Curve flows and soliton equations: the vortex filament case

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Abstract

The Vortex Filament Equation, describing the self-induced motion of a vortex filament in an ideal fluid, is a simple but important example of integrable curve dynamics. These notes are a short introduction about VFE and the goal is to show its connection with the nonlinear Schrödinger equation through the Hasimoto map. This is just the starting point for a wide spectrum of research which is meant to study curve flows in $\mathbb{R}^3$ using powerful results borrowed from soliton theory.

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1 Physical (fluid dynamics) setting: vortices

Let's consider a perfect fluid (incompressible, non-viscous) filling an unbounded domain $\Omega \subseteq \mathbb{R}^3$.

The **vorticity** of a fluid is the rotational tendency of the fluid. Mathematically, the vorticity is defined as

$$\omega = \nabla \times \mathbf{v}$$

If the curl vanishes throughout the domain Ω, no particle of the fluid has a rotational component, and the fluid is called **irrotational**.

A **vortex tube** is a tubular region of the fluid that has a much higher vorticity than that of the surrounding fluid. Common examples are smoke rings and whirlpools.

We will consider a slender vortex tube (with vanishingly small diameter) and approximate it with a space curve γ representing the position of the core of the vortex. When the vorticity inside the tube is very large (infinite vorticity)

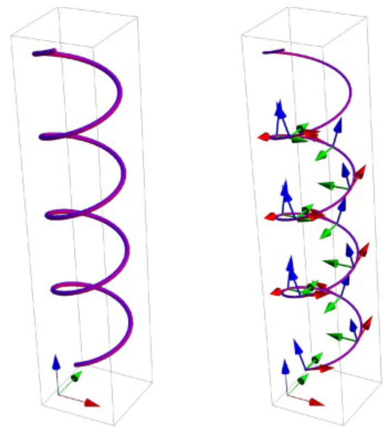
$$\omega = \nabla \times \mathbf{v} = \delta_\gamma(\mathbf{x}),$$

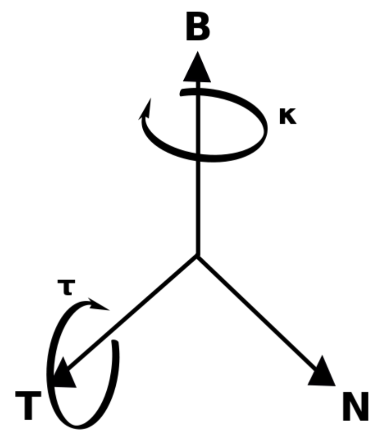
the vortex tube is called a **vortex filament**.

For a more detailed description and derivation of the physical model, we refer to [2, Section 1.1].

2 Mathematical setting: curves and solitons

Curves. A curve in $\mathbb{R}^3$ is a vector-valued function $\gamma(s,t) \in \mathbb{R}^3$ where s is the arclength parameter and t is the time parameter.


(a) (b)



To every point of the curve we can associate the Frenet frame:

$$
\begin{aligned}
\text{Tangent} \qquad & \mathbf{T} := \frac{\mathrm{d}\gamma}{\mathrm{d}s} \\
\text{Normal} \qquad & \kappa\ \mathbf{N} := \frac{\mathrm{d}\ \mathbf{T}}{\mathrm{d}s} \\
\text{Binormal} \qquad & \mathbf{B} :=\ \mathbf{T} \times\ \mathbf{N}
\end{aligned}
$$

with $\kappa = \kappa(s,t)$ the curvature function of γ.

The behaviour of the curve γ is described by the Frenet-Serret equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}$$

with $\tau = \tau(s,t)$ the torsion of the curve.

Solitons. Solitons are solitary wave (localized travelling waves) solution of a nonlinear dispersive wave equation (e.g. KdV: $u_t + u_{xxx} + 6uu_x = 0$).

Their shape and velocity is constant in time (also their total energy) and their mutual interaction is elastic, meaning that when two solitons interact their shape and velocity remain unchanged (but the solution picks up a phase shift). As it was originally noticed in the numerical experiment by Zabusky and Kruskal ('65), they "survive" collisions, despite lack of superposition principle.

Nonlinear integrable equations (giving rise to solitons) can be effectively described by the Lax formalism:

$$\mathcal{L}\left(x,t,u,u_x,u_t,\ldots\right) = 0 \qquad \Leftrightarrow \qquad \begin{cases}\Psi_x = A\Psi\\ \Psi_t = B\Psi\end{cases} \qquad \text{such that } A_t - B_x + [A,B] = 0$$

where $A = A(s,t,u;\lambda)$ and $B = B(s,t,u;\lambda)$ are 2×2 matrices (λ being the spectral parameter).

