega \to \mathbb{R}$) is called **trace**.

The following results hold.

Theorem 24. *$C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for any $p \in [1, +\infty)$, for any k.*

However, in the case of a bounded domain Ω we only achieve a weaker result.

Theorem 25. *Given a bounded domain Ω, sufficiently regular on the boundary, $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for any $p \in [1, +\infty)$.*

The space $C^\infty(\overline{\Omega})$ on the other hand is not our space of test functions and indeed in general we have that $\overline{(C_c^\infty(\Omega))}^{\|\cdot\|_{k,p}} \subsetneq W^{k,p}(\Omega)$ (i.e. $C_c^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$).

Definition 26. The closure of the space $C_c^\infty(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is a subset of $W^{k,p}(\Omega)$ and it is denoted as

$$W_0^{k,p}(\Omega) := \overline{(C_c^\infty(\Omega))}^{\|\cdot\|_{k,p}} \quad \text{for } p \in [1, +\infty).$$

Proposition 27. *For any $p \in [1, +\infty)$, the space $W_0^{k,p}(\Omega)$ is a separable Banach space with respect to the induced norm $\|\cdot\|_{W^{k,p}(\Omega)}$. Furthermore, it is reflexive for $p \in (1, +\infty)$.*

For studying solutions of PDEs, it is usually more convenient to work in the Hilbert spaces $H^k(\Omega)$. We denote

$$H_0^k(\Omega) := \overline{(C_c^\infty(\Omega))}^{(\cdot,\cdot)_{H^k(\Omega)}}$$

It is also possible to prove that $H_0^k(\Omega)$ is a Hilbert space with respect to the induced inner product.

In particular, we have the following results for the space $H_0^1(\Omega)$ (recall that $H^1(\Omega) = \{u \in L^2(\Omega) \mid \partial_j u \in L^2(\Omega), j = 1, \ldots, n\}$).

Theorem 28. *Given a bounded domain $\Omega \subset \mathbb{R}^n$ sufficiently smooth, then*

- $H_0^1(\Omega)$ *is dense in* $L^2(\Omega)$
- $u \in H_0^1(\Omega) \Leftrightarrow u = 0$ *on* $\partial\Omega$.

Remark 29. *u is not necessarily continuous on $\partial\Omega$ or even defined everywhere, but the expression "$u = 0$ on $\partial\Omega$" is claiming that $u|_{\partial\Omega} = 0$ almost everywhere with respect to the $(n-1)$-dimensional Lebesgue measure on $\partial\Omega$.*

We are almost able to define what a restriction on $\partial\Omega$ means for a general function belonging to a Sobolev space. We will focus here only on the space $H^1(\Omega)$ and its subset $H_0^1(\Omega)$.

Suppose that $u \in H^1(\Omega)$ (i.e. $u, Du \in L^2(\Omega)$). Then if we want non-zero boundary conditions, say $u|_{\partial\Omega} = g$ where g is a given function, we need to sacrifice some regularity (to be precise, we loose "half" derivative) when we apply the restriction operation.

Definition 30. Given abounded domain $\Omega \subset \mathbb{R}^n$ and $\partial\Omega =: \Gamma \subset \mathbb{R}^{n-1}$, we define

$$H^{\frac{1}{2}}(\Gamma) := \left\{ u \in L^2(\Gamma) \;\middle|\; \frac{|u(x) - u(y)|}{|x-y|^{\frac{1}{2}+\frac{n}{2}}} \in L^2(\Gamma \times \Gamma) \right\}.$$

Remark 31. *It is possible to define fractional Sobolev spaces $H^\theta(\Gamma)$ for $0 < \theta < 1$ on the regular manifold $\Gamma = \partial\Omega$ by using local charts (diffeomorphism).*

We can now conclude by stating one version of the Trace Theorem that guarantees that the restriction operation for $H^1(\Omega)$-functions is well-defined.

Theorem 32 (Trace Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, sufficiently smooth. Then, there exists the* **trace operator**

$$\begin{aligned} \gamma_0 : H^1(\Omega) &\to H^{\frac{1}{2}}(\Gamma) \\ u &\mapsto u|_\Gamma =: \gamma_0 u \end{aligned} \tag{30}$$

with the following properties

1) γ_0 *is a linear, bounded, continuous and surjective operator*

2) $\ker \gamma_0 = H_0^1(\Omega)$

3) Green's formula holds: $\forall\, u, v \in H^1(\Omega)$

$$\int_\Omega \frac{\partial u}{\partial x_j} v \,\mathrm{d}x = \int_\Gamma (\gamma_0 u)\,(\gamma_0 v)\,\nu_j \,\mathrm{d}\sigma - \int_\Omega u \frac{\partial v}{\partial x_j} \,\mathrm{d}x$$

where ν_j is the j-th component of the outer normal $\vec{\nu}$ and $\mathrm{d}\sigma$ is the Lebesgue measure on $\Gamma \subset \mathbb{R}^{n-1}$.

We state one last Trace Theorem that is fundamental for studying well-known PDEs of second order (wave equation, heat equation, etc.)

Theorem 33 (**Trace Theorem**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, sufficiently smooth. Then, there exists the* **trace operator**

$$\begin{aligned} \gamma : H^2(\Omega) &\to H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \\ u &\mapsto \left(u|_\Gamma, \left.\frac{\partial u}{\partial \vec{\nu}}\right|_\Gamma \right) =: \gamma u \end{aligned} \tag{31}$$

with the following properties

1) γ_0 *is a linear, bounded, continuous and surjective operator*

2) $\ker \gamma = H_0^2(\Omega)$

3) Green's formula holds: $\forall\ u, v \in H^2(\Omega)$

$$\int_\Omega \Delta u \, v \, \mathrm{d}x = \int_\Gamma \frac{\partial u}{\partial \vec{\nu}} v \, \mathrm{d}\sigma - \int_\Omega \nabla u \nabla v \, \mathrm{d}x$$

where ν_j *is the* j*-th component of the outer normal* $\vec{\nu}$ *and* $\mathrm{d}\sigma$ *is the Lebesgue measure on* $\Gamma \subset \mathbb{R}^{n-1}$.

References

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