They can be solved through the (Direct and Inverse) Scattering methods, in particular with the use of Riemann–Hilbert techniques. Other properties (stability, dynamics, asymptotics...) can also be derived from the Riemann–Hilbert problems associated to such equations.

What is a RH problem: Given a set of oriented contours Σ in the complex plane, find a (matrix-valued) function X such that:

1. X is holomorphic in $\mathbb{C} \setminus \Sigma$;
2. jump condition: there exists (finite) the limit of X as λ approaches the contours $X_{\pm}(\lambda)$ such that

$$X_+(\lambda) = X_-(\lambda)J(\lambda) \qquad \lambda \in \Sigma;$$

3. normalization at infinity:

$$X(\lambda) = I + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \lambda \to \infty.$$

Remark 1. *Explicit solutions are extremely rare!*

3 The Vortex Filament Equation (VFE)

The first mathematical model of the evolution of a vortex filament in an ideal fluid was derived in 1906 by Luigi S. Da Rios. The PDE equation reads

$$\gamma_t = \gamma_s \times \gamma_{ss}$$

i.e.

$$\gamma_t = \mathbf{T} \times \kappa \mathbf{N} = \kappa \mathbf{B}$$

in the Frenet-Serret frame.

This implies that, assuming that the curvature never vanishes, the vortex filament curve moves in time in the direction of the binormal.

The simplest non-trivial solutions of the VFE are

- a straight line $\gamma(s,t) = (0,0,s)$ (stationary solution);
- a circular filament with periodic boundary conditions

$$\gamma(s,t) = \left(\frac{1}{\kappa}\cos(\kappa s), \frac{1}{\kappa}\sin(\kappa s), \kappa t\right).$$

with κ the constant curvature (κ^{-1} is the radius of the circle); the binormal vector $\mathbf{B}$ is the constant unit vector $\mathbf{e}_3$, perpendicular to the (x,y)-plane containing the circular filament.

Thus, a planar circle moves in time under the VFE in the direction perpendicular to its osculating plane at speed κ: the smaller the radius of the circle, the faster the filament will travel through the fluid.

We state here a few properties of the VFE.

Theorem 2. *If a curve γ is a vortex filament at $t = 0$, then it remains so for all times (provided it moves in a perfect fluid).*

Theorem 3 (**Helmholtz's second theorem**). *A vortex filament cannot end in a fluid; it must extend to the boundaries of the fluid or form a closed path.*

Theorem 4. *The VFE is a locally arc-length-preserving vector field, i.e.*

$$\frac{\partial}{\partial t}\left(\|\gamma_s\|^2\right) = 0$$

Proof.

$$\frac{\partial}{\partial t}\left(\|\gamma_s\|^2\right) = 2\gamma_s \cdot \gamma_{st} = 2\gamma_s \cdot (\gamma_s \times \gamma_{ss})_s = 2\mathbf{T} \times (\kappa \mathbf{B})_s = 2\kappa_2 \mathbf{T} \times \mathbf{B} - 2\kappa\tau \mathbf{T} \times \mathbf{N} = 0$$

□

Remark 5. *The VFE is actually a Hamiltonian system with Hamiltonian*

$$H(\gamma) = \int_\gamma \|\gamma_s\| \, \mathrm{d}s$$

i.e. the length of the curve.

As a consequence, the total length of the vortex filament is invariant during the evolution. Furthermore, the local preservation property implies that the vortex filament moves in time without stretching and the variables s and t are independent from each other.

3.1 VFE (2D case)

Theorem 6. *γ is a solution to the Planar Filament Equation (PFE)*

$$\gamma_t = \frac{\kappa^2}{2}\boldsymbol{T} + \kappa_s \boldsymbol{N},$$

if and only if κ is a solution to the modified KdV equation

$$\kappa_t = \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss}$$

Proof. Consider the relation $\kappa^2 = \|\mathbf{T}_s\|^2 = \gamma_{ss} \cdot \gamma_{ss}$ and take the derivative w.r.t. time:

$$2\kappa\kappa_t = 2\gamma_{ss} \cdot \gamma_{sst} = 2(\kappa \mathbf{N}) \cdot \left(\frac{\kappa^2}{2}\mathbf{T} + \kappa_s \mathbf{N}\right)_{ss} = 3\kappa^3\kappa_s + 2\kappa\kappa_{sss}$$

where we used the Frenet equations in 2-dimensions. □

Remark 7. *Notice that the MKdV equation is the second equation of the NLS hierarchy.*

3.2 VFE (3D case)

Theorem 8 (Hasimoto, '72)**.** *γ is a solution to the VFE*

$$\gamma_t = \gamma_s \times \gamma_{ss}$$

if and only if the function

$$\psi_\gamma(s,t) = \kappa \exp\left\{ i \int^s \tau(x) \mathrm{d}x \right\}$$

solves the (focusing) nonlinear Schrödinger equation (NLS)

$$i\psi_t = -\psi_{ss} - \frac{1}{2} |\psi|^2 \psi$$

(the map $\mathcal{H} : \gamma \mapsto \psi_\gamma$ is the Hasimoto map).

3.2.1 Sketch of the proof

Introduce a new frame, called **Relatively Parallel Frame**:

$$\langle\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}\ \rangle$$

with

$$\begin{cases} \mathbf{T} = \mathbf{T} \\ \mathbf{U} = \cos\theta\ \mathbf{N} - \sin\theta\ \mathbf{B} \\ \mathbf{V} = \sin\theta\ \mathbf{N} + \cos\theta\ \mathbf{B} \end{cases}$$

(this corresponds to a rotation of the normal and binormal vectors). To uniquely identify **U** and **V**, we impose that their s-derivative are proportional to the tangent vector **T**.

It follows that the right rotational angle θ is such that $\partial_s \theta = \tau$, meaning

$$\theta = \int^s \tau(\eta, t)\, \mathrm{d}\eta$$

and the Frenet-Serret equations become

$$\frac{\mathrm{d}}{\mathrm{d}s} \begin{bmatrix} \mathbf{T} \\ \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{U} \\ \mathbf{V} \end{bmatrix}$$

where $\kappa_1 = \kappa \cos\theta$, $\kappa_2 = \kappa \sin\theta$ are called "natural curvatures" (these equations are sometimes called Darboux equations).

Lemma 9. *Given the curve γ in the Relatively Parallel Frame, then*

$$\psi_\gamma = \kappa e^{i \int^s \tau(x)\, \mathrm{d}x} = \kappa_1 + i\kappa_2.$$

Lemma 10. *The VFE in the Relatively Parallel Frame reads*

$$\gamma_t = -\kappa_2 \boldsymbol{U} + \kappa_1 \boldsymbol{V}.$$

Given the VFE in the new frame

$$\gamma_t = -\kappa_2\ \mathbf{U} + \kappa_1\ \mathbf{V}$$
$$\gamma_{ss} = \kappa_1\ \mathbf{U} + \kappa_2\ \mathbf{V}$$

the compatibility condition $(\gamma_{ss})_t = (\gamma_t)_{ss}$ yields

$$(\gamma_t)_{ss} = -\kappa_{2;ss}\mathbf{U} + \kappa_{1;ss}\mathbf{V} + (\kappa_1\kappa_{2;s} - \kappa_2\kappa_{1;s})\,\mathbf{T}$$
$$(\gamma_{ss})_t = \kappa_{1;t}\mathbf{U} + \kappa_{2;t}\mathbf{V} + \kappa_1\mathbf{U}_t + \kappa_2\mathbf{V}_t$$

On the other hand, we notice that

1. taking the s-derivative of the VFE, we have

$$\mathbf{T}_t = -\kappa_{2;s}\mathbf{U} + \kappa_{1;s}\mathbf{V};$$

2. since $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$ are mutually orthogonal, then $\mathbf{U}_t \cdot \mathbf{T} = -\mathbf{U} \cdot \mathbf{T}_t$ and $\mathbf{U}_t \cdot \mathbf{V} = -\mathbf{U} \cdot \mathbf{V}_t$;

3. using the previous two points, we have

$$(\mathbf{U}_t \cdot \mathbf{V})_s = -\kappa_1\kappa_{1;s} - \kappa_2\kappa_{2;s} = -\frac{1}{2}\left(\kappa_1^2 + \kappa_2^2\right)_s$$

   meaning

$$\mathbf{U}_t \cdot \mathbf{V} = -\frac{1}{2}\left(\kappa_1^2 + \kappa_2^2\right) + A(t)$$

   where $A(t)$ is a real-valued function of t.

Getting bak to our derivatives $(\gamma_t)_{ss} = (\gamma_{ss})_t$ and projecting them over the vectors $\mathbf{U}$ and $\mathbf{V}$, we obtain

$$\begin{cases} -k_{2;ss} = \kappa_{1;t} + \kappa_2\left[\frac{1}{2}\left(\kappa_1^2 + \kappa_2^2\right) - A(t)\right] \\ k_{1;ss} = \kappa_{2;t} - \kappa_1\left[\frac{1}{2}\left(\kappa_1^2 + \kappa_2^2\right) - A(t)\right] \end{cases}$$

where we used the fact that $\mathbf{U}_t \cdot \mathbf{U} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{U}\|^2 \equiv 0$ ($\mathbf{U}$ is a unit vector at all times).

Therefore, if we consider the quantity

$$\psi = \kappa_1 + i\kappa_2$$

the equations above are simply the real and imaginary part of the PDE for ψ, which reads

$$\frac{1}{i}\psi_t = \psi_{ss} + \psi\left(\frac{1}{2}|\psi|^2 + A(t)\right)$$

The function $A(t)$ can be easily eliminated from the PDE by the gauge transformation

$$\psi \mapsto \psi \exp\left\{-i\int A(z)\mathrm{d}z\right\}.$$

4 Geometric interpretation of the VFE

The idea is that the NLS and the VFE are the same Hamiltonian systems written in different Poisson structure.

4.1 Connection with $\mathfrak{su}(2)$

Consider the Lie algebra $\mathfrak{su}(2) = \langle E_1, E_2, E_3 \rangle$ of skew-Hermitian (i.e. $M^* = -M$) 2×2 matrices where

$$E_1 = i\sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad E_2 = i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad E_3 = i\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

(σ_i's are the Pauli matrices) with structure constants

$$[E_1, E_2] = -2E_3, \qquad [E_2, E_3] = -2E_1, \qquad [E_3, E_1] = -2E_2$$

and endowed with scalar product and vector product

$$(A, B) = -\frac{1}{2}\operatorname{Tr}(AB), \qquad A \wedge B = -\frac{1}{2}[A, B].$$

We can "translate" a curve $\gamma \subset \mathbb{R}^3$ into a curve in $\mathfrak{su}(2)$ via the isometry

$$\begin{aligned} \mathcal{I} : \left(\mathbb{R}^3, \cdot, \times\right) \quad &\longrightarrow \quad \left(\mathfrak{su}(2), (\cdot,\cdot), \wedge\right) \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad &\mapsto \quad \sum_{i=1}^{3} x_i E_i = \begin{bmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{bmatrix} \end{aligned}$$

Thanks to the transitivity action of $SU(2)$ (the Special Unitary Group, i.e. the Lie group of unitary matrices $U^*U = I$ with determinant $= 1$) on $\mathfrak{su}(2)$, $\exists\ \Phi_0 \in SU(2)$ such that

$$\begin{aligned} \mathcal{I}(\mathbf{T}) &= T = \Phi_0^{-1} E_1 \Phi_0 \\ \mathcal{I}(\mathbf{U}) &= U = \Phi_0^{-1} E_2 \Phi_0 \\ \mathcal{I}(\mathbf{V}) &= V = \Phi_0^{-1} E_3 \Phi_0 \end{aligned}$$

The Darboux equations in the $\mathfrak{su}(2)$ representation become

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi_0 = \begin{bmatrix} 0 & i(\kappa_2 - i\kappa_1) \\ i(\kappa_2 + i\kappa_1) & 0 \end{bmatrix} \Phi_0 = \begin{bmatrix} 0 & i\overline{\psi} \\ i\psi & 0 \end{bmatrix} \Phi_0$$

We recognize the 1st equation of the NLS Lax pair when $\lambda = 0$:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}s} = \begin{bmatrix} i\lambda & i\overline{\psi} \\ i\psi & -i\lambda \end{bmatrix} \Phi \qquad \lambda \in \mathbb{R}$$

with prescribed initial conditions $\Phi(0, 0; \lambda) = I_{SU(2)}$ and $\Phi(s, t; 0) = \Phi_0$.

4.2 From NLS to VFE

Given $\psi(s,t)$ solution to the NLS equation

$$i\psi_t = -\psi_{ss} - 2\,|\psi|^2\,\psi,$$

(this can be deduced from the previous version of the NLS equation by the rescaling $\psi \mapsto \frac{1}{2}\psi$) with associate the Lax pair

$$\frac{\mathrm{d}\Phi}{\mathrm{d}s} = A\Phi = \left(i\lambda\sigma_3 + \begin{bmatrix} 0 & i\overline{\psi} \\ i\psi & 0 \end{bmatrix}\right)\Phi$$
$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = B\Phi = \left((2i\lambda^2 - i\,|\psi|^2)\sigma_3 + \begin{bmatrix} 0 & 2i\lambda\overline{\psi} + \psi_s \\ 2i\lambda\psi - \psi_s & 0 \end{bmatrix}\right)\Phi$$

we can use standard techniques borrowed from spectral analysis (solve the AKNS spectral problem) to find the fundamental solution matrix Φ (Jost solution).

Theorem 11. *Define*

$$\Gamma(s,t) = \Phi^{-1}\big|_{\lambda=0}\, \frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \in \mathfrak{su}(2).$$

Then, Γ is a curve in $\mathfrak{su}(2)$ with tangent vector $\Gamma_s := \frac{\mathrm{d}\Gamma}{\mathrm{d}s}$ equal to

$$\Gamma_s = \Phi_0^{-1}\,\frac{\mathrm{d}A}{\mathrm{d}\lambda}\bigg|_{\lambda=0}\Phi_0 = \Phi_0^{-1}E_1\Phi_0 = T.$$

Moreover, the time evolution of the curve Γ is

$$\Gamma_t = \Phi_0^{-1}\,\frac{\mathrm{d}B}{\mathrm{d}\lambda}\bigg|_{\lambda=0}\Phi_0 = -\kappa_2 U + \kappa_1 V.$$

where we can already recognize the VFE in the Relatively Parallel Frame.

Remark 12. *It is a matter of straightforward calculations to also obtain the VFE (in $\mathfrak{su}(2)$):*

$$\Gamma_t = -\frac{1}{2}\,[\Gamma_s, \Gamma_{ss}] \quad \in \mathfrak{su}(2).$$

4.3 Example - The solitary wave

Consider the solution of the NLS equation which describes a soliton propagating steadily with constant velocity c along the filament which is straight at ∞:

$$\psi(s,t) = \frac{\kappa(s-ct)}{2}\exp\left\{i\int_0^s \tau(\eta - ct)\mathrm{d}\eta\right\},\ \text{with } \kappa = 0 \text{ as } s \to \infty.$$

Then, solving the real and imaginary part of the NLS equation, we get

$$\begin{cases} -c\kappa\,[\tau(s-ct) - \tau(-ct)] = \kappa'' - \kappa\tau^2 - \frac{1}{2}\left(\kappa^2 + A(t)\right)\kappa \\ c\kappa' = 2\kappa'\tau + \kappa\tau' \end{cases}$$

where the $'$ means the derivative with respect to the variable $z = s - ct$.

The second equation is equivalent to $\left((c-2\tau)\frac{\kappa^2}{2}\right)' = 0$, which implies

$$\tau = \frac{c}{2} \quad \text{constant torsion}$$

(we used the boundary condition $\kappa = 0$ as $s \to \infty$). With this value of τ, the first equation becomes

$$\kappa'' + \frac{1}{2}\left(\kappa^2 - 2\nu^2\right)\kappa = 0$$

by choosing $A(t) = 2\left(\tau^2 - \nu^2\right)$, with $\nu \in \mathbb{R}$. The equation can be explicitly integrated, giving

$$\kappa = 2\nu\,\text{sech}\,(\nu(s - ct))$$

Knowing κ and τ, the VFE solution can be now explicitly recovered and numerically integrated.

5 Further models

- Fukumoto-Miyazaki model (finite vortex core and axial flow within the core):

$$\gamma_t = \gamma_s \times \gamma_{ss} + \alpha\left[\gamma_{sss} + \frac{3}{2}\gamma_{ss} \times (\gamma_s \times \gamma_{ss})\right] \quad \alpha \in \mathbb{R}$$

  Using Hasimoto map, it can be shown that γ is a solution of the above equation if and only if ψ_γ is a solution to the Hirota equation (NLS hierarchy)

$$\psi_t = i\psi_{ss} + \frac{i}{2}|\psi|^2\psi + \alpha\left(\psi_{sss} + \frac{3}{2}|\psi|^2\psi_s\right)$$

- Gross-Pitaevskii Equation for Bose-Einstein Condensates: knots in superfluids are identified with closed vortex lines, which can be described through the defocusing Gross-Pitaevskii equation

$$2i\psi_t + \Delta\psi - |\psi|^2\psi = 0$$

- Sine-Gordon equation: given a solution $\psi(s,t)$ of the Sine-Gordon equation

$$\psi_{st} = \sin\psi$$

  then the evolution of the corresponding curve $\Gamma := \Phi^{-1}\Phi_\lambda|_{\lambda=\tau_0}$ is a curve of constant torsion τ_0 and it describes a family of asymptotic lines of a pseudo-spherical surface.

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